THEORY OF ELECTRON STATES IN HEAVILY DOPED SEMICONDUCTORS

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A theory is proposed making it possible to describe the electron states in a field of charged impurities for all energies in the conduction band and not very deep in the forbidden band. Such a description is possible because owing to large-scale fluctuations of the concentration of impurities the states with large kinetic energies give the major contribution to the density of states in the energy region under consideration. A diagram technique is developed which permits one to take this circumstance into account. A justification for the method of "distorted bands" in the theory of the density of states is given, the region of its applicability is indicated, and the first correction to it is calculated.

1. INTRODUCTION

LET us consider semiconductors with shallow impurities of one type (the compensation is small) at low temperatures. In this case the electron states are characterized by one dimensionless parameter β, which represents the ratio of the average distance between impurities to the Bohr radius a $(a = \hbar^2 \epsilon_0 / me^2$ where ϵ_0 is the dielectric constant and m is the effective mass). If $\beta = (Na^3)^{-1/3} \gg 1$ (N denotes the concentration of impurities), then at zero temperature all electrons are found on impurities. In the opposite case, $\beta \ll 1$, the electrons form a weakly nonideal Fermi gas of large density and possess a metalic conductivity. This case is also considered in the present article.

If $\beta \ll 1$ one can regard the electrons near the Fermi surface as free; however, electron states with small energies are substantially distorted by the random potential of the impurities. In addition, states appear in the forbidden band, which have an effect on interband absorption and emission of light.

First let us consider the one-electron problem, assuming the positions of the impurities to be random and their potential to be a Coulomb potential with Debye screening (this approximation is discussed at the end of Sec. 2). In a number of articles[1-3] the idea has been expressed that in a calculation of the density of states in a heavily doped semiconductor one can regard the electrons as classical. In this connection the electrons "experience" only the potential of that point at which they are located, and the density of states is expressed by the simple formula

$$\rho_{cl}(\varepsilon) = \frac{(2m)^{3/2}}{2\pi^2\hbar^3} \int\limits_{-\infty} \sqrt{\varepsilon - u} F(u)\, du, \tag{1}$$

where ε denotes the energy reckoned from the bottom of the conduction band (for simplicity we assume the spectrum of the electrons in an ideal crystal to be isotropic and quadratic), F(u) denotes the distribution function of the potential u,

$$u = \sum_j \varphi(\mathbf{r} - \mathbf{r}_j), \quad \varphi(r) = \frac{e^2}{\epsilon_0 r} e^{-\kappa r}, \tag{2}$$

where $\kappa = (2/a)(3/\pi)^{1/6}\beta^{-1/2}$ is the reciprocal of the screening radius. For a random distribution of impurities and a small value of u

$$F(u) = \frac{1}{\gamma\sqrt{\pi}} e^{-u^2/\gamma^2}, \quad \gamma^2 = 2N \int \varphi^2(r)\, d^3r. \tag{3}$$

A theory is developed in the present article which makes it possible to determine the spectrum of the electrons in a field of random impurities for all positive energies and for not very large negative energies (satisfying the condition (5), see below). The range of validity of formula (1) is determined and the corrections to it are calculated.

2. QUALITATIVE DISCUSSION OF THE RESULTS

It is easy to obtain[1] formula (1) by assuming the potential to be smooth, i.e., by making an expansion of the density of states in powers of Planck's constant. Formula (1) is obtained in the zero-order approximation (it is purely classical), where F(u) is not necessarily expressed by formula (3). However one can verify that the next term of this expansion, taking the gradient of the potential into account, is larger than the preceding term in the ratio $\kappa a \approx \beta^{-1/2} \gg 1$. One can easily understand this since in order to expand in powers of Planck's constant it is necessary that each potential well, described by the potential $\varphi(r)$, should contain many levels.[4] The condition that a screened Coulomb well contains many levels is $(\kappa a)^{3/2} = \beta^{-3/4} \ll 1$, which contradicts the condition for heavy doping, $\beta \ll 1$. Nevertheless, we shall show that formula (1) has a region of validity. Our conclusion is not due to the expansion with respect to Planck's constant and is based on an ultra-quantum property—the absence of a level in a strongly screened Coulomb well.

