SPHERICALLY SYMMETRIC SOLUTIONS OF THE EINSTEIN EQUATIONS FOR A UNIVERSE FILLED WITH RADIATION

T. V. RUZMAĬKINA and A. A. RUZMAĬKIN

Applied Mathematics Institute, U.S.S.R. Academy of Sciences

Submitted April 7, 1970

Zh. Eksp. Teor. Fiz. 59, 1576–1582 (November, 1970)

Spherically symmetric inhomogeneous solutions of the Einstein equations for an expanding universe filled with radiation with the equation of state $p = \epsilon/3$ are considered. For the case of small perturbations imposed on the background of a homogeneous and isotropic Friedmann model a method of solution is employed which differs from the method of Lifshitz [4] who utilized a Fourier expansion. The system of Einstein equations in this approximation reduces to a single third order equation for the perturbation of the density which by simple substitution is reduced to the wave equation describing the propagation of perturbations with the velocity of sound $c/\sqrt{3}$. The general solution of this equation enables one to investigate simply the problem of the behavior of a local perturbation. An analysis is given also of the self-preserving solutions with the variable $\xi = r\sqrt{3}/2\sqrt{t}$. A system of ordinary differential equations is derived for finding the self-similar solutions. An analytic solution of this system is obtained in the linear approximation and its properties are investigated.

1. INTRODUCTION

IN order to investigate the question important in connection with the problem of the formation of galaxies concerning the development of inhomogeneities at an early stage of the expansion of the universe when the density of radiation exceeds considerably the density of matter inhomogeneous cosmological solutions are needed. Of exceptional interest is the verification of the possibility of a strong inhomogeneity of an early stage in the evolution of the universe, in particular of the hypothesis of Novikov concerning nuclei delayed in the expansion.[1] In the hot model accretion of radiation by the delayed nuclei is inevitable. If it turns out to be catastrophically large then this hypothesis will contradict observations.[2]

The only known inhomogeneous solution in the case of expanding matter with the equation of stage $p = \epsilon/3$ is the approximate solution obtained by Lifshitz of the linearized equations of the gravitational field for small perturbations of the Friedmann universe. However, within the framework of the Lifshitz method it is difficult to go over to the investigation of nonlinear inhomogeneous solutions. Moreover, Lifshitz' solution is obtained in a form convenient for the investigation of spectral properties, but not directly applicable to the investigation of the behavior in time of an individual perturbation with given characteristics.

In the present paper a different approach is utilized for finding inhomogeneous solutions for matter with an ultrarelativistic equation of state. We consider a spherically symmetric problem. Unfortunately, the solution of the gravitational equations even in the spherically symmetric case with the equation of state $p = \epsilon/3$ is associated with great mathematical difficulties and, apparently, will be effectively carried out with the aid of electronic computers. However, this does not exclude the possibility of obtaining special and approximate analytic solutions, and the present paper is devoted to this undertaking.

2. SMALL PERTURBATIONS

An arbitrary spherically symmetric gravitational field can be described in a synchronous frame of reference by the metric

$$ds^2 = dt^2 - te^{\lambda(r,t)}dr^2 - tr^2e^{\mu(r,t)}(d\theta^2 + \sin^2\theta\, d\varphi^2). \tag{1}$$

The metric coefficients are so defined that $\lambda = \mu = 0$ corresponds to a Friedmann solution with a flat comoving space filled with matter with the equation of state[1)] $p = \epsilon/3$.

The equations for the gravitational field and the equations of relativistic hydrodynamics contained in them in this case represent a complex system of nonlinear partial differential equations in five unknowns: the metric coefficients λ and μ; the energy density $\epsilon = \epsilon_0(1+\delta)$, $\epsilon_0 = 3/4t^2$ in the units adopted; the temporal u_0 and the radial u_1 components of the 4-velocity. Without writing out in full this system of equations we go directly to special cases when the problem becomes simplified.

