A SIMILARITY HYPOTHESIS IN THE STRONG INTERACTIONS. I. MULTIPLE HADRON PRODUCTION IN e^+e^- *ANNIHILATION*

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A hypothesis of asymptotic scale invariance is proposed for the study of the properties of the strong interactions at small distances. A direct consequence of this hypothesis is power-law asymptotic behavior of the amplitudes at very large momenta and very far from the mass shell. It is shown that such asymptotic forms agree with the conditions of unitarity and analyticity. Exact equations describing the interaction of strongly virtual particles are formulated. Application of these equations to the process $e^+e^- \to \underline{\text{hadrons}}$ at high energies allows us to determine the total cross section for this process [Eq. (1.4)], to find the multiplicity of the hadron production [Eq. (1.5)], and to examine the properties of the cross sections for the production of given numbers of hadrons [Eqs. (1.2) and (1.3)] for cases in which the number is large. In conclusion we consider an analogy between the proposed theory and the theory of second-order phase transitions, and discuss the question of the role of SU(3) × SU(3) symmetry in this scheme.

1. INTRODUCTION

FIFTEEN years ago Landau and Pomeranchuk[1] showed that in the framework of quantum field theory with logarithmic divergences it is not possible to describe a strong interaction. This is due to the fact that in renormalized theories the physical coupling constant g^2 is determined by the formula[1]

$$g^2 = 3\pi / \ln (\Lambda^2 / m^2). \tag{1.1}$$

Here m is the mass of the particle occurring in the theory, and Λ is the maximum momentum up to which the theory has meaning. For the theory to have a region of applicability it is necessary that $\Lambda \gg m$. It follows that $g^2 \ll 1$, which makes a strong interaction impossible in this theory.

We have recently come to understand that the logarithmic situation is not the only possibility in quantum field theory. Gribov and Migdal,[2] considering the problem of the interaction of reggeons, showed that one possibility is a "strong coupling" case in which the many-particle Green's functions are power-law, not logarithmic, functions of their arguments. It later turned out that strong coupling and a power-law situation exist also in many-particle systems near points of second-order phase transitions.[3-5] The problem arising here is formally identical with that of the relativistic theory. It is extremely interesting to study the possible existence of a "strong coupling" regime in hadron interactions and to find some sort of observable manifestations of this situation.

Power-law asymptotic forms of the Green's functions at large momenta mean that the theory has an invariance with respect to change of the space-time scale which is manifested at very small distances. In a paper by Wilson[6] such an invariance was postulated independently of the idea of strong coupling, on a purely phenomenological level. Even with this approach Wilson succeeded in deriving a number of important qualitative results regarding current algebra. These results convincingly show the appropriateness of the approach based on the similarity hypothesis.

For a quantitative analysis of the strong interactions, however, the similarity hypothesis by itself is insufficient, and one must use dynamical equations based on unitarity and analyticity. In the present paper we study the consequences for the strong interactions of hadrons of the application of three ideas—unitarity, analyticity, and similarity. The problem is made easier by the fact that the required apparatus has already been applied in the solution of other problems,[2-5] where its connection with scale invariance was pointed out. The contents of the present paper are arranged as follows.

In Sec. 2 we derive a purely methodological result which is necessary for the subsequent analysis. Namely, the usual unitarity condition for the Green's function $G(k^2)$, which contains an expansion in terms of $\mathrm{Im}\,(k^2 - m^2 + i0)^{-1}$ (the phase volume), is reformulated in such a way that it involves $\mathrm{Im}\, G(k^2)$.

In Sec. 3 it is shown that the strong-coupling case requires that there be no subtractive constants in the dispersion relations and means that the theory is of the "bootstrap" type, and that neither bare fields nor bare interactions are essential in it. It is shown that the equations derived in Secs. 2 and 3 possess the group of scale transformations.