As is clear from Eq. (3), fluctuations of order κ^{-1} give the major contribution to the mean-square potential γ (values of $r \sim \kappa^{-1}$ give the major contribution to the integral). Let us assume that the random potential which we are studying is smoothed out with regard to fluctuations of smaller scale. A typical well for a smoothed-out potential has a width $\sim\kappa^{-1}$ and a depth $\sim\gamma$ (Fig. 1).

In order for such a well to have many levels, the condition $\gamma m/\hbar^2\kappa^2 = \beta^{-1/4} \gg 1$ is necessary. We shall assume that this condition is satisfied. Then all the
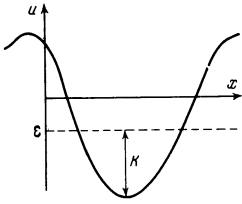
FIG. 1. Random potential which is smoothed-out with respect to fluctuations of scale smaller than κ^{-1}.

electron states of such a well, having a kinetic energy $K \gg \hbar^2 \kappa^2/2m$, may be described classically.

Let us now remember the very small-scale fluctuations. For a given kinetic energy K the fluctuations having a scale larger than the wavelength $\hbar/\sqrt{mK}$ may be regarded as classical. Fluctuations of smaller scale are averaged over the wavelength of the electron. One can estimate the level shift of the electron, to which these fluctuations lead, by having calculated the mean-square potential of these fluctuations γ_K. It is expressed by a formula analogous to (3) but the integration goes over all values of $r < \hbar/\sqrt{mK}$:

$$\gamma_K{}^2 = N \int_0^{\hbar (mK)^{-1/2}} \varphi^2 r^2\, dr \approx \frac{e^4 N \hbar}{\varepsilon_0{}^2 \sqrt{mK}}.$$

(In approximate equations we do not write numerical factors). It is obvious that the very small-scaled fluctuations do not have any influence on the spectrum of the electrons if $K > \gamma_K$. Let us define K_c by the equation

$$K_c = \gamma_{K_c}; \qquad K_c = \frac{e^2}{\alpha \varepsilon_0} \frac{1}{\beta^{6/5}}. \tag{4}$$

Since $K_c > \hbar^2 \kappa^2/m$ $(K_c m/\hbar^2 \kappa^2 = \beta^{-1/5})$ the quasiclassical description is valid provided $K > K_c$.

Expression (1), being purely classical, takes into account only states with $\epsilon - u = K > 0$. Substituting $F(u)$ from (3) into (1), one can easily see that for $\epsilon > \gamma$ values of $K = \epsilon$ are essential in the integral, for $-\gamma < \epsilon < \gamma$ values of $K \approx \gamma$ are essential, and finally for $\epsilon < 0$, $|\epsilon| \gg \gamma$ values of $K \approx \gamma^2/\epsilon$ are essential. In this last case $F(u)$ is small, $u \approx \epsilon$, and the states near the bottom of the well (Fig. 1) are important. Therefore, it is obvious that for $\epsilon < 0$ the condition for the validity of (1) is

$$|\varepsilon| < \varepsilon_c = \gamma^2/K_c. \tag{5}$$

In the region $\epsilon < 0$ the density of states ρ_{cl} decreases like $\exp(-\epsilon^2/\gamma^2)$. However, since $\epsilon_c/\gamma = \beta^{-1/20} \approx 1$, the Gaussian law of decrease in the quasiclassical region is absent in practice.

It is easy to see that formula (3) for $F(u)$ is valid for $u \le \epsilon_c$. As already stated, the fluctuations of order κ^{-1} give the major contribution to it. The potential created by such a fluctuation is $e^2 \kappa Z/\varepsilon_0$, where Z is the excess number of impurities in the fluctuation. Gaussian statistics are applicable if Z is smaller than the average number of impurities in a given volume, which is equal to $N\kappa^{-3}$, i.e., provided $u < u_c$ $\approx (e^2\kappa/\varepsilon_0) N\kappa^{-3}$. But $u_c/\epsilon_c \approx \beta^{-7/10} \gg 1$. Thus the quasiclassical approximation ceases to work sooner than Gaussian statistics.

As shown in the articles by Shklovskiĭ and the author,[5] the rearrangement of the density of states for $|\epsilon| \approx \epsilon_c$ only involves the pre-exponential factor.