First of all we consider small perturbations of the Friedmann universe λ, μ, $\dot\lambda$, $\dot\mu$, λ', μ', δ, $u_1 \ll 1$. In the linear approximation in terms of small quantities the three equations: (0^0)-component of the equations of the gravitational field written in form

$$R_i^k = T_i^k - {}^1/_2\delta_i^kT, \tag{2}$$

and also the equations of hydrodynamics in the form analagous to the equation of continuity $u^iT^k_{i;k} = 0$ and to the Euler equation $(u_iu^k - \delta_i^k)T^l_{k;l} = 0$ form a closed system of equations for the quantities $\delta(r, t)$, $u_1(r, t)$, $h(r, t) \equiv \lambda + \mu$:

$$t^2\ddot h + t\dot h = -{}^3/_2\delta, \tag{3a}$$

$$\sqrt{t}r^2\left(\dot h + \frac{3}{2}\dot\delta\right) = 2\left(\frac{r^2}{t}u_1\right)', \tag{3b}$$

1) We introduce the arbitrary constant appearing in the Friedman solution $a = \sqrt{2a_0ct}$ [3] into the definition of r. The system of units is adopted in which $c = 8\pi G = 1$ (c is the velocity of light, G is the gravitational constant).

$$\left(\frac{u_1}{\sqrt{t}}\right)^{\cdot} = \frac{1}{4\sqrt{t}}\delta'. \tag{3c}$$

Eliminating from (3a) and (3b) the quantity h, and then dividing the equation so obtained by $\sqrt{t}$ and differentiating it with respect to time we obtain with the aid of (3c) the equation for $\delta(r, t)$:

$$t^3\dddot{\delta} + \frac{5}{2}t^2\ddot{\delta} - t\left(\frac{1}{2} + \frac{t}{3}\Delta\right)\dot{\delta} + \left(\frac{1}{2} - \frac{t}{3}\Delta\right)\delta = 0;$$
$$\Delta = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right). \tag{4}$$

Among the solutions of Eq. (4) there are contained solutions which describe the perturbation of the energy density associated with the arbitrariness in the choice of the initial time-like hypersurface t = const. As is well known,[4] the form of such a solution can be established without solving Eq. (4). For $p = \epsilon/3$ the coordinate perturbations of the density have the form $\delta = F(r)/t$ where F is an arbitrary function of the spatial variables. This enables us to reduce the order of Eq. (4).

Introducing the new unknown function

$$z = \frac{1}{\sqrt{t}}\frac{\partial}{\partial t}(t\delta) \tag{5}$$

and the independent time variable $\eta = 2\sqrt{t}$ we obtain in place of (4) the wave equation[2)]

$$\frac{\partial^2 z}{\partial \eta^2} - \frac{1}{3}\Delta z = 0,$$

which describes the propagation of perturbations with the velocity of sound $c/\sqrt{3}$ (c = 1). As is well known, this equation for the function rz reduces to the one dimensional wave equation which can be easily integrated if for the independent variables one chooses the characteristics $r - \eta\sqrt{3}$ and $r + \eta/\sqrt{3}$. The general solution for z has the form

$$z(r, \eta) = \frac{f_1(r - \eta/\sqrt{3}) + f_2(r + \eta/\sqrt{3})}{r}. \tag{6}$$

Thus, the function z represents the sum of a diverging and a converging spherical wave with an amplitude falling off as r^{-1}. If at the initial instant the functions $f_1(r)$ and $f_2(r)$ differed from zero only for $0 < r < r_0$ then after a sufficiently long interval of time has elapsed the perturbation $z(r, \eta)$ will represent a diverging wave with a leading and a trailing wave front.

The behavior of the perturbations of the density arising at time t_0 in accordance with (5) and (6) is described by the formula

$$\delta(r, t) = \frac{3\sqrt{3}}{4rt}\left\{\int\limits_{2\sqrt{t_0/3}-r}^{2\sqrt{t/3}-r} f_1(x)(x+r)^2\,dx + \int\limits_{2\sqrt{t_0/3}+r}^{2\sqrt{t/3}+r} f_2(x)(x-r)^2\,dx\right\} + \frac{t_0}{t}\delta_0(r, t_0). \tag{7}$$

(The last term describes the aforementioned perturbations of the density associated with coordinate transformations.)

