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STATIONARY NONLINEAR THEORY OF PARAMETRIC EXCITATION OF WAVES

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The stationary amplitude and phase distributions of parametric waves in a nonlinear dissipative medium in a homogeneous monochromatic (pumping) field are calculated in the case in which the reaction on the pumping source can be neglected. The limitation of the amplitude is due to the collective interaction between pairs of waves, which leads to a self-consistent change of the total phase of each pair, and consequently to a weakening of the coupling with the pumping field. From the set of stationary states that satisfy the energy balance, those are selected which are stable against small perturbations. For a given pumping power the distribution of the pairs in k space depends on the form of the coefficients in the Hamiltonian which describe the interactions of the pairs with the pumping and with each other. The dependences of the nonlinear susceptibilities of the system on the pumping power are calculated; these agree qualitatively with experiments on spin waves in ferromagnetics.

In the present paper we study the phenomena that arise in a nonlinear medium when one applies to it a field periodic in the time, which we shall call the pumping. It is well known[1] that if the amplitude of the pumping is sufficiently large (in comparison with the damping of the waves) an instability develops in the medium, and waves coupled together in pairs will be parametrically excited. They will increase exponentially with time until nonlinear effects appear which limit the amplitude to a certain level.

In cases in which the pumping is one of the natural oscillations of the medium—for example, in Raman scattering of light[2] or in the excitation of spin waves by a uniform precession[1]—the main nonlinear effect, at least for not too large excesses over threshold, is the reaction of the excited waves back on the pumping. In other cases, when the pumping is an external field, the back reaction is unimportant; for example, this is the case in the excitation of spin waves by the method of parallel pumping.

Here the experimental situation is as follows: To a specimen of a ferromagnetic dielectric placed in a constant magnetic field $H \approx 10^3$ Oe one applies a UHF alternating field of frequency $\omega_p \approx 2\pi \times 10^{10}$ sec^{-1} parallel to the constant field. With this geometry ($\mathbf{h} \parallel \mathbf{H}$) no uniform precession of the magnetization is excited. However, beginning at a threshold amplitude $h_c \sim 1$ Oe, absorption of the energy of the UHF field occurs, caused by parametric excitation of spin waves.[3] The experiment can be done on many crystals at room temperature, and the threshold amplitude h_c can be exceeded tens of times over.

In the following we consider the situation in which the back reaction on the pumping field can be neglected. It is interesting because a limitation of the amplitude of the excited waves can occur only owing to the interaction between them. We also confine ourselves to the case of spatially uniform pumping.

From our point of view, the main features of the above-threshold behavior are explained by the fact that a phase correlation is established between waves with equal and oppositely directed wave vectors; in analogy with superconductivity we call this correlation "pairing."

As is well known, the waves parametrically excited by a uniform field are located in k space near the surface $\omega_{\mathbf{k}} = \omega_p/2$ (ω_p is the pumping frequency), and waves with $\mathbf{k}$ and $-\mathbf{k}$ are coupled.[1] The sums of the phases of such waves are in a definite relation with the phase of the pumping (exactly on the surface they differ from it by $\pi/2$), so that the phases in each pair are completely correlated. This correlation is also preserved under nonlinear conditions, so that we must speak not of the interaction of individual waves but of the interaction of pairs.

An investigation of this interaction shows that it can be represented as the appearance of an additional pumping proportional to the total of the squares of the amplitudes of the pairs. It is essential, however, that the phase of this pumping, which coincides with the total phase of the pairs, is different from the phase of the external pumping. The appearance of the additional pumping with a different phase changes the total phase of the waves, and this leads to a decrease of the coupling with the external pumping, and in the final analysis to a limitation of the amplitude.[4]

The fundamental nonlinear equations describing the "pairing" are derived in Sec. 1. The limits of their applicability are also considered there. In Secs. 2–6 we study the stationary states described by these equations and their stability. We shall show that for sufficiently small excesses over threshold a condition is established in which there is one pair of monochromatic waves. This result seems to us to be interesting from the point of view that it allows us to hope that parametric generation of monochromatic waves of a different physical nature will be experimentally possible.

With further increase above threshold a stationary condition appears in the form of two monochromatic pairs, and thereafter the picture rapidly becomes more complicated, showing a resemblance to the pattern of the development of turbulence as predicted by Landau,[5] and at sufficiently large pumping amplitudes it becomes stochastic. The correlation of the phases of each pair is preserved throughout, while the phases of individual waves become statistically independent.

In Sec. 6 we consider cases in which the behavior beyond threshold depends essentially on the nonlinear damping of the waves.[6] We take it into account phenomenologically by introducing into the equations of motion antihermitian additions to the coefficients of the Hamiltonian. In the concluding section some conclusions of the theory are compared with experiment.

1. THE FUNDAMENTAL EQUATIONS

For simplicity we shall describe nonlinear media by means of canonical variables, the complex amplitudes $a_{\mathbf{k}}$ of running waves. Let the Hamiltonian function of the medium be of the form

$$\mathcal{H} = \sum \omega_{\mathbf{k}} a_{\mathbf{k}} a_{\mathbf{k}}^* + \mathcal{H}_p + \mathcal{H}_{int}. \tag{1a}$$

Here $\omega_{\mathbf{k}}$ is the dispersion law of the waves, $\mathcal{H}_p$ describes the interaction of the waves with the external periodic field (pumping), and $\mathcal{H}_{int}$ is the Hamiltonian for the interaction of the waves. For spatially uniform pumping

$$\mathcal{H}_p = \frac{1}{2}\sum_{\mathbf{k}} [V_{\mathbf{k}}^* h^*(t) a_{\mathbf{k}} a_{-\mathbf{k}} + \text{c.c.}]. \tag{1b}$$

The pumping field is $h(t) = h\exp(i\omega_p t)$, and $V_{\mathbf{k}} = V_{-\mathbf{k}}$.

The equations of motion of the medium are

$$\partial a_{\mathbf{k}} / \partial t + \gamma_{\mathbf{k}} a_{\mathbf{k}} = i\delta\mathcal{H} / \delta a_{\mathbf{k}}^*.$$

In these equations we have phenomenologically introduced the term $\gamma_{\mathbf{k}} a_{\mathbf{k}}$ to describe the damping of the waves.