The Gaussian law of decrease extends down to $|\epsilon| \approx \gamma^{4/3} \cdot (e^2/a\varepsilon_0)^{-1/3} < u_c$, after which the density of states decreases like $\exp\{-\sqrt{|\epsilon|}\,\ln|\epsilon|\}$. In the article by Andreev[6] the impurity potential is assumed to be smooth but the terms containing first derivatives are taken into account exactly, but the expansion is carried out to the second derivative. In this connection, however, a model potential is used which satisfies a normal distribution, owing to which such an expansion turns out to be convergent independently of the relation between κ^{-1} and a. In the article by Kasamanyan[7] an attempt is made to obtain formula (1) by a diagram technique. For small energies Kasamanyan's result differs from (1). As will be shown below, the reason for the difference is due to the fact that at small energies the "interference" diagrams which were not considered by Kasamanyan are more important than all the others.

In the investigation presented above the role of the free electrons reduced to a screening of the ions and the establishment of electrical neutrality. Let us discuss the conditions for the validity of this approximation. First of all we note that the maximum potential of the cluster of ions ϵ_c which we are considering is small in comparison with the Fermi energy μ. This means that Debye screening is applicable to such a cluster, and the inhomogeneity of the electron density is small. The number of electron states in a typical well of width κ^{-1} and depth γ is of the order of $\rho_{cl}(0)\gamma\kappa^{-3}$. The ratio of this quantity to the number $(N\kappa^{-3})^{1/2}$ of ions in such a fluctuation is of the order of $\beta^{3/4} \ll 1$, i.e., the excess electrons do not distort the potential well. The minimum value K_c of the kinetic energy of the electrons which we are considering is large in comparison with the interaction energy $e^2 N^{1/3}/\varepsilon_0$ of the electrons with each other, and this interaction does not prevent us from regarding electrons with $K > K_c$ as classical.

Thus, the one-electron problem formulated above makes sense, in the same way as formula (1), which is the most important consequence of it. One can use a similar approach in order to construct a theory of interband emission and absorption of light.[8]
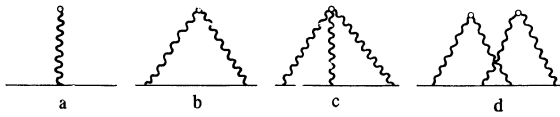
3. DIAGRAM TECHNIQUE

In this section a diagram technique will be developed which makes it possible to determine the corrections to the quasiclassical description. Let us introduce the definitions

$$G = \left\langle \frac{1}{\varepsilon - H + i\delta} \right\rangle, \quad \delta \to +0; \quad H = \frac{p^2}{2m} + \sum_j \varphi(\mathbf{r} - \mathbf{r}_j),$$

$$\rho(\varepsilon) = -\frac{2}{\pi} \operatorname{Im} \operatorname{Sp} G, \tag{6}$$

where φ is determined by formula (2), and $\langle\,\rangle$ denotes averaging over the coordinates of the impurities. The simplest diagrams for G are shown in Fig. 2. Each wavy line corresponds to a factor $4\pi e^2/\varepsilon_0(q^2 + \kappa^2)$, and each point corresponds to a factor N; the law of momentum conservation is satisfied at each point and integration is carried out over all independent momenta. The diagram shown in Fig. 2a leads to an energy shift. From the condition of electrical neutrality this shift is exactly cancelled by the potential of the electrons;

FIG. 2. The simplest diagrams for G.

FIG. 3. Typical diagrams for $\Pi_1(p; q)$ (a) and $\Pi_2(p; q_1, q_2)$ (b). The dotted lines correspond to the propagator $D^0(q)$.

FIG. 4. Typical diagram for $G^{(1)}$ given by Eq. (15). The heavy dotted line corresponds to the propagator $\widetilde{D}(q)$, and the narrow dotted line corresponds to the propagator $D^0(q)$.

therefore we shall not consider it here. The self-energy part corresponding to the graph shown in Fig. 2 b has the form

$$\Sigma^{(1)} = \int \frac{d^3q}{(2\pi)^3} \frac{D(q)}{\varepsilon - \varepsilon_{p-\hbar q} + i\delta},$$ (7)

$$D(q) = N \left[\frac{4\pi e^2}{\varepsilon_0(q^2 + \varkappa^2)} \right]^2,$$ (8)

