THEORY OF STIMULATED EMISSION FROM MOVING BODIES (KINEMATIC MODULATION)

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Expressions for alternating gain and mode-loss coefficients are obtained by solving the wave equation for an open one-dimensional resonator with plane-parallel mirrors containing a moving plane-parallel dielectric layer. Owing to the Doppler effect, the spatial component of the vector potential of the mode field oscillates in time with a period reciprocal to the doubled frequency of the Doppler shift. Solution of kinetic equations with oscillating gain and loss coefficients in a single-mode approximation leads to the kinematic modulation of laser emission.

As a rule stimulated emission intensity of moving bodies fluctuates at a frequency that is constant for a given velocity of the active medium. This phenomenon, which we call kinematic modulation, was first reported in[1]. Kinematic modulation is revealed experimentally by the periodic variation of the generation power level accompanying the motion of the active medium along the axis of a plane resonator as in "traveling medium" lasers[2].

Mukhtarov noted[3] that during the motion of an active medium in a mode field with a spatial period $\lambda/2$, where λ is the emission wavelength, the boundary conditions at the plane end faces of the medium vary with a frequency ν_k:

$$\nu_k = \frac{\omega_k}{2\pi} = \frac{v}{\lambda/2} = \frac{2v}{\lambda}. \tag{1}$$

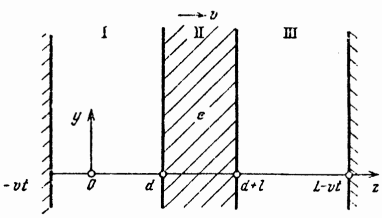
Tursunov measured[4] the velocity dependence of the frequency of kinematic modulation of stimulated emission for various media. It was found that it is determined by (1) with sufficient accuracy.

The stimulated emission theory presented in[5] does not account for the aforementioned motion of the plane boundary of the active medium inside the resonator. Therefore the result of the active-center motion reduces there to the narrowing of the generation spectrum.

This paper presents a theoretical analysis of kinematic modulation. In this connection we consider a problem of the mode field of a plane resonator containing a moving plane-parallel dielectric layer. We show that the layer motion leads to slow pulsations of the spatial component of the electromagnetic field vector-potential of the resonator modes. Inclusion of these pulsations in the kinetic equations of the "traveling medium" laser produces the effect of kinematic modulation of laser emission. In conclusion, kinematic modulation is interpreted as a Doppler effect in electromagnetic standing waves.

1. MODES IN A "TRAVELING MEDIUM" LASER

The figure shows an open plane resonator and a dielectric with plane-parallel end faces moving inside the resonator. Such a compound cavity is the resonator of a "traveling medium" laser.



Naturally, the motion of the dielectric in the axial direction of the resonator should affect primarily the axial structure of the modes. Therefore the problem concerning the motion of the dielectric within the plane resonator as it affects the mode field is reduced to the solution of a wave equation in an infinite (transversely) resonator bounded by specularly reflecting plane-parallel surfaces and containing a moving infinite plane-parallel dielectric layer.

The problem defined in this manner is best solved in a system of coordinates that travel with layer II at a velocity v relative to the plane resonator at rest in the laboratory system of coordinates. Consequently we seek a solution to the wave equation

$$\frac{\partial^2 \mathbf{A}_\alpha}{\partial z^2} - \frac{1}{c_\alpha^2}\frac{\partial^2 \mathbf{A}_\alpha}{\partial t^2} = 0, \qquad \alpha = \mathrm{I}, \mathrm{II}, \mathrm{III}, \tag{2}$$

in a system of coordinates that is fixed with respect to layer II ($\mathbf{A}_\alpha$ and c_α are the vector-potential of the field and velocity of light in medium α, respectively). The following equalities are satisfied at the boundaries of the media (see the figure):

$$\mathbf{A}_{\mathrm{I}}(-vt, t) = 0, \qquad \mathbf{A}_{\mathrm{III}}(L - vt, t) = 0; \tag{3}$$