When the pumping field exceeds the threshold value

$$h_c = \min\{\gamma_{\mathbf{k}} / |V_{\mathbf{k}}|\},$$

the waves with wave vectors $\pm\mathbf{k}$ increase exponentially:

$$a_{\mathbf{k}}(t) = a_{\mathbf{k}} \exp[\tilde{\gamma}_{\mathbf{k}} t + i(^1/_2\omega_p t + {}^1/_2\Psi_{\mathbf{k}})],$$
$$a_{-\mathbf{k}}(t) = a_{\mathbf{k}}^* \exp[\tilde{\gamma}_{\mathbf{k}} t + i(^1/_2\omega_p t + {}^1/_2\Psi_{\mathbf{k}})];$$

here

$$\tilde{\gamma}_{\mathbf{k}} = [|hV_{\mathbf{k}}|^2 - (\omega_{\mathbf{k}} - {}^1/_2\omega_p)^2]^{1/2} - \gamma_{\mathbf{k}},$$
$$\cos(\Psi_{\mathbf{k}} - \alpha_{\mathbf{k}}) = (\omega_{\mathbf{k}} - {}^1/_2\omega_p) / |hV_{\mathbf{k}}|, \tag{2}$$

and $\alpha_{\mathbf{k}}$ is the phase of the pumping. The time origin is chosen so that at the point $\mathbf{k}_0$ where $|V_{\mathbf{k}}|$ has its maximum $\alpha_{\mathbf{k}_0} = 0$. The quantity $\Psi_{\mathbf{k}}$, which is the sum of the phases of the excited waves, is thus completely determined.

The waves that increase most rapidly are those that lie on the resonance surface $\omega_{\mathbf{k}} = \omega_p/2$. It is obvious that also in the nonlinear regime the excited waves will lie near this surface, so that their natural frequencies are almost equal. The Hamiltonian that describes the interaction of such waves is of the form

$$\mathcal{H}_{int} = \frac{1}{2}\sum_{12,34} T_{\mathbf{k}_1\mathbf{k}_2,\mathbf{k}_3\mathbf{k}_4} a_{\mathbf{k}_1}^* a_{\mathbf{k}_2}^* a_{\mathbf{k}_3} a_{\mathbf{k}_4} \Delta(\mathbf{k}_1 + \mathbf{k}_2 - \mathbf{k}_3 - \mathbf{k}_4).$$

If there are terms cubic in the $a_{\mathbf{k}}$ in the total Hamiltonian of the medium they must be taken into account in second-order perturbation theory as a contribution to the quantity $T_{\mathbf{k}_1\mathbf{k}_2,\mathbf{k}_3\mathbf{k}_4}$ (cf.[7]).

The fundamental approximation made in the present paper is a certain simplification of the Hamiltonian $\mathcal{H}_{int}$. What we do is to replace the Hamiltonian $\mathcal{H}_{int}$ by the part of it that is diagonal with respect to the pairs:

$$\tilde{\mathcal{H}}_{int} = \sum_{\mathbf{k}\mathbf{k}'} \left(T_{\mathbf{k}\mathbf{k}'} a_{\mathbf{k}} a_{\mathbf{k}}^* a_{\mathbf{k}'} a_{\mathbf{k}'}^* + \frac{1}{2} S_{\mathbf{k}\mathbf{k}'} a_{\mathbf{k}}^* a_{-\mathbf{k}}^* a_{\mathbf{k}'} a_{-\mathbf{k}'} \right)$$
$$- \frac{1}{2}\sum_{\mathbf{k}} (T_{\mathbf{k}\mathbf{k}} a_{\mathbf{k}} a_{\mathbf{k}} a_{\mathbf{k}}^* a_{\mathbf{k}}^* + 2S_{\mathbf{k}\mathbf{k}} a_{\mathbf{k}}^* a_{-\mathbf{k}}^* a_{\mathbf{k}} a_{-\mathbf{k}}), \tag{1c}$$

where

$$T_{\mathbf{k}\mathbf{k}'} = T_{\mathbf{k}\mathbf{k}',\mathbf{k}\mathbf{k}'}, \quad S_{\mathbf{k}\mathbf{k}'} = T_{\mathbf{k},-\mathbf{k};\mathbf{k}',-\mathbf{k}'},$$
$$T_{\mathbf{k}\mathbf{k}'} = T_{\mathbf{k}'\mathbf{k}}, \quad S_{\mathbf{k}\mathbf{k}'} = S_{\mathbf{k}'\mathbf{k}}^* = S_{-\mathbf{k},-\mathbf{k}'}.$$

There are the following grounds for this approximation. The waves that interact with each other most strongly are those whose frequencies and wave vectors are connected by the relation

$$\omega_{\mathbf{k}_1} + \omega_{\mathbf{k}_2} = \omega_{\mathbf{k}_3} + \omega_{\mathbf{k}_4}.$$

Therefore it is natural to retain in the Hamiltonian $\mathcal{H}_{int}$ the summation over only those vectors that satisfy this condition. Since the frequencies of all the waves are nearly the same, this condition can be satisfied only if

$$\mathbf{k}_1 \approx \pm\mathbf{k}_3, \quad \mathbf{k}_2 \approx \pm\mathbf{k}_4 \tag{3a}$$

or else

$$\mathbf{k}_1 \approx \pm\mathbf{k}_4, \quad \mathbf{k}_2 \approx \pm\mathbf{k}_3. \tag{3b}$$

The approximate equations (3) are satisfied with accuracy up to $\delta k \sim k\delta\omega/\omega$, where $\delta\omega$ is the scatter of the frequencies. In two important cases the approximate equations (3) are to be replaced by exact equations, which corresponds to going over to the Hamiltonian (1c).

First, this can be done when the wave field $a_{\mathbf{k}}$ is a set of monochromatic pairs with wave numbers that are not too nearly equal, so that the difference between them is much larger than δk. It follows from (2) that the frequency is determined up to a quantity of the order of $|hV| - \gamma$. From this we have for the minimal distance between the pairs:

$$\delta k / k \ll (|hV| - \gamma) / \omega_k. \tag{4}$$

We shall call a regime in the form of a finite number of monochromatic pairs a dynamical regime. We note that for a large number of pairs the condition (4) can be very severe. If the individual phases of the waves are random, then all of the terms in $\mathcal{H}_{int}$ for which the conditions (3) are not satisfied exactly will depend explicitly on these phases and will go to zero on averaging. Furthermore the pairs can be distributed arbitrarily densely, even continuously. We shall call a regime in the form of a continuous distribution of pairs with random individual phases a stochastic regime. As the number of pairs increases a dynamical regime goes over into a stochastic regime.