where $\varepsilon_p = p^2/2m$. The basic idea of our expansion consists in the fact that we shall assume the momentum transfer $\hbar q$ in integrals of the type (7) to be small in comparison with p. If one sets $\varepsilon_{p-\hbar q} \approx \varepsilon_p$, then the integral converges and a typical value of $q \sim \varkappa$, i.e., we expand in powers of $\hbar\varkappa/p$. However, such a procedure can be accomplished only in diagrams containing two wavy lines per point. For a large number of lines per point the neglect of momentum transfer leads to divergences. Therefore the addition of a new line corresponds to multiplication by a quantity of the order of $Ge^2 p/\varepsilon_0 \hbar$. For small values of ε values of $G \approx 1/\gamma$, $p \approx \hbar/\sqrt{m\gamma}$ will be essential so that

$$G \frac{e^2 p}{\varepsilon_0 \hbar} \approx \frac{\hbar}{a\sqrt{m\gamma}} \approx \beta^{5/3} \ll \frac{\hbar\varkappa}{\sqrt{m\gamma}} \ll 1.$$

Therefore in the first few orders of the expansion in powers of $\hbar\varkappa/p$ one can consider only diagrams containing two lines per point. Actually this is a consequence of the fact that for the potential $\varphi(r)$ with $\varkappa a \gg 1$ the Born approximation is valid at all energies.

Let us represent the propagator $D(q)$ in the form

$$D(q) = D^0(q) + \widetilde{D}(q), \quad D^0(q) = \tfrac{1}{2}\gamma^2(2\pi)^3\delta(q),$$ (9)

where $\delta(q)$ denotes the three-dimensional δ-function, and

$$\gamma^2 = 2 \int D(q) \frac{d^3q}{(2\pi)^3},$$

which agrees with the definition (3). Here $\int \widetilde{D}(q)d^3q = 0$. Substitution of the first term of (9) instead of $D(q)$ is equivalent to neglecting the momentum transfer. The Dyson equation, taking into consideration only diagrams with two lines per point, has the form

$$\varepsilon - \frac{p^2}{2m} - G_p^{-1} = \int \Gamma(p, p - \hbar q, q) G_{p-\hbar q} D(q) \frac{d^3q}{(2\pi)^3},$$ (10)

where Γ is the vertex part. Let us substitute the first term of (9) in place of $D(q)$ and we use the Ward identity:

$$\Gamma(p, p, 0) = dG_p^{-1}/d\varepsilon.$$

Then we obtain a closed equation for G_p:

$$\frac{dG}{dx} \frac{\gamma^2}{2} + Gx - 1 = 0,$$ (11)

where $x = \varepsilon - (p^2/2m) + i\delta$, whose solution has the form

$$G_{cl} = \frac{1}{\gamma\sqrt{\pi}} \int_{-\infty}^{\infty} \frac{e^{-u^2/\gamma^2} du}{\varepsilon - p^2/2m + u + i\delta}.$$ (12)
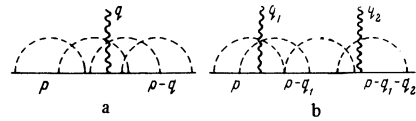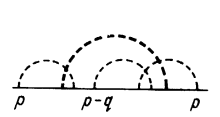
Calculation of the density of states according to Eqs.

(12) and (6) leads to formula (1). This is the zero-order approximation of our theory. This approximation was obtained in a similar way by L. Keldysh (unpublished). In order to calculate the corrections it is necessary to take $\widetilde{D}(q)$ and diagrams with a larger number of lines per point into account.

Let us introduce the n-point diagram $\pi_n(p; q_1, \dots, q_n)$, defined as the sum of all diagrams having two lines per point with n outgoing wavy lines (Fig. 3). In this connection in the internal lines by definition only the first term of the propagator (9), $D^0(q)$, is inserted, where we represent this propagator by a dotted line, but the propagators corresponding to the external wavy lines in π_n are generally not included. In Appendix 1 we prove a theorem, according to which

$$\Pi_n(p; q_1, \dots, q_n) = \frac{1}{\gamma\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-u^2/\gamma^2} g_p(\varepsilon - u) \cdot g_{p-\hbar q_1}(\varepsilon - u) \dots$$
$$\dots g_{p_n}(\varepsilon - u) du,$$ (13)

where

$$g_p(\varepsilon - u) = \left(\varepsilon - \frac{p^2}{2m} - u + i\delta \right)^{-1}, \quad p_n = p - \hbar q_1 \dots - \hbar q_n.$$ (14)