$$\frac{\partial \mathbf{A}_{\mathrm{I}}}{\partial t}(d-0, t) = \frac{\partial \mathbf{A}_{\mathrm{II}}}{\partial t}(d+0, t),$$
$$\frac{\partial \mathbf{A}_{\mathrm{II}}}{\partial t}(d+l-0, t) = \frac{\partial \mathbf{A}_{\mathrm{III}}}{\partial t}(d+l+0, t); \tag{4}$$

$$\frac{\partial \mathbf{A}_{\mathrm{I}}}{\partial z}(d-0, t) = \frac{\partial \mathbf{A}_{\mathrm{II}}}{\partial z}(d+0, t), \quad \frac{\partial \mathbf{A}_{\mathrm{II}}}{\partial z}(d+l-0, t) = \frac{\partial \mathbf{A}_{\mathrm{III}}}{\partial z}(d+l+0, t). \tag{5}$$

Vector $\mathbf{A}_\alpha$ lies in a plane perpendicular to the z axis according to the transverse-direction rule div $\mathbf{A}_\alpha = 0$. By selecting the polarization plane of vector $\mathbf{A}_\alpha$ as the coordinate plane (z, y), we can consider (3)–(5) valid without change for the values of $\mathbf{A}_\alpha$.

An approximate solution with an accuracy to terms of the order of v/c, satisfying the wave equation (2), is

$$A^j_{\mathrm{I,III}}(z,t) \approx \mu(t)\{\exp[ik^j(z+vt)] - \exp[-ik^j(z+vt)]\}\exp(-i\omega^j t), \quad (6)$$

$$A_{\mathrm{II}}^j(z,t) \approx \mu(t)\{D(t)\exp[ik_\varepsilon^j(z+vt)] - C(t)\exp[-ik_\varepsilon^j(z+vt)]\} \times \exp(-i\omega^j t), \quad (7)$$

where $k^j_\epsilon = \sqrt{\epsilon}\omega^j/c$ is the wave number of the j-th mode inside layer II at rest relative to the resonator, and D(t) and C(t) are slowly varying functions so that $\mu(t)$, a normalizing factor, is also a slowly varying function. The slow variation of the above functions is obtained by postulating that the "traveling medium" laser modes be quasi-stationary:

$$\frac{v}{c} \ll \frac{\delta\omega}{\omega^j} \sim \frac{\lambda}{L}, \quad (8)$$

where $\delta\omega$ is the frequency difference between neighboring modes. Here (6) and (7) describe the mode field of the "traveling medium" laser in the interaction representation[6] based on the eigenfunctions of a resonator with an immobile dielectric layer. The limitation of velocity v in (8) means that the time taken by layer II to traverse a characteristic length λ of the spatial inhomogeneity of a mode should be much longer than the time of flight of a photon along the resonator length L. The physical interpretation of this is that field follows the motion of layer II. Condition (8) narrows down the velocity range since it is a much stronger inequality than the slowmotion condition $v/c \ll 1$ (because it then follows that $v/c \ll \lambda/L$). The usual values for lasers are $\lambda \sim 10^{-4}$ cm and $L \sim 10^2$ cm, so that $v \ll 300$ m/sec.

Considering that owing to slow motion $\partial A_\alpha/\partial t \approx -i\omega^j A_\alpha$, boundary conditions (4) can be written in the form

$$A_{\mathrm{I}}(d-0,t) = A_{\mathrm{II}}(d+0,t), \quad A_{\mathrm{II}}(d+l-0,t) = A_{\mathrm{III}}(d+l+0,t). \quad (4^1)$$

Hence it follows that $C(t) = D^*(t)$, and thus stating D(t) in the form

$$D(t) = \rho(t)\exp[i\psi(t)], \quad (9)$$

we have

$$A_{\mathrm{II}}^j(z,t) \approx \mu(t)\rho(t)\{\exp(i[k_\varepsilon^j(z+vt)+\psi(t)]) - \exp(-i[k_\varepsilon^j(z+vt)+\psi(t)])\}\exp(-i\omega^j t). \quad (10)$$

