THE PROPAGATION OF ELECTROMAGNETIC WAVES IN A RIEMANNIAN SPACE

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We here derive and study equations which describe the propagation of electromagnetic waves in a Riemannian space with an arbitrary metric, as if in an anisotropic "medium." The character of the anisotropy is determined by the metric tensor. An analysis of the equations shows that there is never any double refraction of electromagnetic waves in an arbitrary gravitational field. A necessary and sufficient condition for the absence of double refraction in an anisotropic medium is that the tensors of dielectric and of magnetic permeability be proportional. A general expression is found which permits determination of the change of the parameters characterizing the polarization of an electromagnetic wave when it is propagated in a space with a given metric.

INTRODUCTION

THE problem indicated in our title was essentially formulated in papers by Mandel'shtam[1] and by Tamm,[2] who generalized the Minkowski equations of macroscopic electrodynamics to the case of an arbitrary anisotropic medium in translational motion in an inertial reference system. It was shown that the equations describing the propagation of electromagnetic waves in an electrically and magnetically anisotropic (i.e., a bianisotropic) system without dispersion coincide in form with the Maxwell equations written in covariant form. Here the metric is characterized by an interval which always has a null value, and which in the general case is biquadratic,

$$ds^4 = h_{\alpha\beta\gamma\nu}dx^\alpha dx^\beta dx^\gamma dx^\nu,$$

where the components of the fourth-rank tensor $h_{\alpha\beta\gamma\nu}$ are the components of the dielectric tensor ϵ_{ik} and the magnetic permeability tensor μ_{ik} of the medium. In vacuum and in isotropic material media (in an inertial system in the absence of gravitational fields) the tensor $h_{\alpha\beta\gamma\nu}$ breaks up into two identical tensors. In this case the metric is characterized by a quadratic interval. From this it was concluded in[1,2] that a biquadratic interval is associated with double refraction of light. This conclusion, however, is neither necessary nor sufficient.

In the present paper we derive and study equations which describe the propagation of electromagnetic waves in a Riemannian space with an arbitrary metric. The Fresnel equation for the wave-vector surface is derived. In addition it is shown that a necessary and sufficient condition for the absence of double refraction in an arbitrary anisotropic not optically active medium is that the dielectric and magnetic permeability tensors be proportional to each other.

2. THE FUNDAMENTAL EQUATIONS OF ELECTRODYNAMICS IN A RIEMANNIAN SPACE

In the absence of charges and currents the propagation of electromagnetic waves in a Riemannian space with a given metric is described by the covariant Maxwell equations (see, e.g.,[3,4])

$$F_{\alpha\beta;\gamma}+F_{\beta\gamma;\alpha}+F_{\gamma\alpha;\beta}=0, \quad H^{\alpha\beta}{}_{;\beta}=0 \tag{1}$$

where $F_{\alpha\beta}$ and $H^{\alpha\beta}$ are respectively the covariant and the contravariant electromagnetic-field tensors. Indices set off with a semicolon denote covariant derivatives with respect to the corresponding coordinates (here and in what follows Greek indices $\alpha, \beta, \gamma, \ldots$ run through the values 0, 1, 2, 3, and Latin indices i, j, k, ... through the values 1, 2, 3, and summation is taken over any repeated index).

In vacuum the components of the electromagnetic-field tensors $F_{\alpha\beta}$ and $H^{\alpha\beta}$ are connected by the relations[3,4]

$$\sqrt{-g}\,H^{\alpha\beta}=\sqrt{-g}\,g^{\alpha\gamma}g^{\beta\nu}F_{\gamma\nu} \text{ and } F_{\alpha\beta}=\frac{1}{\sqrt{-g}}g_{\alpha\gamma}g_{\beta\nu}\sqrt{-g}\,H^{\gamma\nu}, \tag{2}$$

where $g = \det \| g_{\alpha\beta} \|$ is the determinant of the covariant metric tensor.