The first correction to (12) represents the sum of all diagrams with two lines per point where $\widetilde{D}(q)$ participates once in each diagram (Fig. 4). It is easy to write down an analytic expression for this correction by using expression (13):

$$G^{(1)} = \int \frac{d^3q}{(2\pi)^3} \widetilde{D}(q) \Pi_2(p; q, -q)$$

or

$$G^{(1)} = \frac{1}{\gamma\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-u^2/\gamma^2} g_p^2(\varepsilon - u) du \int \widetilde{D}(q) g_{p-\hbar q}(\varepsilon - u) \frac{d^3q}{(2\pi)^3}.$$ (15)

One can also write down similar expressions in the following orders, having taken the appropriate π_n and having chosen the wavy lines into the points in a different way.

We note that if the integration over u is neglected and if the energy $t = \varepsilon - u$ is introduced instead of ε, then (13) corresponds to the usual product of free Green's functions. Based on this one can formulate a new diagram technique in which an expansion is carried out in powers of the potential of the small-scale fluctuations. Let us introduce the Green's function $\widetilde{G}_p(t)$ such that

$$G_p(\varepsilon) = \frac{1}{\gamma\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-(\varepsilon - t)^2/\gamma^2} \widetilde{G}_p(t) dt.$$ (16)

Then, in accordance with what was said above, for $\widetilde{G}_p(t)$ there exists the usual diagram technique with only the property that one should insert not $D(q)$ but

$\tilde{D}(q)$ into points containing two lines. The variable t plays the role of the energy. The free Green's function has the usual form $\tilde{G}_p^{(0)} = (t - p^2/2m + i\delta)^{-1}$. Its substitution into (16) leads to (12). The diagram of Fig. 2 b for $\tilde{G}_p(t)$ leads to Eq. (15).

Formula (16) reflects the qualitative picture set forth in sec. 2. In fact, for $\epsilon = 0$ values of $t \approx \gamma$ are important in Eq. (16), i.e., it is not necessary to find the Green's function $\tilde{G}_p(t)$ at the bottom of the band, but it is sufficient to know it at the energy γ. In addition, for its determination it is only necessary to know the short-wavelength part of the propagator $D(q)$, and the long-wavelength part is taken into account classically.

4. QUANTUM CORRECTIONS TO THE DENSITY OF STATES

First let us find the self-energy part $\tilde{\Sigma}_p(t)$ corresponding to the diagram shown in Fig. 2 b for the Green's function $\tilde{G}_p(t)$. In order to do this, in Eq. (7) it is sufficient to replace the propagator $D(q)$ by $\tilde{D}(q)$. Evaluating the integral we obtain

$$\tilde{\Sigma}_p(t) = \frac{\gamma^2 \theta}{2} \Big/ \Big(t - \frac{p^2}{2m} - \theta \Big)\Big(t - \frac{p^2}{2m} + i\delta \Big), \quad (17)$$

where

$$\theta = \frac{\hbar^2 \varkappa^2}{2m} - i \frac{\hbar \varkappa}{m} \sqrt{2mt}. \quad (18)$$

A simple analysis of higher-order diagrams indicates that formula (17) is valid provided

$$|t - p^2/2m| > (\gamma^2|\theta|)^{1/3}, \quad t > (\gamma^2|\theta|)^{1/3}. \quad (19)$$

Substituting (18) into the second inequality (19), one can reduce it to the form $t > K_c$, where K_c is determined by formula (4). (In this connection the second term in (18) is essential). For smaller values of t perturbation theory becomes inapplicable for the function $\tilde{G}_p(t)$. This is in complete agreement with what was said in Section 2.

Let us apply the obtained results in order to calculate the quantum corrections to the density of states. For this purpose the first correction to the function $\tilde{G}_p(t)$ turns out to be sufficient:

$$G_p^{(1)}(t) = \tilde{\Sigma}_p(t) \Big(t - \frac{p^2}{2m} + i\delta \Big)^{-2}. \quad (20)$$

Using the definition (6) we obtain

$$\rho(\epsilon) = -\frac{2}{\pi} \int \frac{d^3 p}{(2\pi\hbar)^3} \, \text{Im} \, G_p(\epsilon). \quad (21)$$

Substituting (16) and (20) into (21), we find an expression for the quantum corrections to the density of states $\rho'(\epsilon)$:

$$\rho(\epsilon) = \rho_{cl} + \rho'$$
$$\rho' = -\frac{(2m)^{3/2}}{2\pi^3 \hbar^3} \int_0^\infty \sqrt{x}\, dx \, \text{Im} \int_{-\infty}^\infty e^{-(\epsilon - t)/\gamma^2} \frac{\tilde{\Sigma}_x(t)}{(t - x + i\delta)^2} \frac{dt}{\gamma\sqrt{\pi}}, \quad (22)$$

where $\tilde{\Sigma}_x(t)$ is given by formula (17), and $x = p^2/2m$.