To determine the function $\rho(t) = |D(t)|$ we must take into account the fact that the field energy is always dissipating in the resonator because, for example, of the finite transmission of the right-hand mirror. But then the mode frequency should be complex and equal to

$$\Omega^j = \omega^j - i\gamma_0^j/2. \quad (11)$$

According to wave equation (2), the wave numbers are also complex:

$$k^j = K^j - i\varkappa^j, \quad k_\varepsilon^j = \sqrt{\varepsilon}k^j = K_\varepsilon^j - i\varkappa_\varepsilon^j, \quad K^j = \omega^j/c, \quad \varkappa^j = \gamma_0^j/2c. \quad (12)$$

Considering that $\kappa^j/K^j = \gamma^j_0/2\omega^j = 1/Q^j \ll 1$, i.e., that the Q-factor of the mode is $Q^j \gg 1$, which is always the case in lasers, we find

$$\rho^2(t) = \tfrac{1}{4}\exp(-2\delta(t)\varkappa_\varepsilon^j)\{(1+1/\sqrt{\varepsilon})^2\exp(2\delta(t)\varkappa^j) + (1-1/\sqrt{\varepsilon})^2\exp(-2\delta(t)\varkappa^j) - 2(1-1/\varepsilon)\cos[2K^j\delta(t)]\},$$

where $\delta(t) = d + vt$, or for high resonator Q

$$\rho^2(t) \approx \tfrac{1}{2}(1+\varepsilon^{-1})[1 - m_\varepsilon\cos\Phi_k^j(t)], \quad (13)$$

where

$$m_\varepsilon = (\varepsilon-1)/(\varepsilon+1), \quad \Phi_k^j(t) = \omega_k^j t + \varphi_k^j, \quad \omega_k^j = 2K^j v, \quad \varphi_k^j = 2K^j d. \quad (14)$$

Modes whose wave numbers satisfy the conditions

$$K^j(L-l) = \pi q, \quad q = 1, 2, \ldots; \quad K_\varepsilon^j l = \pi q', \quad q' = 1, 2, \ldots,$$

have constant frequencies and phases (ω^j_k and φ^j_k). In the remaining modes the frequencies and phases of the kinematic modulation of the field are weakly time dependent.

Expressions (6) and (10) contain the normalizing factor $\mu(t)$ that can be determined from the energy conservation law. The field energy of the j-th mode in the volume $L\sigma$ inside the resonator equals

$$W_{L\sigma^j} = \frac{\sigma(\omega^j)^2}{4\pi c^2}\left[\int_{-vt}^{d}|A_{\mathrm{I}}|^2dz + \int_{d}^{d+l}\varepsilon|A_{\mathrm{II}}|^2dz + \int_{d+l}^{L-vt}|A_{\mathrm{III}}|^2dz\right]. \quad (15)$$

Substituting (6) and (10) into (15) we find

$$W_{L\sigma^j} = \frac{\sigma(\omega^j)^2}{4\pi c^2}[L + \tfrac{1}{2}(\varepsilon-1)l]\mu^2(t)[1 - m\cos\Phi_k^j(t)]\exp(-\gamma_0^j t), \quad (16)$$

where

$$m = \frac{(\varepsilon+1)l}{2L+(\varepsilon-1)l}m_\varepsilon, \quad (17)$$

and ω^j_k and φ^j_k are determined by (14). It is clear that $m \le m_\epsilon$, where the equality corresponds to the limiting case of $l \to L$. On the other hand if $l \ll L$, we have

$$m \approx \frac{1}{2}(1+\varepsilon)\frac{l}{L}m_\varepsilon, \quad (18)$$

i.e., $m \ll m_\epsilon$ in this case.