In Sec. 4 the dynamical equations so obtained are applied to the process $e^+e^- \to \underline{\text{hadrons}}$ at high energies. It is shown that the cross section for production of $n \gg 1$ hadrons, $\sigma_n(s)$ is given by the formula

$$\sigma_n(s) \propto \frac{\alpha^2}{s^{1+\delta}} \chi\left(\frac{n}{s^\delta}\right), \tag{1.2}$$

where s is the square of the energies in the c.m.s., $\alpha = 1/137$, and $0 < \delta < 1/2$. The function $\chi(\nu)$ has the asymptotic behavior

$$\chi(\nu)\,|_{\nu\to 0} \to 0, \qquad \chi(\nu)\,|_{\nu\to\infty} \to \exp\left(-\nu^{1/(1-2\delta)}\right). \tag{1.3}$$

The total annihilation cross section $\sigma_t = \sum_n \sigma_n$ is of the form

$$\sigma_t(s) \infty \alpha^2/s \quad \text{for} \quad s \to \infty. \tag{1.4}$$

The hadron multiplicity $\bar{n}(s)$ is determined by the formula

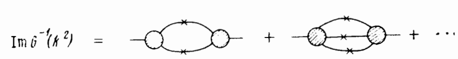
$$\bar{n}(s) = \sum n\sigma_n(s) \Big/ \sum \sigma_n(s) \infty s^\delta. \tag{1.5}$$

In Sec. 5 a possible physical meaning of the scale invariance is brought out, and an analogy is established between the proposed theory and the theory of second-order phase transitions. The question of including the SU(3) × SU(3) symmetry and the weak interactions in the scheme is also considered.

2. REFORMULATION OF THE UNITARITY CONDITION

To prove the asymptotic scale invariance of the amplitudes at small distances or large momenta it is desirable to have equations connecting the amplitudes with each other. The unitarity condition in its usual form is inconvenient for these purposes, since it involves the amplitudes on the mass shell, for which the asymptotic scale invariance is not valid. Therefore we shall reformulate the system of unitarity conditions in such a way that the expansion will go not in terms of the imaginary parts of the poles, $\mathrm{Im}(k^2 - m^2 + i0)^{-1}$, but in terms of the imaginary parts of the Green's functions, $\mathrm{Im}\, G(k^2)$. Furthermore the amplitudes will be taken off the mass shell and will go out to the region of large momenta.

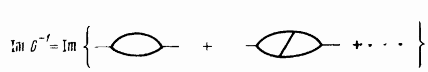
The usual unitarity condition is of the form[7]


(2.1)

Here the quantity $\theta(k_0)\delta(k^2 - m^2)$ is assigned to a line with a cross, and the shaded blocks are exact vertex parts. To simplify the figures we assume that there is only one type of particle. In an actual case with different hadrons all of the amplitudes are matrices in the space of SU(3) and the spin space and have indices denoting the number of the SU(3) multiplet.

Equation (2.1) can be derived by calculating the imaginary part of the perturbation-theory series and using the "separation rule"[7]: one carries out in turn all possible dissections of a diagram, and at each cut replaces the propagator $(k^2 - m^2 + i0)^{-1}$ by $2\pi\theta(k_0)\delta(k^2 - m^2)$. Combination of the diagrams then leads to (2.1).

To reformulate (2.1) we consider instead of the perturbation-theory expansion an expansion in terms of exact Green's functions and calculate the imaginary part of each term. We have

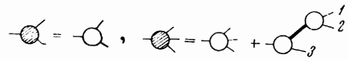

(2.2)

Here the lines correspond to G(k) and the points to bare vertices, which can be arbitrary.

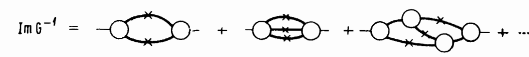
The rule for calculating the Im in (2.2) is again the "separation rule," the only difference being that instead of $\theta(k_0)\delta(k^2 - m^2)$ one uses $\theta(k_0)\mathrm{Im}\, G(k^2)$. This becomes obvious if in (2.2) we substitute the Lehmann expansion for each line. Here, however, the combination of the diagrams does not lead to complete vertex parts, as in (2.1), because in (2.2) there are no diagrams of types such as

(2.3)

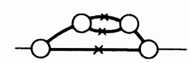
It is convenient to write the resulting equations in the following way.[1)] We introduce nodal vertices—sums of diagrams that cannot be divided by cutting one line. We shall denote them by unshaded circles. Their connection with the complete vertices is obvious:


+ permutations of (1, 2, 3) (2.4)

and so on. Collection of the diagrams in (2.2) leads to the relation


(2.5)

Equation (2.5) is obtained by substituting (2.4) in (2.1), replacing thin lines with a cross by thick lines with a cross, and dropping extra terms of the type


(2.6)

[thick lines with a cross correspond to $\mathrm{Im}\, G(k^2)\theta(k_0)$].