Now it is necessary to expand (17) in powers of $\varkappa$. In the linear approximation we obtain

$$\tilde{\Sigma}_x(t) = -i \frac{\hbar \varkappa \gamma^2}{2m} \frac{\sqrt{2mt}}{(t - x + i\delta)^2}. \quad (23)$$

The lowest-order correction to the density of states is determined from Eqs. (23) and (22):

$$\rho'(\epsilon) = \frac{m\varkappa}{6\pi^{7/2}\hbar^2} Q(\Delta), \quad (24)$$

where $\Delta = \epsilon/\gamma$ and

$$Q(\Delta) = \int_0^\infty \sqrt{x}\, dx \frac{d^3}{dx^3} \int_0^\infty \frac{e^{-(t-\Delta)^2} \sqrt{t}\, dt}{t - x}. \quad (25)$$

Evaluation of the integral (25) is given in Appendix 2. As a result we have

$$Q(\Delta) = \frac{3\pi^2}{4} \Delta e^{-\Delta^2}. \quad (26)$$

According to Eq. (1), for $\Delta < 0$ and $|\Delta| \gg 1$

$$\rho_{cl} = \frac{m^{3/2}\sqrt{\gamma}}{4\pi^2\hbar^3|\Delta|^{3/2}} e^{-\Delta^2}. \quad (27)$$

In this case the pre-exponential factor in the correction increases with $|\Delta|$ more rapidly than in ρ_{cl}. Comparing relations (24) and (27) one can easily verify that $\rho' < \rho_{cl}$ provided $\epsilon > -\epsilon_c$ where ϵ_c is defined in Eq. (5). For small energies values of $t < K_c$ are important in the integral (25), and our theory does not work.

As is evident from Eqs. (24)–(26), for $\Delta \gg 1$ the correction to the density of states which is linear in $\varkappa$ becomes exponentially small, and it is necessary to calculate the quadratic correction. Expanding (17) we obtain

$$\tilde{\Sigma}_x'(t) = \frac{\hbar^2\varkappa^2\gamma^2}{4m} \frac{1}{(t-x+i\delta)^2} - \frac{\hbar^2\varkappa^2\gamma^2}{m} \frac{t}{(t-x+i\delta)^3}. \quad (28)$$

Then it is necessary to substitute (28) into (22) and perform a simple integration. The correction $\rho'(\epsilon)$ which is thus obtained is large in comparison with (24) only for $\Delta \gg 1$. In this case it has the form

$$\rho'(\epsilon) = \frac{3(2m)^{1/2}\gamma^2\varkappa^2}{2^7 \epsilon^{5/2}\hbar}, \quad \frac{\epsilon}{\gamma} \gg 1. \quad (29)$$

The terms of order $\varkappa^2$ in $\tilde{\Sigma}_x(t)$ may be obtained from the diagrams of higher order which we have not considered; however their contribution to the density of states is small according to the parameter $\gamma/\epsilon \ll 1$ our technique is essentially not required. The correction (29) may be obtained from ordinary perturbation theory. It is small in comparison with the classical correction to the density of states of the free electrons, which may be calculated from Eqs. (1) and (2). Both of these corrections fall off rapidly with increasing energy. As indicated in[1], this leads to the result that the correction to the Fermi energy due to the exchange interaction of the electrons is larger than that due to their interaction with impurities.

In conclusion I wish to take this opportunity to express my gratitude to V. N. Gribov for a number of considerations which turned out to be very important for this work. I also wish to thank V. I. Perel' and B. I. Shklovskii for reading the manuscript and for very useful comments.