We write the equations of motion corresponding to the Hamiltonian (1) in terms of the variables $c_{\mathbf{k}} = a_{\mathbf{k}}\exp(-i\omega_p t/2)$:

$$\partial c_{\mathbf{k}} / \partial t + (\gamma_{\mathbf{k}} + i\tilde{\omega}_{\mathbf{k}}) c_{\mathbf{k}} = -iP_{\mathbf{k}} c^*_{-\mathbf{k}}, \tag{5a}$$

$$\tilde{\omega}_{\mathbf{k}} = \omega_{\mathbf{k}} - \frac{\omega_p}{2} + 2\sum_{\mathbf{k}'} T_{\mathbf{k}\mathbf{k}'} c_{\mathbf{k}'} c_{\mathbf{k}'}^* - T_{\mathbf{k}\mathbf{k}} c_{\mathbf{k}} c_{\mathbf{k}}^* - 2S_{\mathbf{k}\mathbf{k}} c_{-\mathbf{k}} c_{-\mathbf{k}}^*, \tag{5b}$$

$$P_{\mathbf{k}} = hV_{\mathbf{k}} + \sum_{\mathbf{k}'} S_{\mathbf{k}\mathbf{k}'} c_{\mathbf{k}'} c_{-\mathbf{k}'}. \tag{5c}$$

In what follows we shall consider only the physically interesting case in which the amplitudes are equal in each pair, i.e., $|c_{\mathbf{k}}| = |c_{-\mathbf{k}}|$.

For the case of chaotic phases we must average the equations (5). Two correlation functions then appear,

$$n_{\mathbf{k}} = \langle c_{\mathbf{k}} c_{\mathbf{k}}^* \rangle, \quad \sigma_{\mathbf{k}} = \langle c_{\mathbf{k}} c_{-\mathbf{k}} \rangle,$$

which must satisfy the equations

$$-{}^1/_2 i\dot{\sigma}_{\mathbf{k}} = \sigma_{\mathbf{k}}(-\tilde{\omega}_{\mathbf{k}} + i\gamma_{\mathbf{k}}) + P_{\mathbf{k}} n_{\mathbf{k}}, \quad {}^1/_2 \dot{n}_{\mathbf{k}} + \gamma_{\mathbf{k}} n_{\mathbf{k}} = \operatorname{Im}(P_{\mathbf{k}}^* \sigma_{\mathbf{k}}). \quad \text{(6a)}$$

In the derivation we have used the relation $n_{\mathbf{k}} = n_{-\mathbf{k}}$. It is easily verified that the equations (6a) admit the solutions $n_{\mathbf{k}} = |\sigma_{\mathbf{k}}|$ and reduce to one equation for $\sigma_{\mathbf{k}}$. This solution means that the phases are the same for all pairs that lie in the neighborhood of a given point in $\mathbf{k}$ space; in this case the statistical problem reduces to a dynamical problem.

It is interesting that Eq. (6a) could have been derived from the exact Hamiltonian $\mathcal{H}_{\text{int}}$, with a definite method for decoupling the fourth-order correlations, namely

$$\langle a_{\mathbf{k}}^* a_{\mathbf{k}_1}^* a_{\mathbf{k}_2} a_{\mathbf{k}_3} \rangle = n_{\mathbf{k}_1} n_{\mathbf{k}_2} [\Delta(\mathbf{k}-\mathbf{k}_2)\Delta(\mathbf{k}_1-\mathbf{k}_3) + \Delta(\mathbf{k}-\mathbf{k}_3)\Delta(\mathbf{k}_1-\mathbf{k}_2)] + \sigma_{\mathbf{k}}^* \sigma_{\mathbf{k}_2} \Delta(\mathbf{k}+\mathbf{k}_1)\Delta(\mathbf{k}_2+\mathbf{k}_3),$$

$$\langle a_{\mathbf{k}}^* a_{\mathbf{k}_1} a_{\mathbf{k}_2} a_{\mathbf{k}_3} \rangle = n_{\mathbf{k}} \sigma_{\mathbf{k}_1} [\Delta(\mathbf{k}-\mathbf{k}_2)\Delta(\mathbf{k}_1+\mathbf{k}_3) + \Delta(\mathbf{k}-\mathbf{k}_3)\Delta(\mathbf{k}_1+\mathbf{k}_2)] + n_{\mathbf{k}} \sigma_{\mathbf{k}_2} \Delta(\mathbf{k}-\mathbf{k}_1)\Delta(\mathbf{k}_2+\mathbf{k}_3).$$

The equations (6a) can be written in real variables by introducing $\sigma_{\mathbf{k}} = n_{\mathbf{k}} \exp(i\Psi_{\mathbf{k}})$:

$${}^1/_2 \dot{n}_{\mathbf{k}} = [-\gamma_{\mathbf{k}} + \operatorname{Im}(P_{\mathbf{k}}^* e^{i\Psi_{\mathbf{k}}})] n_{\mathbf{k}}, \quad {}^1/_2 \dot{\Psi}_{\mathbf{k}} = \tilde{\omega}_{\mathbf{k}} + \operatorname{Re}(P_{\mathbf{k}}^* e^{i\Psi_{\mathbf{k}}}). \quad \text{(6b)}$$

We note that (6b) is completely equivalent to (5a). To verify this, we set

$$c_{\mathbf{k}} = \sqrt{n_{\mathbf{k}}} \exp(i\varphi_{\mathbf{k}} t/2 + i\Psi_{\mathbf{k}}/2), \quad c_{-\mathbf{k}} = \sqrt{n_{\mathbf{k}}} \exp(-i\varphi_{\mathbf{k}} t/2 + i\Psi_{\mathbf{k}}/2).$$

Then for $n_{\mathbf{k}}$ and $\Psi_{\mathbf{k}}$ we get (6b), and also we get the further equation

$$\dot{\varphi}_{\mathbf{k}} = 0,$$

which means that in the framework or our model the individual phases of the waves are in neutral equilibrium and can be randomized by an arbitrarily small additional interaction.