The density at the point r at time t is determined in the diverging wave by the values of the function $f_1(x)$ for all x from $2\sqrt{t_0/3} - r$ to $2\sqrt{t/3} - r$ and in the converging wave by the values of $f_2(x)$ in the interval from $2\sqrt{t_0/3} + r$ to $2\sqrt{t/3} + r$. If the perturbation has reached a given spatial point, then it no longer stops at that point. But the amplitude δ in the region encompassed by the perturbation for sufficiently large values of the time falls off as t^{-1}, since if $f_1(x)$ and $f_2(x)$ are not equal to zero in a finite volume, then for any fixed value of r the integrals (7) depend on the time only as long as $2\sqrt{t/3} - r < r_0$. After that $\delta \sim t^{-1}$, i.e., the small local perturbations that arise will finally be damped out.

The solution obtained above agrees with the results of E. Lifshitz[3] as can be easily verified by solving the wave equation for z by the Fourier method or by investigating directly the solution (6). For example, in the case of a large scale perturbation, the dimensions of which $r\sqrt{t}$ exceed the dimensions of the horizon of t ($r \gg \sqrt{t}$), expansion of the function z in a series in terms of the parameter $\sqrt{t}/r$ with an accuracy up to terms of the first order of smallness and subsequent integration yield

$$\delta \approx A_1(r)t + A_2(r)\sqrt{t} + \frac{A_3(r)}{t}$$

(A_1, A_2, A_3 are expressed in terms of $f_1(r)$, $f_2(r)$, their derivatives and $\delta_0(r)$.) This agrees with the time dependence obtained by Lifshitz for the longwave Fourier components of the perturbations of the density. The growth of perturbations described by this formula is restricted by the time $\sqrt{t} \sim r$.

For a local perturbation, in which at the initial instant the perturbation of the density and its time derivative differ from zero within a finite volume, this growth, as can be seen from the preceding, will be replaced by damping,[3)] if, of course, the perturbation δ has not reached the value of unity when nonlinear effects become essential. The phenomenon of catastrophic accretion of radiation by a small inhomogeneity which manifests itself in its uninterrupted growth does not occur within the framework of the linear problem.

3. SELF-SIMILAR SOLUTIONS

Another possible simplification of the problem is a search for self-similar solutions depending on a single variable. Since in this case the coordinate scale of the region encompassed by the perturbation varies proportionally to t, while lengths increase with time as $\sqrt{t}$, the self-similar variable, as has been pointed out by Zel'dovich[2] must have the form $\xi = r\sqrt{3}/2\sqrt{t}$. The coefficient of proportionality is chosen to satisfy the condition that ξ would be equal to unity on the sound horizon, i.e., at the point where the velocity of the Friedmann ($\lambda = \mu = 0$) reference frame $v = \sqrt{t}\,dr/dt$ becomes equal to the velocity of sound $1/\sqrt{3}$.

We write out the system of the ordinary differential equations of the gravitational field and of hydrodynamics (2) for the self-similar solutions $\lambda(\xi)$, $\mu(\xi)$, $\delta(\xi)$, $u_0(\xi)$, $\sqrt{u_1 u^1}(\xi)$:

[2)]One can show that this equation, as well as (4), is also valid for perturbations of arbitrary form (not necessarily spherically symmetric).

[3)]In contrast to a shortwave monochromatic perturbation which represents a sound wave with an amplitude independent of the time [3].

$$\xi^2(\lambda''+2\mu'')+\xi(\lambda'+2\mu')+{}^1/_2\xi^2(\lambda'^2+2\mu'^2)=-6\delta+8u_1u^1(1+\delta), \tag{8a}$$

$$\xi^2\lambda''-\xi(\lambda'+2\mu')+\frac{\xi^2}{2}(\lambda'^2+\lambda'\mu')-3e^{-\lambda}\Big[2\mu'' + \frac{2}{\xi}(2\mu'-\lambda')+\mu'^2-\lambda'\mu'\Big]=2\delta-8u_1u^1(1+\delta), \tag{8b}$$