Let us introduce electromagnetic-field vectors according to the scheme[3,5]

$$F_{\alpha\beta}\to(\mathbf{E},\mathbf{B}), \quad H^{\alpha\beta}\to(-\mathbf{D},\mathbf{H}).$$

Writing Eqs. (1) in terms of vectors, we get a system of differential equations identical in form with the Maxwell equations for the electromagnetic field in a material medium:

$$\operatorname{rot}\mathbf{E}+\frac{1}{c}\frac{\partial\mathbf{B}}{\partial t}=0, \quad \operatorname{rot}\mathbf{H}-\frac{1}{c}\frac{\partial\mathbf{D}}{\partial t}=0,$$

$$\operatorname{div}\mathbf{B}=0, \quad \operatorname{div}\mathbf{D}=0. \tag{3}$$

In this case the equations giving the connection between the "inductions" D, B and the field intensities E, H are determined by the relations (2) and are of the form

$$D_i=\sqrt{-g}\{g^{00}\gamma^{im}E_m+g^i(\mathbf{gE})-\gamma^{im}[\mathbf{gB}]_m\},$$
$$H_i=\sqrt{-g}\,e_{ijk}g^k\gamma^{jm}E_m+\frac{g_{00}}{\sqrt{-g}}\gamma_{im}B_m, \tag{4*}$$

where

$$\gamma^{im}=-g^{im}, \quad \gamma_{im}=-g_{im}+\frac{g_{0i}g_{0m}}{g_{00}},$$

$\mathbf{g} = (-g^{01}, -g^{02}, -g^{03})$, and e_{ikm} is the completely antisymmetric pseudotensor of the third rank ($e_{123} = 1$).

It is convenient to write the relations (4) in the following form:

$$D_i=\epsilon_{ik}E_k-[\mathbf{GH}]_i, \quad B_i=\mu_{ik}H_k+[\mathbf{GE}]_i. \tag{5}$$

*$[\mathbf{gB}] \equiv \mathbf{g}\times\mathbf{B}$.

Here ϵ_{ik}, μ_{ik}, and G_i denote the quantities

$$\varepsilon_{ik} = \mu_{ik} = \sqrt{-g}(g_{00})^{-1}\gamma^{ik}, \quad G_i = -g_{0i}(g_{00})^{-1}. \tag{6}$$

The differential equations (3), in which **D** and **H** denote the quantities (4), completely describe the propagation of electromagnetic waves in a Riemannian space with an arbitrary metric. The relations (4) or (5) play the role of the constitutive equations, and Eqs. (6) define the tensors of the effective electric and magnetic permeabilities. The electromagnetic properties of the "medium" are determined by the metric tensor $g_{\alpha\beta}(x, y, z, t)$.

It must be pointed out that the equations (1)—(2) and (3)—(5) are equivalent in any coordinate system, since they differ only in the forms in which they are written. The covariant Maxwell equations (1) can always be written in the form of the Maxwell equations for a material medium. The peculiarities of the noninertial reference system and of the geometry of the space in question affect only the form of the "constitutive" equations which give the connection between the field vectors **E**, **B**, **D**, and **H**. Equations (1)—(5) are written in the reference system in which the components of the metric tensor are defined. The differences between the metric defined by the $g_{\alpha\beta}$ and the pseudoeuclidean metric are due both to the actual gravitational field produced by gravitating bodies and to the use of a noninertial reference system.

In our notation the components of the energy-momentum tensor of the electromagnetic field

$$4\pi T_\alpha{}^\beta = -F_{\alpha\gamma}H^{\beta\gamma} + {}^1/_4 F_{\gamma\nu}H^{\gamma\nu}\delta_\alpha{}^\beta$$

take the form

$$\sqrt{-g}\,T_0{}^0 = \frac{1}{8\pi}\{\mathbf{ED} + \mathbf{BH}\} = W,$$

$$\sqrt{-g}\,T_0{}^i = \frac{1}{4\pi}[\mathbf{EH}]_i = \frac{1}{c}S_i, \quad \sqrt{-g}\,T_i{}^0 = \frac{1}{4\pi}[\mathbf{DB}]_i,$$

$$\sqrt{-g}\,T_i{}^k = \frac{1}{4\pi}\left\{E_iE_k + H_iB_k - \frac{1}{2}[\mathbf{ED} + \mathbf{BH}]\delta_i{}^k\right\}.$$