2. THE STATIONARY STATES. THE CONDITION FOR EXTERNAL STABILITY

The stationary states are described by the equation

$$(P_{\mathbf{k}} e^{-i\Psi_{\mathbf{k}}} + \tilde{\omega}_{\mathbf{k}} + i\gamma_{\mathbf{k}}) n_{\mathbf{k}} = 0, \quad (7)$$

which follows immediately from (6b). Using the fact that the possible solutions are concentrated in a narrow layer near the surface $\omega_{\mathbf{k}} = \omega_p/2$, we introduce the following variables in the $\mathbf{k}$ space: κ is the deviation from the surface in the normal direction, and Ω is the coordinate along the surface. We also replace the quantities $\gamma_{\mathbf{k}}$, $V_{\mathbf{k}}$, $T_{\mathbf{kk}'}$, and $S_{\mathbf{kk}'}$ by their values on the surface, and we expand the quantity $\tilde{\omega}_{\mathbf{k}}$ in a series up to the first term:

$$\tilde{\omega}_{\kappa\Omega} = 2\sum_{\kappa'\Omega'} T_{\Omega\Omega'} n_{\kappa'\Omega'} - (T_{\Omega\Omega} + 2S_{\Omega\Omega}) n_{\kappa\Omega} + \kappa \frac{\partial\omega}{\partial\kappa}. \quad (8)$$

We get

$$(P_{\Omega} e^{-i\Psi_{\kappa\Omega}} + \tilde{\omega}_{\kappa\Omega} + i\gamma_{\Omega}) n_{\kappa\Omega} = 0, \quad (9)$$

where

$$P_{\Omega} = hV_{\Omega} + \sum_{\kappa'\Omega'} S_{\Omega\Omega'} n_{\kappa'\Omega'} \exp(i\Psi_{\kappa'\Omega'}).$$

We see from this that at the points where the solution is concentrated $\operatorname{Im}[P_{\Omega} \exp(-i\Psi_{\kappa\Omega})] = \gamma_{\Omega}$, and consequently $\Psi_{\kappa\Omega}$ does not depend on κ. Since P_{Ω} is also independent of κ, $\operatorname{Re}[P_{\Omega} \exp(-i\Psi_{\kappa\Omega})]$ also does not depend on κ. Then it follows from (9) that

$$\tilde{\omega}_{\kappa\Omega} = \operatorname{Re}(P_{\Omega} e^{i\Psi_{\Omega}}) \quad (10)$$

for those values of κ at which the solution is different from zero.

For a continuous distribution of the pairs in Ω, when the term $(T_{\Omega\Omega} + 2S_{\Omega\Omega}) n_{\kappa\Omega}$ in (8) can be neglected, it follows from (10) that the solutions are located on a certain surface in $\mathbf{k}$ space.

In the dynamical case, when the number of pairs is small and we cannot neglect the term $(T_{\Omega\Omega} + 2S_{\Omega\Omega}) n_{\kappa\Omega}$, we shall also assume that for each Ω there exists not more than one pair. This cannot be proved, however, in the framework of a Hamiltonian diagonal in the pairs. In fact, assuming that in a given direction there are q pairs, and fixing their amplitudes n_j arbitrarily, we get from (8) and (10) an expression for the "distance" between pairs:

$$\frac{\partial\omega}{\partial\kappa}(\kappa_j - \kappa_{j'}) = (T_{\Omega\Omega} + 2S_{\Omega\Omega})(n_j - n_{j'}).$$

It is easily verified that these distances do not satisfy the condition (4) for the applicability of the diagonal Hamiltonian. Therefore the problem of the structure of the distribution in κ must be treated in the framework of the exact Hamiltonian. It can be shown that a state with one pair often possesses stability also in the framework of the exact Hamiltonian; this serves as grounds for the assumption made above.

Accordingly, in a stationary state all of the pairs are on the surface (10). However, there exists an infinite set of stationary states differing in the form of the surface (10) and in the distribution n_{Ω} on it. In fact, we see from (9) that we can assign arbitrarily directions Ω in which n_{Ω} is equal to zero. In reality, out of all the possible states the ones that are realized are those that are stable with respect to small perturbations.

We must distinguish between internal stability—against perturbations of the amplitudes and phases of pairs already existing—and external stability—against the production of new pairs.

The condition of external stability is most easily formulated:

$$\gamma_{\Omega} \geqslant \operatorname{Im} P_{\Omega}^* e^{-i\Psi_{\kappa\Omega}} \quad (11)$$

in the entire $\mathbf{k}$ space, the equals sign holding at the points where the solution is concentrated. Since P_{Ω} does not depend on κ, the maximum (with respect to κ) value is $\operatorname{Im}[P_{\Omega}^* \exp(-i\Psi_{\kappa\Omega})] = |P_{\Omega}|$ and corresponds to the value of κ at which $\operatorname{Re}[P_{\Omega}^* \exp(-i\Psi_{\kappa\Omega})] = 0$. Therefore there is a unique surface in the $\mathbf{k}$ space on which the condition (11) can be satisfied, namely

$$\tilde{\omega}_{\Omega} = 0. \quad (12)$$

The distribution of the pairs on the surface (12) is determined from the condition that the stationary state be stable with respect to the production of new pairs on this same surface. According to (10)–(12) this condition is

$$\gamma_{\Omega} \geqslant |P_{\Omega}|. \quad (13)$$

The condition $\widetilde{\omega}_\Omega = 0$ has a simple physical meaning. It can be seen from Eq. (2) that in the linear regime the waves most strongly coupled to the pumping are those for which the frequency detuning $\omega_{\mathbf{k}} - \omega_p/2 = 0$. The quantity $\widetilde{\omega}_\Omega$ is the frequency detuning with nonlinear terms taken into account. If this quantity is not zero at the points where the solution is located, then there exists a region in $\mathbf{k}$ space where pairs located in it would be more strongly coupled to the pumping than the pairs already in existence, and it is possible for such pairs to be parametrically excited.