$$\xi^2\mu''-\xi(\lambda'+2\mu')+\frac{\xi^2}{2}(2\mu'^2+\lambda'\mu')-3e^{-\lambda}\Big[\mu''+\frac{4}{\xi}\mu' -\frac{\lambda'}{\xi}-\frac{\lambda'\mu'}{2}\Big]+\frac{3}{\xi^2}(e^{-\mu}-e^{-\lambda})=2\delta, \tag{8c}$$

$$\xi\mu''+2\mu'-\lambda'+{}^1/_2\xi(\mu'^2-\mu'\lambda')=2(1+\delta)u_1\sqrt{1-u_1u^1}, \tag{8d}$$

$$\xi^3[(1+\delta)^{3/4}\sqrt{1-u_1u^1}e^{\lambda/2+\mu}]'+\sqrt{3}[\xi^2(1+\delta)^{3/4}u_1e^{-\lambda/2+\mu}]'=0, \tag{8e}$$

$$\frac{2}{\sqrt{3}}u_1'\xi\sqrt{1-u_1u^1}-2u_1'u^1+2u_1u^1\lambda' +\Big[1+\frac{\xi}{\sqrt{3}}u_1\sqrt{1-u_1u^1}-u_1u^1\Big]\frac{\delta'}{2(1+\delta)}=0, \tag{8f}$$

$$u_0^2+u_1u^1=1. \tag{8g}$$

Of course, this system is still sufficiently complicated. But its solution by means of electronic computers no longer presents any difficulties in principle. In the linear approximation we can find an analytic solution of this system.[4)]

In a manner similar to the one employed above we obtain from equations (8a), (8e), (8f) the differential equation for $\delta(\xi)$:

$$(1-\xi^2)\xi\delta'''+2(1-2\xi)\delta''+2(\xi-1/\xi)\delta'+4\delta=0. \tag{9}$$

It has two special power solutions: $\delta\sim\xi^{-1}$, $\delta\sim\xi^2$. (The second of them $\sim r^2/t$ is associated with the choice of the coordinate system.) With the aid of these solutions we can by two consecutive substitutions reduce (9) to a linear equation of the first order with respect to the quantity

$$Q=[\xi^4(\delta/\xi^2)']',$$

which has the simple form

$$(1-\xi^2)Q'-2\xi Q=0$$

and can be easily integrated: $Q=-6C_1/|\xi^2-1|$. Returning to δ we obtain

$$\delta(\xi)=C_1\Big[1+\Big(\xi^2-\frac{1}{\xi}\Big)\ln|1-\xi| +\Big(\xi^2+\frac{1}{\xi}\Big)\ln(1+\xi)\Big]+\frac{C_2}{\xi}+C_3\xi^2. \tag{10}$$

This solution is contained in the general solution of the linearized equations (9) and corresponds to the choice

$$f_1(x)=-3C_1\ln|x|+3C_2,\quad f_2(x)=3C_1\ln|x|. \tag{11}$$

Asymptotically for large ξ in frame of reference in which $C_3=0$, δ falls off as ξ^{-1}, if $C_2\neq 0$, and as ξ^{-2} in the opposite case. Thus, formula (10) describes a perturbation that encompasses all of space. On the contrary, of physical interest are localized solutions which outside a certain finite volume coincide with the Friedmann solution. Such models enable one to construct a model of the universe which is homogeneous only on the average, and, in particular, to describe the picture of nuclei delayed in the expansion of a Friedmann universe.

4)In the linear problem there exist self-similar solutions also of a more general form $\delta=t^\gamma\delta(\xi)$, $\gamma>-½$. In the nonlinear case—only with $\gamma=0$.