When we use the "constitutive" relations (5) the energy density of the electromagnetic field can be easily put in the following form:

$$\sqrt{-g}\,T_0{}^0 = W = \frac{\varepsilon_{ik}}{8\pi}\{E_iE_k + H_iH_k\} + \frac{1}{c}\mathbf{GS}.$$

The additional term in this expression can be given a simple physical interpretation. If $\mathbf{G} = c^{-1}\mathbf{\Omega} \times \mathbf{r}$, then for the integral quantity $U = \int W dv$ we find $U = U_0 + \mathbf{\Omega N}$, where

$$U_0 = \frac{1}{8\pi}\int \varepsilon_{ik}(E_iE_k + H_iH_k)\,dv,$$

and $\mathbf{N} = \int [\mathbf{r} \times \mathbf{P}]\,dv$ is the angular momentum of the electromagnetic field. As is well known, the expression for U determines the law of transformation of the energy on going over to a reference system rotating uniformly with the angular velocity $\mathbf{\Omega}$.

3. THE FRESNEL EQUATION

In order to find the equation which gives the relation between the effective index of refraction of the "medium" and the quantities ϵ_{ik} and μ_{ik} and the direction of propagation of an electromagnetic wave, we choose a coordinate system which is convenient for this purpose. By a spatial rotation of the coordinate system one can always bring the metric tensor into a form in which the only nonvanishing off-diagonal components are g^{01}, g^{02}, g^{03}. In a coordinate system oriented in this way the tensor $\epsilon_{ik} = \mu_{ik}$ is diagonal.

Plane electromagnetic waves exactly satisfy the Maxwell equations only in the case in which the components $g_{\alpha\beta}$ do not depend on the coordinates. If, however, the components $g_{\alpha\beta}(x, y, z, t)$ of the metric tensor change only slightly in a wavelength λ and in an oscillation period T, i.e., if the conditions $\lambda\,|\,\mathrm{grad}\, g_{\alpha\beta}\,| \ll 1$ and $T\,|\,\partial g_{\alpha\beta}/\partial t\,| \ll 1$ are satisfied for the given values $g_{\alpha\beta}$, the idea of plane waves is entirely applicable in restricted regions of space-time. In this approximation, equivalent to Rytov's first approximation,[3,6] we have

$$\mathbf{E}, \mathbf{D}, \mathbf{B}, \mathbf{H} \sim \exp\{i(\mathbf{kr}n - \omega t)\},$$

and the Maxwell equations (3) take the form

$$n[\mathbf{eE}] = \mathbf{B}, \quad n[\mathbf{eH}] = -\mathbf{D}, \qquad \mathbf{eB} = 0, \quad \mathbf{eD} = 0, \tag{7}$$

where $\mathbf{e} = \mathbf{k}/k$ is the unit vector in the direction of propagation of the wave, and n is the effective index of refraction.

It follows from Eqs. (5) and (7) that

$$\varepsilon_{ik}E_k + [\mathbf{n} - \mathbf{G}, \mathbf{H}]_i = 0,$$

$$\mu_{ik}H_k - [\mathbf{n} - \mathbf{G}, \mathbf{E}]_i = 0, \quad \mathbf{n} = n\mathbf{e}.$$

Eliminating the vector **H** from these last equations, we arrive at a system of three linear homogeneous equations for the components of the vector **E**:

$$\{\varepsilon_{ik} + e_{isq}e_{kpm}(n_s - G_s)\mu_{qp}{}^{-1}(n_m - G_m)\}E_k = 0. \tag{8}$$

An analogous equation holds for the vector **H**.