3. THE DYNAMICAL REGIME

The condition of external stability (13) can be given the following geometrical interpretation. The expressions $\gamma = \gamma_\Omega$ and $P = |P_\Omega|$ are the equations of surfaces in $\mathbf{k}$ space. The condition (13) means that the surface P is entirely inclosed within the surface γ and touches it at the points Ω where the solution is concentrated. In virtue of the relations $V_{\mathbf{k}} = V_{-\mathbf{k}}$ and $S_{\mathbf{kk}'} = S_{-\mathbf{k},-\mathbf{k}'}$ both of these surfaces have a center of symmetry. Contact of the surfaces γ and P can occur either at a discrete set of points, or on a continuum—a line, or even a piece of the surface. In the former case we have a dynamical regime consisting of a finite number of monochromatic pairs, and in the latter case, a stochastic regime with random individual phases. An intermediate case is also possible, in which the surfaces touch each other at an isolated pair of points and also on a certain line. In this case a monochromatic pair and a stochastic "background" are simultaneously present in the system.

The problem of investigating the structure of the set of points of contact is extremely complicated in the general case. It can be solved, however, for sufficiently small excesses over threshold. For definiteness let γ = const, so that the surface γ is a sphere. We shall further assume that the surface V_Ω has only one pair of symmetrically situated maxima. At the first stage we also assume that the coefficients V_Ω and $S_{\Omega\Omega'}$ are real. Then we get from (9)

$$|P_\Omega| = \Big[\Big(hV_\Omega + \sum_{\Omega'} S_{\Omega\Omega'} n_{\Omega'} \cos\Psi_{\Omega'}\Big)^2 + \Big(\sum_{\Omega'} S_{\Omega\Omega'} n_{\Omega'} \sin\Psi_{\Omega'}\Big)^2\Big]^{1/2}.$$

For sufficiently small excesses over threshold, when the amplitude of the wave is small, the surface P differs from the surface hV_Ω by a small quantity. This is also true of the second derivatives with respect to Ω, so that the surface P also has only one pair of isolated maxima. It follows that the surfaces γ and P can touch only in one pair of symmetrically situated points. This means that for sufficiently small excesses over threshold the regime will consist of one monochromatic pair of waves. The location of this pair must be found from the condition (13). Differentiating $|P_\Omega|$ with respect to Ω, we find from the condition that the differential be zero that $\partial V_\Omega/\partial\Omega = 0$.

Accordingly, the pair is located at the maximum of the function V_Ω. For the amplitude and phase of the pair we have

$$n_1 = \frac{\sqrt{h^2V_1^2 - \gamma^2}}{|S_{11}|}, \quad \sin\Psi_1 = \frac{\gamma}{hV_1}. \tag{14}$$

Here V_1 is the value of the function V_Ω at the point where it is a maximum, and $S_{11} = S_{\Omega_1\Omega_1}$ at this point. If $\gamma \neq$ const, then, as is easily verified, the pair will be excited at the point where the function V_Ω/γ_Ω has its maximum.

The state with one pair is externally stable only for not too large excesses over threshold. For the function $|P_\Omega|$ we have

$$|P_\Omega|^2 = n_1^2\Big(S_{11}\frac{V_\Omega}{V_1} - S_{\Omega 1}\Big)^2 + \Big(\frac{V_\Omega}{V_1}\Big)^2\gamma^2,$$

where n_1 is given by Eq. (14). The amplitude $h(\Omega)$ of the external field at which external stability is violated can be determined from the condition $P_\Omega = \gamma$, which gives

$$\frac{h^2(\Omega)}{h_1^2} = 1 + S_{11}^2\frac{V_1^2 - V_\Omega^2}{(S_{11}V_\Omega - V_1S_{\Omega 1})^2}$$

where $h_1 = \gamma/V_1$.

To obtain the threshold h_2 for production of a second pair we must minimalize the expression for $h(\Omega)$. In the case $V_\Omega/V_1 = S_{\Omega_1}/S_{11}$ this threshold is infinite and the pair remains stable at any amplitude of the external field. We note further that the minimum of the function cannot lie too close to the point where the first pair is located. For small $\Omega - \Omega_1$ we have $V_\Omega \approx V_1 - V - V'(\Omega - \Omega_1)^2$, $S_{\Omega_1} \approx S_{11} + S'(\Omega - \Omega_1)^2$. Also here $h(\Omega) \sim \Omega^{-2}$. The simplest case to consider is S_{Ω_1} = const. Then we have

$$h^2(\Omega)/h_1^2 = 2V_1/(V_1 - V_\Omega),$$

and the threshold for excitation of the second pair is minimal at the minimum point of V_Ω.

Let us consider the behavior of the amplitudes of the first and second pairs near h_2. The equations for the amplitudes and phases of the two pairs are

$$\begin{aligned} &i\gamma e^{i\Psi_1} + hV_1 + S_{11}n_1e^{i\Psi_1} + S_{12}n_2e^{i\Psi_2} = 0, \\ &i\gamma e^{i\Psi_2} + hV_2 + S_{12}n_1e^{i\Psi_1} + S_{22}n_2e^{i\Psi_2} = 0, \end{aligned} \tag{15}$$

where V_2 is the value of V_Ω at the point of production of the second pair with $\Omega = \Omega_2$, $S_{12} \equiv S_{\Omega_1\Omega_2}$, $S_{22} \equiv S_{\Omega_2\Omega_2}$.

It is easy to calculate from these equations the derivatives above and below the threshold h_2:

$$\left.\frac{\partial n_1}{\partial h}\right|_{h=h_2+0} = 0, \quad \left.\frac{\partial n_2}{\partial h}\right|_{h=h_2+0} = \frac{h_2V_1^2}{S_{12}^2n_1},$$
$$\left.\frac{\partial n_1}{\partial h}\right|_{h=h_2-0} = \frac{h_2V_1^2}{S_{11}^2n_1}, \quad \left.\frac{\partial n_2}{\partial h}\right|_{h=h_2-0} = 0.$$

Although the amplitudes of the pairs change continuously at the point $h = h_2$, there are discontinuities of the derivatives with respect to h. For $S_{11} \neq S_{12}$ the derivatives of the total amplitude are also discontinuous.