According to the law of conservation of energy

$$dW_{L\sigma}^j/dt = -\sigma S_n^j, \quad (19)$$

where S^j_n is the normal component of the Poynting vector on the internal surface of the right-hand mirror. Computation yields

$$S_n^j = \frac{(\omega^j)^2}{4\pi c^2}\gamma_0^j L\mu^2(t)\exp(-\gamma_0^j t). \quad (20)$$

Substituting (16) and (20) into (19) and solving the resulting equation for $\mu^2(t)$ we obtain

$$\mu^2(t) = \frac{\mu_0^2}{1 - m\cos\Phi_k^j(t)}\exp\left[-\int\frac{\gamma_0^{j\prime}dt}{1 - m\cos\Phi_k^j(t)} + \gamma_0 t\right], \quad (21)$$

where

$$\gamma_0^{j\prime} = \gamma_0^j/[1 - \tfrac{1}{2}(\varepsilon-1)l/L]. \quad (22)$$

Inserting (21) into (16) we find the loss coefficient of the j-th mode:

$$\gamma^j = \gamma_0^{j\prime}/(1 - m\cos\Phi_k^j(t)). \quad (23)$$

The mode losses of a "traveling medium" laser thus vary periodically. The period of oscillation of the loss coefficient is $T^j_k = 2\pi/\omega^j_k = \lambda^j/2v$. Since usually (see (18)) $m \ll 1$, it follows that

$$\gamma^j \approx \gamma_0^{j\prime}[1 + m\cos\Phi_k^j(t)]. \quad (24)$$

The obtained solution shows that the motion of layer II is accompanied by oscillations of the spatial component of the electromagnetic potential of the mode and of the loss coefficient. In the dielectric layer II these oscillations are defined by the product $\mu(t)\rho(t)$, and outside layer II within the resonator by the function $\mu(t)$. If the layer is very thin (plate) or the resonator is relatively long, the field modulation in gap regions I and III, and consequently the loss coefficient modulation, is very weak since the modulation coefficient m (18) contains the factor l/L. At the same time the field modulation in the active layer II is fairly large and is determined by the modulation coefficient m_ϵ.

Physically the kinematic oscillations are due to the beats of opposed plane waves with frequencies $\omega^j - K^j v$ and $\omega^j + K^j v$, forming the modes of the "traveling medium" laser, as is apparent from the expressions for mode potentials. Doppler shifts $|\Delta\omega_D| = K^j v$ of the opposed traveling-wave frequencies cause an oscillation of the field amplitude, with a fundamental frequency ω_k^j, called the frequency of kinematic modulation and determined by one of the formulas (14), which coincides with (1):

$$\omega_k^j = 2K^j v = 2|\Delta\omega_D| = 2\pi\frac{2v}{\lambda}, \tag{25}$$

so that kinematic field oscillations essentially represent a Doppler effect in standing waves.

It is noted that an inclined position of layer II to the resonator axis also lends itself to mode field modulation analysis and yields corresponding values of the coefficients m_ϵ and m. In particular, if the normal to layer II is inclined at the Brewster angle to the resonator axis and the vector E lies in the plane of incidence, then $m_\epsilon = m = 0$, i.e., there is no kinematic modulation of the field.

2. KINEMATIC MODULATION OF STIMULATED EMISSION

The generation of stimulated emission by moving bodies is analyzed by the kinetic method[5]. In the case of solid media the initial kinetic equations are conveniently written in terms of photon numbers N^j per mode and inverted population n of the active centers as follows:

$$\frac{dn}{dt} = -\frac{n-n_0}{\tau_p} - s\sum_k Dg^k N^k n|A_{ac}^k|^2,$$

$$\frac{dN^j}{dt} = -\gamma^j N^j + \int_{V_{ac}} Dg^j N^j n|A_{ac}^j|^2\,dz, \tag{26}$$

where n_0 is the density of population inversion set up in the active medium by the given pumping, τ_p is the characteristic pumping time that at mode-rate pump levels is close to the spontaneous decay time of the excited state of the active center, D is a coefficient proportional to the probability of stimulated radiative transition between the final and excited states of the active center, g^j is a function describing the probability distribution for the interaction between active centers and photons as a function of their frequency, $|A_{ac}^j|^2$ is the squared modulus of the spatial component of the electromagnetic potential to the j-th mode in the active medium, and γ^j is the loss coefficient of the j-th mode ($|A_{ac}^j|^2$ and γ^j can vary with time at a frequency that is much lower than the homogeneous width of a laser transition); the factor s = 1 for four-level systems and s = 2 for three-level systems.