Equating the determinant of the system of homogeneous equations (8) to zero and expanding it, we find the generalized Fresnel equation for the wave-vector surface in the anisotropic "medium":

$$\sum(n_1 - G_1)^4\varepsilon_1\mu_1 + \sum(n_1 - G_1)^2(n_2 - G_2)^2(\varepsilon_1\mu_2 + \varepsilon_2\mu_1) - \sum(n_1 - G_1)^2\varepsilon_1\mu_1(\varepsilon_2\mu_3 + \varepsilon_3\mu_2) + \varepsilon_1\mu_1\varepsilon_2\mu_2\varepsilon_3\mu_3 = 0, \tag{9}$$

where ϵ_i and μ_i are the components of the diagonal tensors ϵ_{ik} and μ_{ik}, and the sign Σ indicates summation of the three terms obtained by cyclic permutation of the indices 1, 2, 3 in the expression following it.

In the expansion of the determinant the terms of sixth degree in the n_i cancel each other. Equation (9) is of the fourth degree in n_i. To the two positive values of n there correspond two directions of polarization of the electromagnetic waves. The connection between the components of the fields, and the character of the polarization of the field, are determined by Eq. (8).

Equation (9) is a generalization of the Fresnel equation derived in Tamm's paper,[2] and is identical with it in the case G = 0. For given $g_{\alpha\beta}$ Eq. (9) determines the wave-vector surface.

In the general case of ordinary biaxial anisotropic material media at rest in an inertial reference system, the wave-vector surface and the ray surface are two-sheeted self-intersecting surfaces of fourth order. Plane waves propagated in an anisotropic medium are completely linearly polarized in definite planes (cf., e.g.,[7]). The Fresnel equation (9) for an arbitrary

Riemannian space with a given metric in vacuum also determines a two-sheeted fourt-order surface. But for this "medium" we always have $\epsilon_{ik} = \mu_{ik}$. The fourth-order equation (9) breaks up into two identical second-order equations:

$$\left\{\frac{(n_1-G_1)^2}{\varepsilon_2\varepsilon_3}+\frac{(n_2-G_2)^2}{\varepsilon_1\varepsilon_3}+\frac{(n_3-G_3)^2}{\varepsilon_1\varepsilon_2}-1\right\}^2=0. \tag{10}$$

Accordingly, in this case the wave-vector surface consists of two identical and coinciding triaxial ellipsoids, the respective semiaxes being $(\epsilon_2\epsilon_3)^{1/2}$, $(\epsilon_1\epsilon_3)^{1/2}$, $(\epsilon_1\epsilon_2)^{1/2}$. With respect to both electromagnetic waves (the "ordinary" and the "extraordinary") the Riemannian space behaves like an anisotropic medium in which there is no double refraction. Consequently, in a gravitational field there is no double refraction of electromagnetic waves. The polarizations of the waves are superposed, giving in general an elliptically polarized wave.

It is easily seen from Eq. (10) that the wave ellipsoid is displaced relative to the origin of coordinates by the amounts G_i along the axes of n_i. This means that the index of refraction of the gravitational "medium" (and also the wave vector and the speed of propagation of light) is different not only for electromagnetic waves travelling along different lines, but also for waves propagated in opposite directions along the same line. In the general case the space (the "medium") is nonreciprocal with respect to oppositely directed plane wave trains. In this way the gravitational "medium" is essentially different from a material anisotropic medium which is at rest in an inertial reference system with no gravitational field. The wave ellipsoid (10) has six principal directions. The principal values of the index of refraction are

$$n_{1\pm}=\left[\varepsilon_2\varepsilon_3\left(1-\frac{G_2^2}{\varepsilon_1\varepsilon_3}-\frac{G_3^2}{\varepsilon_1\varepsilon_2}\right)\right]^{1/2}\pm G_1, \tag{11}$$

where the "+" sign corresponds to an electromagnetic wave propagated in one direction and the sign "-" to a wave in the opposite direction. To get the principal values $n_{2\pm}$ ($n_{3\pm}$) one must replace the index 1 by the index 2 (3) and the index 2 (3) by the index 1 everywhere in (11).