Proceeding to the study of the case with complex coefficients V_Ω and $S_{\Omega\Omega'}$, we first note that without loss of generality we can assume that hV_Ω is real and positive. To make this so we must go over from the phases Ψ_Ω to the "invariant" phases $\widetilde{\Psi}_\Omega = \Psi_\Omega - \alpha_\Omega$ and make the substitution

$$S_{\Omega\Omega'} \to \widetilde{S}_{\Omega\Omega'} \equiv S_{\Omega\Omega'} \exp[i(\alpha_\Omega - \alpha_{\Omega'})],$$

so that $\widetilde{S}_{\Omega\Omega'}$ remains Hermitian. Dropping the tilde, we get for the monochromatic regime

$$|P_\Omega| = n_1^2\left(S_{11}\frac{V_\Omega}{V_1} - S_{\Omega 1}'\right)^2 + \left(n_1 S_{\Omega 1}'' - \frac{V_\Omega}{V_1}\gamma\right)^2,$$
$$S_{\Omega 1} = S_{\Omega 1}' + iS_{\Omega 1}''.$$

The location of the pair is determined from the condition $\partial|P_\Omega|/\partial\Omega = 0$, which gives

$$n_1\frac{\partial S_{\Omega 1}''}{\partial\Omega}\Big|_{\Omega=\Omega_1} = \frac{\gamma}{V_1}\frac{\partial V_\Omega}{\partial\Omega}\Big|_{\Omega=\Omega_1} \qquad (16)$$

so that, unlike the previous case, we have $\partial V_\Omega/\partial\Omega = 0$ only at the threshold point. Afterward there is a "drift" of the pair Ω_1 which is proportional to n_1 ($\partial V_\Omega/\partial\Omega = 0$ for $\Omega = 0$). The drift away from the point $\Omega = 0$ causes an additional impairment of the interaction with the pumping and is a further cause, besides the loss of phase matching, for the limitation of the amplitude of the pair. Solving (14) and (16) simultaneously, we get for small n_1

$$n_1^2 = [(hV_{max})^2 - \gamma^2]\left[S_{11}^2 + \left(\frac{hV_{max}}{\gamma}\right)^2\left(\frac{\partial S''}{\partial\Omega}\right)^2\frac{V_{max}}{\partial^2 V/\partial\Omega^2}\right]^{-1}.$$

Comparing this result with (14) at $V_1 = V_{max}$ we see that the "drift" leads to an additional term in the denominator, in general of the same order of magnitude.

Let us now qualitatively consider the behavior of the system for further increase of the pumping amplitude. Substituting in P_Ω the solutions of the equations (15), we can conclude that at some h_3 the state with two pairs becomes unstable and a third pair appears at $\Omega = \Omega_3$. For $h = h_4$ the next pair appears at Ω_4, and so on. This process of production of new pairs can be thought of as a succession of contacts of the surface P_Ω from inside with the surface γ_Ω at the points $\Omega_1, \Omega_2, \ldots$. We estimate the "dynamics" of this process from the following considerations. Let the number of pairs be q. Then the mean deviation of the surface P from γ is $\overline{P-\gamma} \sim (hV)^2/\gamma q$. When hV changes by an amount $V\delta h \sim \overline{P-\gamma}$ the number of pairs roughly doubles. We thus have

$$dq / Vdh \sim q^2\gamma / (hV)^2$$

and

$$q \approx h / (h_s - h) \quad (Vh_s \sim \gamma).$$

Accordingly, there is a sequence of external-field amplitudes h_q at which production of new pairs occurs, and this sequence converges to a limit, $h_q \to h_s$, for $q \to \infty$. For $h > h_s$ the surface P completely "adheres" to the surface γ, and a stochastic regime arises.

This description of the picture corresponds completely to the Landau hypothesis about the origin of turbulence.[5] The quantity that here plays the part of the Reynolds number is the dimensionless external-field amplitude hV/γ.

4. THE STOCHASTIC REGIME

The behavior of a system in the stochastic regime is most simply described in the limiting case of very large pumpings ($hV \gg \gamma$). In zeroth approximation in the parameter γ/hV we will have $P_\Omega = 0$, and then we have from (9) for real $S_{\Omega\Omega'}$

$$hV_\Omega + \int S_{\Omega\Omega'}x_{\Omega'}\,d\Omega' = 0, \qquad (17)$$

$$\int S_{\Omega\Omega'}y_{\Omega'}\,d\Omega' = 0, \qquad (18)$$

where

$$x_\Omega = n_\Omega\cos\psi_\Omega, \quad y_\Omega = n_\Omega\sin\psi_\Omega.$$

The problem thus reduces to linear Fredholm equations of the first kind. For simplicity we shall assume that in the absence of pumping the medium is isotropic, i.e., V_Ω depends on the polar angle θ, and $S_{\Omega\Omega'}$ depends only on the angle α between the directions Ω and Ω'.

Expanding all of the functions in (17) and (18) in terms of Legendre polynomials

$$S(\cos\alpha) = \sum_{n=0}^{\infty} S_n P_n(\cos\alpha) \qquad \text{etc.}$$

and using the addition theorem

$$P_n(\cos\alpha) = \sum_{m=n}^{-n} P_n^m(\cos\theta)P_n^m(\cos\theta')\exp[im(\varphi - \varphi')],$$

we get after an elementary integration over θ' and φ' the formal solution

$$x(\theta) = -\frac{1}{4\pi}\sum_{n=0}^{\infty}(2n+1)\frac{hV_n}{S_n}P_n(\cos\theta). \qquad (19)$$

In a general investigation of the solutions of Eqs. (17), (18) there are three possible cases.

1. The equations (18) have nontrivial solutions not orthogonal to V_Ω. Then Eq. (17) has no solutions, and the theory based on the diagonal Hamiltonian does not give a limitation of the amplitude. We shall call this a singular case. For an isotropic model a singular case occurs when for at least one $n = j$ we have $S_j = 0$ and $V_j \neq 0$. It was shown in Sec. 3 that for $V \neq$ const and not too large excesses over threshold a solution in the form of one pair or several pairs exists even in the singular case. Accordingly there is some critical pumping amplitude h_c, on reaching which there is a "breakoff" of the stationary state, and above which the limitation is due only to weaker nonlinear mechanisms.

It will be shown below that in the most singular case, when $S_0 = 0$ and V = const, there is no limitation of the amplitude for any excess over threshold at all, i.e., $h_c = \gamma/V = h_1$. In less singular cases $h_c > h_1$.