One can also verify that Eqs. (2.5) and (2.1) are equivalent by separating out from (2.5) the terms with n-particle intermediate states. An equation analogous to (2.5) can be written for the case of any number of external lines.

3. THE SIMILARITY HYPOTHESIS AND THE DISPERSION RELATIONS

According to[3,6], the condition of asymptotic scale invariance means that the transformation

$$x \to x' = \lambda x, \quad \varphi_i(x) \to \varphi_i'(x') = \lambda^{-\Delta_i}\varphi_i(x) \tag{3.1}$$

(φ_i are the operators for the various fields, and Δ_i are constants, which we shall call the dimensionalities of the fields φ_i) leaves the correlation functions

$$G_{a_1\ldots a_n}(x_1\ldots x_n) = \left\langle 0 \left| T\prod_{k=1}^{n} \varphi_{a_k}(x_k) \right| 0 \right\rangle \tag{3.2}$$

unchanged, provided that $(x_i - x_j)^2 \ll m^{-2}$. If we translate this requirement into the language of the amplitudes $G_{a_1a_2}(k)$ and $\Gamma_{a_1\ldots a_n}(k_1, \ldots, k_n)$,[3] it means that for $k_ik_j \gg m^2$

$$G_{a_1a_2}(k) = \mathrm{const}\cdot k^{\Sigma_2-4},$$
$$\Gamma_{a_1\ldots a_n}(k_1\ldots k_n) = k_1^{4-\Sigma_n}\gamma_{a_1\ldots a_n}\left(\frac{k_1}{\sqrt{k_1^2}}, \ldots, \frac{k_n}{\sqrt{k_1^2}}\right), \tag{3.3}$$

where $\Sigma_n = \Delta_{a_1} + \Delta_{a_2} + \ldots + \Delta_{a_n}$.

We now want to know under what conditions the asymptotic form (3.3) agrees with the unitarity conditions and the dispersion relations. We begin by finding out about the unitarity conditions. To do so we substitute (3.3) in (2.5) and examine, for example, the first term of this expansion. We have

$$\mathrm{Im}\, G^{-1}(q) = \int |\Gamma(k, q)|^2\theta(k_0)\theta(q_0 - k_0)\,\mathrm{Im}\, G(k)\,\mathrm{Im}\, G(q-k)\, d^4k + \ldots \tag{3.4}$$

Introducing the dimensionless variables

$$\varkappa = k/\sqrt{q^2}, \tag{3.5}$$

we get $\mathrm{Im}\, G^{-1} \infty q^{4-2\Delta}$, in agreement with (3.3).

1)This form was suggested by V. N. Gribov and A. A. Migdal (private communication).

In the derivation of this formula it was assumed that the important regions for the integral are $k^2 \sim q^2 \gg m^2$ and $(k-q)^2 \sim q^2 \gg m^2$. Larger values of k^2 and $(k-q)^2$ are impossible because of conservation of energy:

$$\sqrt{q^2} \geqslant \sqrt{k^2} + \sqrt{(k-q)^2},$$

and the contribution from smaller values is small because the integral converges for small k.

Examination of the n-th term gives the following relation between the dimensions of Γ_n and G:

$$[G^{-1}] = [\Gamma_n^2 G^{n-1} (k^4)^{n-2}]. \tag{3.6}$$

Here square brackets denote dimensions, i.e., the power of k by which the function of the dimensionless arguments is multiplied. Equation (3.6) follows from the fact that the n-th term of (2.5) contains n − 1 Green's functions, two vertices[2)] Γ_n, and n − 2 integrations over d^4k. A comparison of (3.3) and (3.6) shows that these formulas are equivalent.

An examination of the unitarity conditions for the Γ_n themselves again gives (3.6) and (3.3).

This analysis shows that all of the terms of the unitarity condition (2.5) are of the same order for large q^2. If we assume that the number of important terms in (2.5) is of the order of unity, i.e., that the series (2.5) converges, then the sum of the series will be of the same order of magnitude, and the equations (3.3) will be consistent with unitarity.