APPENDIX 1

First let us consider the 3-point diagram $\pi_1(p; q)$, a typical diagram for which is shown in Fig. 3 a. Since by definition $D^0(q)$ is inserted into the internal lines, all electron Green's functions are carried out from under the integrals, and their momentum is equal to p up to the point of attachment of a wavy line, and this is replaced by $p - \hbar q$. Any arbitrary diagram for

$\pi_1(p; q)$ may be obtained if one introduces a wavy line into the corresponding diagram for G_{c1}. Let us consider one of the diagrams for G_{c1} containing s Green's functions (it is obvious that the number s is necessarily odd). The simple expression

$$G_{cl}^{(s)}(x_p) = g_p{}^s(\gamma^2/2)^{(s-1)/2}, \quad s = 1, 3, 5, \ldots, \quad (1.1)$$
$$g_p = (x_p + i\delta)^{-1},$$

corresponds to it, where $x_p = \epsilon - p^2/2m$. Let us denote by $\pi_1^{(S)}(p; q)$ the totality of diagrams for $\pi_1(p; q)$ which we obtained from G_{c1} by inserting a wavy line everywhere where it is possible to do so. One can easily verify that

$$\Pi_1^{(s)}(p; q) = \left(\frac{\gamma^2}{2}\right)^{(s-1)/2} \sum_{s'=1}^{s} g_p{}^{s'} g_{p-\hbar q}^{s-s'+1}. \quad (1.2)$$

Summing the geometrical progression, we obtain

$$\Pi_1^{(s)}(p; q) = \left(\frac{\gamma^2}{2}\right)^{(s-1)/2} \frac{g_p{}^s - g_{p-\hbar q}^s}{x_{p-\hbar q} - x_p} = \frac{G_{cl}^{(s)}(x_p) - G_{cl}^{(s)}(x_{p-\hbar q})}{x_{p-\hbar q} - x_p}. \quad (1.3)$$

Formula (1.3) is valid for any of the diagrams which correspond to the function G_{c1}. Having summed all of the diagrams we obtain

$$\Pi_1(p; q) = \frac{G_{cl}(x_p) - G_{cl}(x_{p-\hbar q})}{x_{p-\hbar q} - x_p}. \quad (1.4)$$

Substituting (12) into (1.4) we obtain

$$\Pi_1(p; q) = \frac{1}{\gamma\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-u^2/\gamma^2} g_p(\epsilon - u) g_{p-\hbar q}(\epsilon - u) du, \quad (1.5)$$

i.e., we have proved theorem (13) for the case $n = 1$. Now let us assume that (13) is satisfied for $\pi_n(p; q_1, \ldots, q_n)$ and let us prove an analogous equation for $\pi_{n+1}(p; q_1, \ldots, q_n, q_{n+1})$. With (1.5) taken into account this will be a complete proof of the theorem. One can obtain all possible graphs for π_{n+1} by inserting a wavy line with momentum q_{n+1} in all diagrams for π_n, in all possible places to the right of the wavy line q_n.

Let us consider one of the diagrams for π_n, for which there are s Green's functions to the right of q_n. Let us write the expression corresponding to this diagram in the form

$$\Pi_n^{(s)}(p; q_1, \ldots, q_n) = R^{(s)} g_{p_n}{}^s. \quad (1.6)$$

Since by definition the momentum of the Green's functions changes only upon a transition through an external wavy line, all of these s-functions have the same momentum $p_n = p - \hbar q_1 - \ldots - \hbar q_n$. The quantity $R^{(S)}$ contains $(\gamma^2/2)^{(S-1)/2}$, and also everything which corresponds to the left (with respect to q_n) side of the diagram. Let us denote by $\pi_{n+1}^{(S)}(p; q_1, \ldots, q_{n+1})$ the totality of diagrams which are obtained by means of the indicated prescription from the diagram $\pi_n^{(S)}(p; q_1, \ldots, q_n)$. In analogy with (1.2) and (1.3) we obtain

$$\Pi_{n+1}^{(s)}(p; q_1, \ldots, q_{n+1}) = R^{(s)} \sum_{s'=1}^{s} g_{p_n}^{s'} g_{p_n-\hbar q_{n+1}}^{s-s'+1} =$$
$$= \frac{\Pi_n^{(s)}(p; q_1, \ldots, q_n) - \Pi_n^{(s)}(p; q_1, \ldots, q_n + q_{n+1})}{x_{p_{n+1}} - x_{p_n}}, \quad (1.7)$$

where $p_{n+1} = p_n - \hbar q_{n+1}$. Dealing with all of the diagrams for π_n in a similar fashion we obtain

$$\Pi_{n+1}(p; q_1, \ldots, q_n, q_{n+1}) = \frac{\Pi_n(p; q_1, \ldots, q_n) - \Pi_n(p; q_1, \ldots, q_n + q_{n+1})}{x_{p_{n+1}} - x_{p_n}}. \quad (1.8)$$