Let us consider in more detail the case in which S = const and V varies monotonically from V_1 to V_2. In this case the surface $|P_\Omega|$ has not more than two maxima. Therefore in the stationary state there cannot be more than two pairs. Solving the equations (15), we get the following expression for the total amplitude:

$$n_1 + n_2 = \frac{\gamma}{S}\frac{h(V_1+V_2)}{[4\gamma^2 - h^2(V_1-V_2)^2]^{1/2}}.$$

It can be seen from this that

$$h_c|_{S=\text{const}} = 2\gamma / (V_1 - V_2).$$

2. The equations (18) have nontrivial solutions orthogonal to V_Ω. Then the equations (17) have solutions defined up to the addition to them of solutions of (18), and the theory based on the diagonal Hamiltonian does not determine the stationary state uniquely. We shall call this an indefinite case. For an isotropic model an indefinite case occurs when for at least one $n = j$ we have $S_j = 0$ and $V_j = 0$. As in the singular case, if $V \neq$ const, then for small excesses h_1 over threshold there is a unique solution in the form of a single pair. Consequently there is some critical pump-

ing amplitude h_u above which the stationary state loses its definiteness. The solution is arbitrary to different extents depending on the number of harmonics for which $S_n = V_n = 0$. In the most indefinite case S = const, V = const we have $h_u = h_1$, and any distribution is admissible that satisfies the condition[4]

$$\int n_\Omega \, d\Omega = \frac{\sqrt{h^2V^2 - \gamma^2}}{|S|}. \qquad (20)$$

3. The equations (18) have no nontrivial solutions. We shall call this the regular case. In the regular case all $S_n \neq 0$. Then Eq. (17) has a unique solution (19), which, however, has a physical meaning only subject to certain restrictions. First it is necessary that the sequence V_n/S_n converge sufficiently rapidly to zero for $n \to \infty$. This condition, however, is not a sufficient one.

In fact, let us allow for the finiteness of the damping γ in Eqs. (17) and (18). To do so we use Eqs. (9) and (12):

$$\gamma \sin \Psi_\Omega = hV_\Omega + \int S_{\Omega\Omega'} n_{\Omega'} \cos \Psi_{\Omega'} \, d\Omega',$$

$$-\gamma \cos \Psi_\Omega = \int S_{\Omega\Omega'} n_{\Omega'} \sin \Psi_{\Omega'} \, d\Omega'.$$

For an isotropic model, linearizing these equations against the "background" of the solution (19), we get

$$-\gamma \delta\Psi_\Omega = \int S(\cos \alpha) \, \delta n_\Omega \, d\Omega,$$

$$-\gamma = \int S(\cos \alpha) \, n_\Omega \delta\Psi_\Omega \, d\Omega.$$

From this we get

$$\delta\Psi_\Omega = -\frac{\gamma}{4\pi S_0 n_\Omega}, \quad \delta n_m = -\gamma \frac{2m+1}{4\pi S_m} \delta\Psi_m.$$

In order for a region of applicability of the solutions (19) to exist it is necessary that the distribution n_Ω be nowhere zero.

We note that for V = const we can obtain an exact solution of the equations with damping for any degree of excess over threshold. In this case we have

$$\sin \Psi = \frac{\gamma}{hV}, \quad \int n \, d\Omega = \frac{\sqrt{h^2V^2 - \gamma^2}}{|S_0|}. \qquad (21)$$

In the regular case we also have n_Ω = const. For S = const (the indefinite case) Eq. (20) follows from (21).

5. INTERNAL STABILITY

We have so far been studying stationary states having external stability, i.e., stability against the production of new pairs. These states, however, will be realized only when they also have internal stability, i.e., stability against changes of the natural amplitude and phase.

We shall study the question of internal stability with the simple example of an isotropic two-dimensional medium with V = const (regular case). In this case, according to Sec. 4, the stationary state for any excess over threshold is a stochastic regime with a uniform distribution around the circle, satisfying the conditions (21). Choosing the perturbations of the amplitude and phase in the form

$$n_\varphi = n_0 + \sum_{p=0}^{\infty} \delta n_p e^{ip\varphi}, \quad \Psi_\varphi = \Psi_0 + \sum_{p=0}^{\infty} \delta\Psi_p e^{ip\varphi},$$

we get from (6b) and (12):

$$\tfrac{1}{2}\delta\dot{n}_p = -S_p n_0^2 \delta\Psi_p, \quad \tfrac{1}{2}\delta\dot{\Psi}_p = (2T_p + S_p)\delta n_p - \gamma\delta\Psi_p.$$

Here S_p and T_p are Fourier components of the functions $S(\varphi - \varphi')$ and $T(\varphi - \varphi')$. Taking δn_p, $\delta\Psi_p$ $\backsim \exp(2\nu_p t)$, we find that the growth increment of the p-th harmonic is

$$\nu_p = -\frac{2S_p(2T_p + S_p)}{4\pi^2 S_0^2}(hV - \gamma)$$

for small excesses over the threshold h_1. If it turns out that even a single one of the increments ν_p is greater than 0, the system has no stationary states at all. The question of the nonlinear stage of the development of internal instabilities is not considered here. There can be either a divergence of the total amplitude to infinity (breakoff of the stationary state), or a situation of periodic oscillations of the stationary amplitude.

6. THE NONLINEAR DAMPING

Generally speaking, the interaction between the pairs is not the only factor determining the behavior of the system beyond threshold. As the result of the interaction of the waves with the thermostat a nonlinear damping is produced, which also is of some importance. We have investigated this question in detail in a preprint.[8] Here we shall only present some results.

1. The nonlinear damping decreases the total amplitude of the waves. For example, instead of (14) [or (21)] for one pair (or for the total amplitude of a group of pairs with the same V) we get

$$\int n \, d\Omega = \frac{-\gamma\eta + [h^2V^2(S^2 + \eta^2) - \gamma^2 S^2]^{1/2}}{\eta^2 + S^2},$$

where η is a phenomenological coefficient which describes the nonlinear damping and is introduced into the equation of motion (6) as an antihermitian term added to S.