Let us now find out what choice of subtractive terms in the dispersion relations will correspond to the asymptotic forms (3.3). We first consider the Green's function $G(q^2) \equiv G(s)$; the most general form of the dispersion relation for it is[8]

$$G^{-1}(s) = \frac{s-m^2}{\pi} \int \frac{\operatorname{Im} G^{-1}(s')\, ds'}{(s'-s-i\delta)(s'-m^2)} + \sum \frac{R_n (s-m^2)}{(\mu_n^2 - m^2)(\mu_n^2 - s - i\delta)} + c(s-m^2), \tag{3.7}$$

where $R_n \geq 0$ and $c \geq 0$.

For the existence of similarity in the theory it is required that $\operatorname{Im} G^{-1}(s) \infty G^{-1}(s) \infty s^{\alpha}$ $(\alpha = 2 - \Delta)$ for $s \to \infty$. For (3.7) to converge we must have $\alpha < 1$ or $\Delta > 1$. But then for the condition $G^{-1} \infty s^{\alpha}$ to hold it is necessary that c = 0, since otherwise $G^{-1} \approx cs$ for $s \to \infty$.

The physical meaning of c = 0 is that the term associated with the bare Green's function, $c(s - m^2)$, is absent.[3)] Quite analogously, in the dispersion relations for Γ_n the subtractive terms associated with the bare interaction must be absent, since otherwise it turns out that $\Gamma_n \to c_n$ for $k_i \to \infty$ (where c_n is the constant coefficient of φ^n in the original Lagrangian), in contradiction with (3.3). We postpone a discussion of the question of the possible physical meaning of the similarity conditions (3.3) until Sec. 5.

4. THE PROCESS $e^+e^- \to$ hadrons

The results which were obtained in[6], and which we have reconstructed from a dynamical point of view here, are not directly observable and do not make possible an experimental test of the correctness of the proposed theory. We shall remedy this shortcoming and show that the application of the idea of scale invariance to the process $e^+e^- \to$ hadrons at high energy allows us to make a number of definite predictions which cannot be confused with those of other theories.[9-12]

As is well known,[9] the total cross section for this process is connected with the imaginary part of the hadron contribution to the vacuum polarization, $\operatorname{Im} \Pi_{\mu\mu}(s)$:

$$\sigma_t \infty \frac{\alpha^2}{s^2} \operatorname{Im} \Pi_{\mu\mu}(s). \tag{4.1}$$

Let us calculate the dimensionality of $\operatorname{Im} \Pi_{\mu\mu}$, which determines the behavior of the quantity for $s \gg m^2$. To do so we note that by an expansion of the type of (2.5) we have the relation

$\operatorname{Im} \Pi_{\mu\mu} \sim$  $\sim \Gamma_\mu^2 G^2 r^4$ (4.2)

(here Γ_μ is the vertex part for the emission of a photon).

The terms not written out in (4.2) give a contribution of the same order, owing to the scale invariance. The Ward identity determines the dimensionality of the quantity Γ_μ (see the analogous arguments in[3])[4)]:

$$[\Gamma_\mu] = [G^{-1}] / [q]. \tag{4.3}$$

Substituting (4.3) in (4.2), we find

$$\operatorname{Im} \Pi_{\mu\mu} \infty q^2 \tag{4.4}$$

or

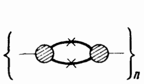
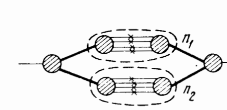
$$\sigma_t(s) \infty \alpha^2 / s \text{ for } s \to \infty. \tag{4.5}$$

The law (4.5) has been proposed on the basis of other arguments in[9], and therefore testing it is not of critical importance for our theory.

Let us proceed to the treatment of more interesting quantities than σ_t, namely the cross sections $\sigma_n(s)$ for the production of n hadrons, and the multiplicity $\bar{n}(s)$.

An important fact for the calculation of $\sigma_n(s)$ is that although for $s \to \infty$ the number of particles produced increases, so that the number of important terms in the unitarity condition taken "by particles" [Eq. (2.1)] increases, the number of "jets", i.e., the number of important terms in the unitarity condition (2.5) taken "by jets" remains of the order of unity. This is because owing to scale invariance all of the terms in (2.5) are of the same order (3.6). In view of the importance of a small number of jets, we can get from (2.5) an equation for the quantity $\Lambda(n, s)$, which is the n-particle contribution to $\operatorname{Im} G^{-1}(s)$, i.e., the probability of finding n particles in a jet whose mass is $s^{1/2}$.