The linear self-preserving solution found above can, on giving up the continuity of the first derivatives of density and of velocity, be localized by means of joining it at the sound horizon $\xi=1$ with the Friedmann solution which is valid for $\xi>1$. Such a solution can describe the picture of the development of a local inhomogeneity. Indeed, if in homogeneously distributed matter a local increase of density occurs, then around it there will be formed a region of reduced density. At the boundary between this region and the unperturbed substance a wave of load redistribution arises which is propagated with the velocity of sound and in which, as is well known, the density and the velocity of the substance remain continuous, while their derivatives become discontinuous. The condition (10) of joining at $\xi=1$ to the Friedmann solution ($\delta(1)=0$) imposes one relation on three arbitrary constants

$$C_1(1+2\ln 2)+C_2+C_3=0.$$

However, the requirement of the change of sign of the perturbation of the density at $\xi<1$ brings out one unpleasant characteristic of solution (10). It can easily be seen that δ is a negative near the sound horizon for $C_1<0$. If this inequality is satisfied then the positiveness of δ for small ξ, where $\delta\approx C_1(3-2\xi^2\ln\xi)+C_2/\xi+C_3\xi^2$ can be attained only under the condition $C_2>0$. But the presence in the expression for the density of a term which increases for $\xi\to 0$ as ξ^{-1}, denotes the presence at the center of a source of particles.

Indeed, we write the equation $u^iT^k_{i;k}=0$, which is analogous to the equation of continuity in nonrelativistic hydrodynamics in integral form which describes the variation of the number of particles inside the sphere of radius R:

$$\frac{d}{dt}N(R)=-P(R).$$

Here

$$N(R)=\int_0^R(1+\delta)^{3/4}u_0e^{\lambda/2+\mu}r^2\,dr$$

is the number of particles inside a sphere of radius R,

$$P(R,t)=R^2(1+\delta)^{3/4}u^1e^{-\lambda/2+\mu}$$

is the flux of particles over the surface of the sphere.

In the linear approximation $P(R,t)=-u_1(t,R)R^2/t$. The velocity appearing here can be obtained from equations (8):

$$u_1(\xi)=-\frac{\sqrt{3}}{8}\Big\{C_1\Big[-\frac{2}{\xi}+\Big(4\xi-3-\frac{1}{\xi^2}\Big)\ln|1-\xi| +\Big(4\xi+3+\frac{1}{\xi^2}\Big)\ln(1+\xi)-8\xi\ln\xi\Big]+\frac{C_2}{\xi^2}+4C_3\xi+C_4\Big\}.$$

For $\xi\to 0$ the principal term in $u_1(\xi)\sim\xi^{-2}$ and the flux $P(R,t)\approx C_2\sqrt{t}/2\sqrt{3}$ does not vanish if $C_2\neq 0$, and this is what denotes the presence of a source at $r=0$.

The presence of a source in the linear self-similar solution does not permit us to use it directly for describing the development of a local inhomogeneity. However, this singularity of the solution (10) is associated with extrapolating it into the region of small ξ where the validity of the linear approximation is violated. Apparently the nonlinear solution will not contain such a singularity. In order to construct a nonlinear self-similar solution it will be necessary to solve the system (8) numerically. In doing so it is useful to keep in mind that in the neighborhood of $\xi = 1$ the nonlinear terms in (8) are not essential and the linear approximation is valid.[5)] In the case of a numerical integration of the system (8) in the range $1 > \xi > 0$ the region $\xi \sim 1$ will be described by the linear solution.

[5)]From the continuity of the density and of the velocity on the sound horizon, as can be seen from the equations for the gravitational field in (8), follows the continuity of λ, μ, λ', μ', as $\xi \to 1$. Therefore, in the neighborhood of $\xi \sim 1$ the nonlinear terms in the equations are not essential ($\lambda' \gg \lambda'^2$).

We use this opportunity to express our gratitude to Ya. B. Zel'dovich, A. G. Doroshkevich, and I. D. Novikov for valuable advice and for critical comments.

[1]I. D. Novikov, Astron. Zh. 41, 1075 (1964) [Sov. Astron.-AJ 8, 857 (1965)].

[2]Ya. B. Zel'dovich and I. D. Novikov, Relyativistskaya astrofizika (Relativistic Astrophysics), Fizmatgiz, 1967.

[3]L. D. Landau and E. M. Lifshitz, Teoriya polya (Field Theory), Fizmatgiz, 1967.

[4]E. M. Lifshitz, and I. M. Khalatnikov, Usp. Fiz. Nauk 80, 391 (1963) [Sov. Phys.-Usp. 6, 495 (1964)].

Translated by G. Volkoff
183