For small excesses over threshold, when $\rho \equiv (hV - \gamma)/\gamma < (\eta/S)^2$, this formula becomes simpler, and we have

$$\int n \, d\Omega = \frac{hV - \gamma}{\eta},$$

from which it follows that the limitation of the amplitude is due to the nonlinear damping. In the other limiting case, for $\rho > (\eta/S)^2$,

$$\int n \, d\Omega = \left[\frac{h^2V^2 - \gamma^2}{S^2 + \eta^2}\right]^{1/2}.$$

Hereafter we shall assume that there is no nonlinear damping caused by third-order processes, so that $\eta/S \sim \gamma/\omega \ll 1$. Then the limitation is entirely determined by the mechanism of interaction of pairs.

2. The nonlinear damping increases the indefiniteness of the state beyond threshold, In[8] we considered a two-dimensional medium with axial symmetry, so that V_k = const and $S_{kk'} = S(\varphi - \varphi')$. Without nonlinear damping the unique stationary state corresponds to a stochastic regime with a uniform distribution n_φ = const. Including nonlinear damping makes stable states arise with finite numbers q of pairs. The formula

$$N_c(q) \approx \eta\gamma / qS_{2q}^2$$

gives an estimate of the total amplitude $N_c(q)$ of pairs, below which a state with q or more pairs remains stationary against production of new pairs [S_{2q} is the Fourier coefficient of the expansion of $S(\varphi)$]. This means that states with a finite number q of pairs are always stable ($N_c \to \infty$) if the Fourier series for $S(\varphi)$ contains not more than q terms. In many cases, for example for waves on the surface of a liquid and for spin waves in a ferromagnet,[7] $S_q/S_0 \sim a^{-q}(a \sim 2)$. The threshold $N_c(q)$ then increases exponentially with the number of pairs, so that for $q \approx 10$ the state will be stable for any reasonable excess over the threshold h_1. Of course, states with larger numbers of pairs will be even more stable; in the framework of our model we cannot make a choice between these stationary states. It is important, however, that the total amplitude of the pairs is practically independent of their number.

7. CONCLUSION

In this paper we have calculated the amplitudes n_k and phases Ψ_k of the pairs as functions of the pumping amplitude. The most complete information would be given by a comparison of this theory with distributions n_k measured experimentally with scattering of light, sound, or neutrons by parametric waves. One may also measure integral characteristics of a system, for example the power absorbed by it.

The energy flux entering a collective of pairs from the external pumping is determined from (1b):

$$P_+ = h\omega_p \sum_k V_k \operatorname{Im} \sigma_k.$$

A quantity convenient for experimental determination is the susceptibility χ'', which is the ratio of P_+ to the pumping power $\omega_p h^2/2$:

$$\chi'' = \frac{2}{h} \sum V_k \operatorname{Im} \sigma_k.$$

In a similar way we define the susceptibility

$$\chi' = \frac{2}{h} \sum V_k \operatorname{Re} \sigma_k,$$

which characterizes the frequency "detuning" in the system.

For sufficiently small pumping amplitudes, when one pair (or group of pairs) with $V_k = V_{max}$ is excited in the system, we get for χ' and χ'' the expressions

$$\chi' = \frac{2V_{max} N \cos\psi}{h} = -\frac{2V_{max}^2}{S_0}\left[1 - \left(\frac{\gamma}{hV_{max}}\right)^2\right],$$

$$\chi'' = \frac{2V_{max} N \sin\psi}{h} = \frac{2V_{max}^2}{|S_0|}\left[1 - \left(\frac{\gamma}{hV_{max}}\right)^2\right]\frac{\gamma}{hV_{max}}.$$

At larger amplitudes ($hV \gg \gamma$), when pairs with all the V_k are excited in the system, we get from (19)

$$\chi' = \frac{4\pi}{h}\int V(\theta) x(\theta) \sin\theta \, d\theta = -\sum_{n=0}^{\infty} \frac{2V_n^2}{S_n},$$

$$\chi'' = \frac{4\pi}{h}\int V(\theta) y(\theta) \sin\theta \, d\theta = \frac{2V_0^2}{|S_0|}\frac{\gamma}{hV_0}.$$

Forms like these for χ' and χ'' have been observed experimentally with spin waves in $RbNiF_3$.[9] as we remarked in[4], the order of magnitude of the susceptibility in ferromagnetic substances also agrees with experiment. The ratio χ'/χ'' is ~ 1 for $\rho \sim 1$; in a more accurate calculation it depends on the specific form of the coefficients γ_k, V_k, and $S_{kk'}$. The experimental values of χ'/χ'' in ferromagnetics are of the order of 0.1 to 0.3.[9,10]

More detailed comparisons of the theory with published experimental results will require a detailed calculation of the Hamiltonian coefficients V_k and $S_{kk'}$, and in some cases will call for computer calculations. A different way seems more promising to us—the making of suitable experiments to test the qualitative conclusions of the theory, for example the monochromaticity of the stationary regime for small excesses over threshold. We note by the way that in experiments with ferromagnetics one as a rule observes oscillations of the susceptibility; these oscillations can be ascribed to the nonexistence of stable stationary states, as was already noted in Sec. 5. Another possible explanation of the oscillations is that we have not taken into account the effect of the parametric waves on the thermal waves. In fact, the excitation of parametric waves must change the shape of the distribution function of the thermal waves, which in turn leads to a change of the coefficients ω_k, $T_{kk'}$, and $S_{kk'}$ of the effective Hamiltonian of the parametric waves. This change occurs with a delay by the relaxation time of the thermal waves. Similar ideas about an "inertial" nonlinearity were first used in[11] to explain oscillations in ferromagnetics.

We note that the theory considered here can be applied for very large excesses over threshold, $hV \sim (\gamma\omega)^{1/2}$. In fact, among the terms dropped in simplifying the Hamiltonian $\mathscr{H}_{int}$ there are fourth-order terms describing the interaction between individual parametric waves, which lead to a diffusion of energy in k space from the resonance surface $\tilde{\omega}_k = 0$. It can be shown that owing to this there is excitation of a packet of waves of width $\Delta\omega_k \sim SN(\gamma/\omega)^{1/2}$; for $\rho \ll (\omega/\gamma)^{1/2}$ this width is much smaller than γ and can be neglected.

The behavior of waves with parametric excitation above threshold has been studied in the greatest detail in ferromagnetics. It would be very interesting to carry out such experiments in plasmas, in "two-dimensional" media (waves on liquid surfaces and in ferromagnetic films), and so on. This would make it possible to find general regularities in the behavior of parametric waves in different media and to model the nonlinear effects by means of simpler nonlinear media.

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