Let us examine, for example, the first "two-jet" term in (2.5) and separate out in it the contribution of n particles:

$\Big\{$  $\Big\}_n = \sum_{n_1+n_2=n}$  (4.6)

Here n_1 is the number of particles in the upper jet, and n_2 the number in the lower jet. The analytic expression for the first graph in (4.6) is

[2)]It is easily verified that the dimensions of all the terms in (2.4) are the same, and it may be assumed that the unitarity involves Γ_n^2. For brevity $\Gamma_{a_1} \ldots {}_{a_n}$ is denoted by Γ_n.

[3)]More exactly, the bare Green's function contributes only to the mass renormalization. The quantity c is the usual renormalization constant.

[4)]An analysis of the equations for $\Gamma_\mu(p, q)$ shows that the dimension of this quantity is the same as that of $\Gamma_\mu(p, 0) = \partial G^{-1}/\partial p_\mu$.

$$\sum_{n_1+n_2=n} \int d^4k |\Gamma(k,q)|^2 |G(k)|^2 |G(q-k)|^2 \theta(k_0)\theta(q_0-k_0).$$
$$\times \Lambda(n_1, k^2)\Lambda(n_2, (q-k)^2). \quad (4.7)$$

If we assume, relying on subsequent verification, that for $n \gg 1$ the terms with $n_1 \gg 1$ and $n_2 \gg 1$ in (4.7) are most important, the integration over d^4k in (4.7) will be over the region

$$k^2 > n_1^2 m^2 \gg m^2, \ (q-k)^2 > n_2^2 m^2 \gg m^2, \quad (4.8)$$
$$\sqrt{q^2} > \sqrt{k^2} + \sqrt{(q-k)^2}.$$

In the region (4.8) the vertices Γ_n and the Green's functions G can be replaced by their asymptotic values (3.3). Substituting (3.3) in (4.7) and using dimensional considerations, we can rewrite (4.7) in the form

$$\int_0^\infty \frac{dn_1 dn_2 ds_1 ds_2}{s_1^{1+\alpha/2} s_2^{1+\alpha/2}} K_2\left(\frac{s_1}{s}, \frac{s_2}{s}\right) \Lambda(n_1 s_1)\Lambda(n_2 s_2) \quad (4.9)$$
$$\times \delta(n - n_1 - n_2)\theta(\sqrt{s} - \sqrt{s_1} - \sqrt{s_2}),$$

where

$$\frac{1}{s_1^{1+\alpha/2}} \frac{1}{s_2^{1+\alpha/2}} K_2\left(\frac{s_1}{s}, \frac{s_2}{s}\right) \equiv \int d^4k |\Gamma(k,q)|^2 |G(k)|^2 |G(q-k)|^2$$
$$\times \delta(k^2 - s_1)\delta((q-k^2) - s_2)\theta(k_0)\theta(q_0 - k_0), \quad (4.10)$$

with $s = q^2$, $G(s) \infty s^{-\alpha}$; the θ function in (4.9) appears owing to (4.8) and expresses the law of conservation of energy. We have replaced the sum over n_1 and n_2 in (4.7) by an integral, since $n_1 \gg 1$ and $n_2 \gg 1$.

Any term in (2.5) can be rewritten in analogy with (4.9). The result is the following expression for $\Lambda(n, s)$:

$$\Lambda(n,s) = \sum_{N=2}^\infty \int \prod_{i=1}^N \frac{dn_i ds_i \Lambda(n_i s_i)}{s_i^{1+(N-1)\alpha/N}} K_N\left(\frac{s_1}{s} \dots \frac{s_N}{s}\right) \delta\left(n - \sum n_i\right)$$
$$\times \theta\left(\sqrt{s} - \sum \sqrt{s_i}\right). \quad (4.11)$$

In Eq. (4.11) N is the number of jets, and n_i is the number of particles in the i-th jet; s_i is the square of the mass of the i-th jet, and K_N is connected with the vertex parts and the Green's functions by a relation analogous to (4.10).

From the definition of the quantity $\Lambda(n, s)$ we have the equation

$$\operatorname{Im} G^{-1}(s) = \int_0^\infty dn\, \Lambda(n,s) = \text{const}\cdot s^\alpha. \quad (4.12)$$

The sum rule (4.12) can be verified by integrating (4.11) over n, which brings the equation to the form (2.5).