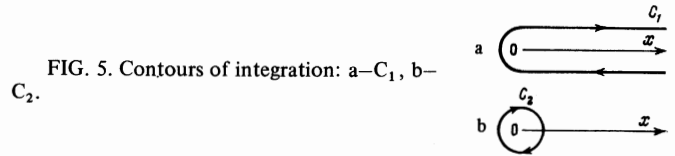FIG. 5. Contours of integration: a—C_1, b—C_2.

Substituting (13) into (1.8) we obtain

$$\Pi_{n+1}(p; q_1, \ldots, q_n, q_{n+1}) = \frac{1}{\gamma\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-u^2/\gamma^2} g_p(\epsilon - u) \cdot g_{p-\hbar q_1}(\epsilon - u) \ldots$$
$$\ldots g_{p_n}(\epsilon - u) g_{p_{n+1}}(\epsilon - u) du, \quad (1.9)$$

q.e.d.

APPENDIX 2

Let us represent the inner integral in (25)

$$I(x) = \int_0^{\infty} \frac{\sqrt{t}\, e^{-(t-\Delta)^2}}{t - x} dt \quad (2.1)$$

in the form of an integral along the contour C_1 (shown in Fig. 5 a)

$$I(x) = \frac{1}{2} \int_{c_1} \frac{\sqrt{t}\, e^{-(t-\Delta)^2}}{t - x} dt. \quad (2.2)$$

In this connection the contributions of the pole coming from different sides of the cut cancel out. Now let us understand the order of integration in (25) be to be the following:

$$Q(\Delta) = -\frac{1}{2} \int_{c_1} \sqrt{t}\, e^{-(t-\Delta)^2} dt \frac{d^3}{dt^3} \int_0^{\infty} \frac{\sqrt{x}}{t - x} dx. \quad (2.3)$$

One can easily evaluate the inner integral:

$$\frac{d^3}{dt^3} \int_0^{\infty} \frac{\sqrt{x}}{t - x} dx = -\frac{3\pi i}{8} \frac{1}{t^{5/2}}, \quad (2.4)$$

the cut of the integrand in (2.3) vanishes and an integral is left along the contour C_2 (see Fig. 5 b):

$$Q(\Delta) = \frac{3\pi i}{16} \int_{c_2} \frac{dt}{t^2} e^{-(t-\Delta)^2} = \frac{3\pi^2}{4} \Delta e^{-\Delta^2}. \quad (2.5)$$

[1] V. L. Bonch-Bruevich, Fiz. Tverd. Tela 4, 2660 (1962) [Sov. Phys.-Solid State 4, 1953 (1963)]; Fiz. Tverd. Tela 5, 1852 (1963) [Sov. Phys.-Solid State 5, 1353 (1964)]. In the collection: Fizika Tverdogo Tela, Institut nauchnoĭ informatsii (Institute of Scientific Information), 1965, p. 174.

[2] Evan O. Kane, Phys. Rev. 131, 79 (1963).

[3] L. V. Keldysh and G. P. Proshko, Fiz. Tverd. Tela 5, 3378 (1963) [Sov. Phys.-Solid State 5, 2481 (1964)].

[4] I. M. Lifshitz, Zh. Eksp. Teor. Fiz. 53, 743 (1967) [Sov. Phys.-JETP 26, 462 (1968)].

[5] B. I. Shklovskiĭ and A. L. Éfros, Fiz. Tekh. Poluprov. 4, 305 (1970) [Sov. Phys.-Semicond. 4, 000 (1970)]; Zh. Eksp. Teor. Fiz. 59, No. 10 (1970) [Sov. Phys.-JETP 32, No. 4 (1971)].

[6] I. V. Andrew, Zh. Eksp. Teor. Fiz. 48, 1437 (1965) [Sov. Phys.-JETP 21, 961 (1965)].

[7] Z. A. Kasamanyan, Fiz. Tekh. Poluprov. 2, 1709 (1968) [Sov. Phys.-Semicond. 2, 1425 (1969)].

[8] M. I. Dyakonov, A. L. Efros, and D. I. Mitchell, Phys. Rev. 180, 819 (1969).

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