We note that if we examine the shape of the sections of the ellipsoid (10) by the coordinate planes (for example, by the xy plane), we get

$$\frac{(n_1-G_1)^2}{\varepsilon_2\varepsilon_3(1-G_3^2/\varepsilon_1\varepsilon_2)}+\frac{(n_2-G_2)^2}{\varepsilon_1\varepsilon_3(1-G_3/\varepsilon_1\varepsilon_2)}-1=0, \tag{12}$$

i.e., the section is an ellipse. The lengths of the semiaxes of this ellipse are smaller than the corresponding semiaxes of the ellipsoid (10). The explanation is that the coordinate plane making the intersection is displaced relative to the center of the ellipsoid. The center of the intersection ellipse is displaced relative to the origin of coordinates by the amounts G_1 and G_2. There is an analogous situation for the other coordinate-plane intersections with the wave-vector surface.

The results obtained above also allow us to draw more general conclusions. Let us consider an anisotropic not optically active material medium which is at rest in an inertial reference system with no gravitational field. Let the principal axes of the dielectric constant tensor ϵ_{ik} and the magnetic permeability tensor μ_{ik} coincide. We choose a coordinate system whose axes coincide with the principal axes of the tensors ϵ_{ik} and μ_{ik}. The Fresnel equation for such a medium is the same as Eq. (10) with $G = 0$. In a material medium ϵ_{ik} and μ_{ik} have dispersion. If in a certain range of frequencies of electromagnetic waves $\epsilon_{ik} = \kappa\mu_{ik}$, then the two-sheeted wave-vector surface of the fourth order degenerates into two equivalent second-order surfaces:

$$\left\{\frac{\varkappa n_x^2}{\varepsilon_y\varepsilon_z}+\frac{\varkappa n_y^2}{\varepsilon_x\varepsilon_z}+\frac{\varkappa n_z^2}{\varepsilon_x\varepsilon_y}-1\right\}^2=0, \quad \varepsilon_i=\varkappa\mu_i.$$

In such a frequency range the medium behaves in the same way with respect to the ordinary and extraordinary electromagnetic waves. The birefringence disappears.

Accordingly, a necessary and sufficient condition for the absence of birefringence in an anisotropic medium (independent of the cause of the anisotropy) is that the dielectric tensor ϵ_{ik} and the magnetic permeability tensor ϵ_{ik} be proportional to each other. Consequently, by regulating the magnetic permeability μ_{ik} of a medium one can eliminate the phenomenon of double refraction in an anisotropic medium.

The Fresnel equation (10) allows us to find the effective index of refraction of a "medium" that imitates a Riemannian space of arbitrary metric, as a function of the direction of propagation of an electromagnetic wave. In first approximation with respect to $|\mathbf{G}|$ we find

$$n=\left[\frac{\varepsilon_1\varepsilon_2\varepsilon_3}{\varepsilon_1\sin^2\theta\cos^2\psi+\varepsilon_2\sin^2\theta\sin^2\psi+\varepsilon_3\cos^2\theta}\right]^{1/2}\pm$$
$$\pm\frac{\varepsilon_1G_1\sin\theta\cos\psi+\varepsilon_2G_2\sin\theta\sin\psi+\varepsilon_3G_3\cos\theta}{\varepsilon_1\sin^2\theta\cos^2\psi+\varepsilon_2\sin^2\theta\sin^2\psi+\varepsilon_3\cos^2\theta}$$

where ψ and θ are angles determining the direction of propagation of the wave relative to the axes of the chosen coordinate system.

If in a gravitational field $G = 0$, then the center of the wave-vector ellipsoid (10) coincides with the origin of coordinates. The gravitational "medium" behaves like a material anisotropic medium at rest in an inertial reference system and having no birefringence.

If $\epsilon_1 = \epsilon_2 = \epsilon_3 = \epsilon$, $G \neq 0$, which is the case, for example, for a rotating centrally symmetric gravitating mass[3,4] $[\epsilon = (-g)^{1/2}(g_{00})^{-1}g^{11}$, $\mathbf{G} = 2k[\mathbf{MR}]/c^3R^3$, where $\mathbf{M}$ is the angular momentum of the body], then the gravitational "medium" has a preferred direction along the vector $\mathbf{G}$. In this case the additional term $(\mathbf{GS})/c$ in the expression for the energy density is equal to $-2k(\mathbf{L}\cdot\mathbf{M})/c^3R^3$, where $\mathbf{L} = \mathbf{R}\times\mathbf{P}$ is the density of orbital angular momentum of the electromagnetic field. If $\mathbf{L}\cdot\mathbf{M} \neq 0$, the effect described in[8] occurs. The wave-vector ellipsoid (10) degenerates into a sphere of radius ϵ, whose center is displaced relative to the origin by the amount $\mathbf{G}$. The index of refraction of such a "medium" has the value found in[9]:

$$n \approx \varepsilon + \mathbf{e}\mathbf{G}.$$

The phase velocity of a wave depends on the direction of propagation:

$$v_{max}=\frac{c}{\varepsilon-G}\approx\frac{c}{\varepsilon}+\frac{cG}{\varepsilon^2}, \quad v_{min}=\frac{c}{\varepsilon+G}\approx\frac{c}{\varepsilon}-\frac{cG}{\varepsilon^2},$$

where $G = |\mathbf{G}|$.

We also arrive at a result of this kind when a dielectric medium is in translational motion relative to an inertial reference system in the absence of a gravitational field. The constitutive relations for such a medium[7] are of the form

$$\mathbf{D}+\frac{1}{c}[\mathbf{vH}]=\varepsilon\left(E+\frac{1}{c}[\mathbf{vB}]\right),$$

$$\mathbf{B}+\frac{1}{c}[\mathbf{Ev}]=\mu\left(\mathbf{H}+\frac{1}{c}[\mathbf{Dv}]\right).$$

Here ϵ and μ are the dielectric and magnetic permeabilities of the medium, and $\mathbf{v}$ is the velocity of motion of the medium relative to the inertial reference system. The wave-vector surface for an isotropic medium moving along the x axis of an inertial reference system,

$$\left\{\left(n_x+\frac{\varepsilon\mu-1}{c}v_x\right)^2+n_y^2+n_z^2-\varepsilon\mu\right\}^2=0,$$

consists of two coincident spheres of radius $(\epsilon\mu)^{1/2}$ with center displaced relative to the origin. The effective refractive index is

$$n=\sqrt{\varepsilon\mu}-\frac{\varepsilon\mu-1}{c}\mathbf{ev}.$$

The speed of light depends on the direction of its propagation. A moving medium is equivalent to an anisotropic medium without double refraction at rest in an inertial reference system with no gravitational field.

From the relations (3) derived here one can easily find the law of the change of the plane of polarization of an electromagnetic wave propagated in the stationary gravitational field of rotating masses[3,10]:

$$\frac{d\varphi}{ds}=\frac{1}{T_\text{K}}+\frac{1}{2}\boldsymbol{\tau}\,\mathrm{rot}\,\boldsymbol{\tau},$$

where φ is the angle between the vector $\mathbf{E}$ and the principal normal to the ray, s is arc length measured along the ray direction τ, and T_t is the radius of torsion of the ray. The effective index of refraction of the "medium" for right-circularly polarized electromagnetic waves is different from the effective index of refraction for left-circularly polarized waves. By the method given in[11] we can easily determine the angle of rotation of the plane of polarization of an electromagnetic wave which is propagated along a given trajectory from a point a in the space to a point b:

$$\Phi=\frac{\omega}{c}\int_a^b\frac{\varepsilon_1G_1\,dl_1+\varepsilon_2G_2\,dl_2+\varepsilon_3G_3\,dl_3}{\varepsilon_1e_1^2+\varepsilon_2e_2^2+\varepsilon_3e_3^2},\qquad(13)$$

where dl = edl, and ω is the frequency of the electromagnetic wave.

Accordingly, it is always possible to reduce any problem about the propagation of electromagnetic waves in a space with a given metric to a problem of the optics of moving anisotropic media. Our formulas (3), (5), (10), and (13) enable us to treat directly the various effects associated with the propagation of light in an arbitrary gravitational field and in noninertial reference systems (for example, the rotation of the plane of polarization, the curvature of the ray, the changes of frequency and of phase of the wave travelling in a rotating ring laser, and other effects). The calculations for some actual cases are presented in[3,11,12].

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