We shall look for the solution of (4.11) in the form

$$\Lambda(n,s) = s^{\alpha-\delta}\psi(n/s^\delta). \quad (4.13)$$

Let us substitute (4.13) in (4.11) and introduce the dimensionless variables

$$\nu_i = n_i s^{-\delta}, \quad z_i = s_i/s. \quad (4.14)$$

The equation for the function ψ takes the form

$$\psi(\nu) = \sum_{N=2}^\infty \int_0^\infty \prod_{i=1}^N \frac{d\nu_i dz_i}{z_i^{1+\delta-\alpha/N}} \psi\left(\frac{\nu_i}{z_i^\delta}\right) \cdot K_N(z_1 \dots z_N)\, \delta\left(\nu - \sum_1^N \nu_i\right)$$
$$\times \theta\left(1 - \sum_1^N \sqrt{z_i}\right). \quad (4.15)$$

We shall first show that the condition of solvability of (4.15) uniquely determines δ. To do so we write the conditions for the quantities

$$\overline{\Phi} = \int_0^\infty \psi(\nu)\, d\nu, \quad \overline{\nu\psi} = \int_0^\infty \nu\psi(\nu)\, d\nu. \quad (4.16)$$

that follow from (4.15).

These relations, easily derived by integrating (4.15), are

$$\overline{\Phi} = \sum_{N=2}^\infty \overline{\Phi}^N \int K_N, \quad (4.17)$$

where

$$\overline{\nu\psi} = \overline{\nu\psi} \sum_{N=2}^\infty \overline{\Phi}^{N-1} \int K_N \sum_1^N z_i^\delta,$$

A necessary condition for consistency between the equations (4.17) is

$$\sum_{N=2}^\infty \overline{\Phi}^N \int K_N \sum z_i^\delta \Big/ \sum_{N=2}^\infty \overline{\Phi}^N \int K_N = 1. \quad (4.18)$$

For a known kernel K_N Eq. (4.18) uniquely determines δ. If it could be shown that all the K_N are nonnegative, it would follow from (4.18) and the condition

$$\sum_1^N \sqrt{z_i} < 1,$$

that $0 < \delta < 1/2$. Despite the lack of a proof of nonnegativeness, we can assume that the inequality $0 < \delta < 1/2$ is satisfied, since by (4.13) the average number of particles in a jet is

$$\bar{n}(s) = \int_0^\infty n\Lambda(n,s)\, dn \Big/ \int_0^\infty \Lambda(n,s)\, dn = s^\delta \frac{\overline{\nu\psi}}{\overline{\Phi}} = \text{const}\cdot s^\delta \quad (4.19)$$

Owing to energy conservation the maximum number of particles is of the order of $s^{1/2}$, so that we must have $\delta \le 1/2$.

Let us now examine the behavior of $\psi(\nu)$ for $\nu \gg 1$, i.e., the probability of finding in a jet many more particles than the average number $\bar{n} \infty s^\delta$. We shall look for a solution of (4.15) in the form

$$\psi(\nu) = a(\nu)\exp(-\nu^\gamma), \quad \nu \gg 1, \quad (4.20)$$

where $a(\nu)$ is a function which varies slowly in comparison with an exponential. We substitute (4.20) in (4.15) and calculate the resulting integral by the method of steepest descent.

The exponential part of the answer is determined by the minimum of the argument of the exponential in the integrand:

$$\min \sum_1^n (\nu_i^\gamma / z_i^{\delta\gamma}), \quad (4.21)$$

under the conditions

$$\sum \nu_i = \nu, \quad \sum \sqrt{z_i} \leqslant 1.$$

This minimum is at the point

$$\nu_i = \nu/N, \quad z_i = N^{-2} \quad (4.22)$$

and is equal to

$$N(\nu N^{-1})^\gamma N^{2\delta\gamma} = N^{1-\gamma(1-2\delta)}\nu^\gamma. \quad (4.23)$$

In order for the asymptotic form (4.20) to reproduce itself, it is necessary that the equation

$$\gamma = 1/(1-2\delta). \quad (4.24)$$

hold.

Accordingly,

$$\psi(\nu)|_{\nu\to\infty} \infty \exp(-\nu^{1/(1-2\delta)}). \quad (4.25)$$

We note that in the limit $\nu \to \infty$ the important region in the integrals is where $\Sigma z_i^{1/2} \approx 1$ or $\Sigma s_i^{1/2} \approx s^{1/2}$, which corresponds to disintegration into almost stationary jets.

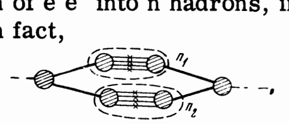
The asymptotic formula (4.25) may have a very narrow range of applicability, since the correction terms (associated with $s_i \sim m^2$) to Eq. (4.11) can decrease more slowly than (4.25) for $n \to \infty$, and finally predominate over it.

The coefficient $a(\nu)$ of the exponential can be calculated with certain assumptions about the behavior of the kernel K_N, and is a monomial. We shall not give these calculations here, since the significance of the result is not worth the complications and the use of a rather special model. We note that besides solutions of the form (4.13) Eq. (4.11) admits solutions of the form[5)]

$$\widetilde{\Lambda}(n, s) = e^{\lambda n} s^{\alpha-\delta} \psi(n / s^{\delta})$$

with an arbitrary value of λ. When, however, we substitute this in the sum rule (4.12) we find that $\lambda = 0$, and we come back to the solution (4.13).

We have here studied in detail the distribution of the particles in a jet. It is clear that it is easy to express the physically interesting quantity $\sigma_n(s)$, the cross section for annihilation of e^+e^- into n hadrons, in terms of this distribution. In fact,

$$\sigma_n(s) \backsim \sum_{n_1+n_2}$$  , (4.26)

i.e., this quantity is directly connected with $\Lambda(n, s)$ and is of the same self-similar form. Using (4.5), we find

$$\sigma_n(s) = \frac{\alpha^2}{s^{1+\delta}} \chi\left(\frac{n}{s^\delta}\right). \tag{4.27}$$

The average number of hadrons produced (the multiplicity) is

$$\bar{n}(s) \equiv \sum n\sigma_n(s) \Big/ \sum \sigma_n(s) = \text{const} \cdot s^\delta. \tag{4.28}$$

When one considers a process with a number of particles larger than the average, the behavior of the cross section is

$$\sigma_n(s) \infty \exp\left[-(ns^{-\delta})^{1/(1-2\delta)}\right], \quad n > s^\delta. \tag{4.29}$$

As has already been mentioned, formulas of the type of Eq. (4.29) obviously do not hold for extremely large values of n, but it is hard to set an exact limit on their validity.

Experimental observation of the self-similar law (4.27) and a power law for the multiplicity in electron-positron annihilation would be an indisputable confirmation of our theory.

5. CONCLUSION. ANALOGIES WITH THE THEORY OF PHASE TRANSITIONS. THE SYMMETRY SU(3) × SU(3) AND THE WEAK INTERACTIONS

An analogy with the theory of phase transitions has great heuristic importance in working with the proposed theory. It is useful to establish an exact correspondence between the concepts of these two theories; in particular, this will help to show how the various symmetries and the interactions of the hadrons fit into our scheme. As the system undergoing a phase transition, let us consider an isotropic Heisenberg ferromagnet. When the temperature approaches the transition point from above, a long-range correlation appears between the spins, extending over distances large in comparison with the interatomic distances. At such distances the crystal lattice does not affect the correlations, and they are isotropic. We may expect that in a relativistic theory the various distortions of the space at very small distances a (the "lattice") will be unimportant at large distances. The scale invariance for $r \gg a$ is physically necessary, since the scale-fixing length a is unknown. All correlations then depend only on relative distances, and not on the orientation and dimensions of the lattice.

Let us look at the question of symmetries in phase transitions and in elementary particles. Above the transition point there necessarily is a symmetry, which is spontaneously destroyed below the transition point. In a Heisenberg ferromagnet this is the symmetry O(3) under simultaneous rotations of all the spins; in a Bose gas it is the symmetry U(1) (the substitution $\psi \to e^{i\alpha}\psi$); and so on. It is natural and, as we shall show, even necessary to assume that the analogous symmetry in elementary-particle theory is SU(3) × SU(3).

We also assume[6,13] that the addition to the Lagrangian that destroys this symmetry is

$$\delta L = \varepsilon w + \varepsilon_0 u_0 + \varepsilon_8 u_8, \tag{5.1}$$

where w is invariant under SU(3) × SU(3), and u_0 and u_8 transform according to the representation (3*, 3) + (3, 3*). In the language of phase transitions the meaning of the hypothesis is as follows. The term ϵw does not change the symmetry of the system, and from the transition point it can take the system either into the ordered or into the unordered phase. In the first case we can say that there is a spontaneous breaking of the conservation of the eight axial currents.[13] The result is that the term ϵw produces a unique finite mass in each SU(3) multiplet and requires that there be eight pseudoscalar mesons with zero mass.[13] The analogy to this in the phase transition is the appearance of a pole $1/k^2$ in the transverse susceptibility below the transition point.

The addition of the term $\epsilon_0 u_0$, where u_0 is a SU(3) singlet, but is a component of the (3*, 3) + (3, 3*) representation of SU(3) × SU(3), is analogous to an external magnetic field in the ferromagnet and produces finite and identical masses of the pseudoscalar mesons. Finally, the term $\epsilon_8 u_8$ gives the octet mass splitting within the SU(3) multiplet. Furthermore, as was pointed out in[13], the masses of the pseudoscalar mesons are of the order of the baryon mass differences, so that $\epsilon_0 \sim \epsilon_8 < \epsilon$.

The second type of symmetry breaking (the system above the transition point) can be treated analogously.

We shall now explain why the symmetry SU(3) × SU(3) must be asymptotically exact (at small distances). To do so we note that some of the 16 currents of SU(3) × SU(3) are weak-interaction currents, and their matrix elements between hadron states are directly observable in the semileptonic decays of hadrons. At the same time, if these currents were not asymptotically conserved, their renormalization would depend on the cutoff radius $\Lambda \sim 1/a$, would contain a factor m/Λ, and would give very small values for the axial and vector weak-decay constants. Owing to the asymptotic conservation of the currents the dependence on Λ drops out (in analogy with the relation $z_1 = z_2$ in quantum electrodynamics), and we get a finite answer.

5) A. A. Migdal called my attention to this fact.

Accordingly it is clear why the weak-interaction currents generate the approximate symmetry group of the strong interactions.

It is tempting to assume that the weak interactions are due to small corrections to our theory owing to very small distances. At present, however, only very speculative arguments can be given on this question.

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[1] L. D. Landau and I. Ya. Pomeranchuk, Dokl. Akad. Nauk SSSR 102, 489 (1955).

[2] V. N. Gribov and A. A. Migdal, Zh. Eksp. Teor. Fiz. 55, 1498 (1968) [Sov. Phys.-JETP 28, 784 (1969)].

[3] A. M. Polyakov, Zh. Eksp. Teor. Fiz. 55, 1026 (1968) [Sov. Phys.-JETP 28, 533 (1969)].

[4] A. A. Migdal, Zh. Eksp. Teor. Fiz. 55, 1964 (1968) [Sov. Phys.-JETP 28, 1036 (1969)].

[5] A. M. Polyakov, Zh. Eksp. Teor. Fiz. 57, 271 (1969) [Sov. Phys.-JETP 30, 151 (1970)].

[6] K. Wilson, Phys. Rev. 179, 1499 (1969).

[7] A. I. Akhiezer and V. B. Berestetskiĭ, Kvantovaya elektrodinamika (Quantum Electrodynamics), Nauka, 1969, p. 609.

[8] B. V. Geshkenbeĭn and B. L. Ioffe, Zh. Eksp. Teor. Fiz. 44, 1211 (1963) [Sov. Phys.-JETP 17, 820 (1963)].

[9] V. N. Gribov, B. L. Ioffe, and I. Ya. Pomeranchuk, Yad. Fiz. 6, 587 (1967) [Sov. J. Nucl. Phys. 6, 427 (1968)].

[10] J. Dooher, Phys. Rev. Letters 19, 600 (1967).

[11] M. Halpern and G. Segrè, Phys. Rev. Letters 19, 611 (1967).

[12] J. D. Bjorkén and S. Brodsky, Preprint SLAC-662 (1969).

[13] M. Gell-Mann, R. Oakes, and B. Renner, Phys. Rev. 175, 2195 (1968).

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