The initial kinetic equations of the laser can be particularized with respect to the specific features of the resonator field in the active medium and of the loss coefficient. A resonator with constant parameters and with an active medium at rest has a potential $A_{ac}^j(z)$ that is constant in time. This potential is slowly oscillating in a moving medium, as shown by the results obtained in Sec. 1. According to (10), taking (13) and (21) into account,

$$|A_{ac}^j|^2 = |A_{II}^j|^2 = \frac{1-m_\epsilon\cos\Phi_k^j(t)}{1-m\cos\Phi_k^j(t)}\{1-\cos[2K_\epsilon^j(z+vt)+\psi(t)]\} \tag{27}$$

(μ_0^2 was selected so as to have $|A_{ac}^j|^2 = 1 - \cos 2K^j(z \pm vt)$ for $\epsilon = 1$) where m_ϵ and m are given by (14) and (17) respectively. Thus the mode gain in a resonator with a moving layer of the active medium oscillates slowly (as compared to the time of flight of a photon along the resonator). The frequency of kinematic field modulation is $\omega_k \sim 10^7\ \mathrm{sec}^{-1}$ when the active medium moves with velocities of $v \sim 1$ m/sec. The homogeneous width of a laser transition in solid media $\Delta\omega \sim 10^{10}$–$10^{12}\ \mathrm{sec}^{-1}$ so that $\omega_k \ll \Delta\omega$. This means that kinetic equations are suitable for the analysis of generation in solid state lasers also in the case of kinematic modulation of losses and mode field in the active medium.

To determine the features of the modulation effect on generation we consider a single-mode problem assuming that $m \ll 1$. Substituting (27) for the potential in the active medium and (24) for the loss coefficient in (26) and averaging over z, we obtain

$$\frac{dn}{dt} = -\frac{n-n_0}{\tau_p} - sDgNn[1-M\cos\Phi_k(t)],$$

$$\frac{dN}{dt} = -\gamma_0'[1+m\cos\Phi_k(t)]N + DgnlN[1-M\cos\Phi_k(t)], \tag{28}$$

where $M = m_\epsilon - m$ (the mode index is omitted).

We linearize (28) by introducing the new variables

$$n = n_{st}\left[1+\frac{\Delta n(t)}{n_{st}}\right] = n_{st}[1+\eta(t)],$$

$$N = N_{st}\left[1+\frac{\Delta N(t)}{N_{st}}\right] = N_{st}[1+\pi(t)],$$

where $N_{st} = (\alpha-1)/sDg\tau_p$, and $n_{st} = n_0/\alpha$ is the solution of the stationary problem at M = m = 0 ($\alpha = Dgn_0l/\gamma_0'$). Eliminating η from the resulting system we then obtain the equation of kinematic modulation of laser intensity:

$$[1+M\cos\Phi_k(t)]\ddot{\pi} + \left[\frac{\alpha}{\tau_p} + m_\epsilon\gamma_0'\cos\Phi_k(t) - M\omega_k\sin\Phi_k(t)\right]\dot{\pi}$$
$$+\left[\frac{\alpha-1}{\tau_p}\gamma_0' - m_\epsilon\omega_k\gamma_0'\sin\Phi_k(t)\right]\pi = m_\epsilon\omega_k\gamma_0'\sin\Phi_k(t). \tag{29}$$

This linearization is acceptable provided the frequency ω_k in (29) is much larger than the natural frequency of the relaxation oscillations of the laser intensity:

$$\omega_k \gg \omega_r = \sqrt{(\alpha-1)\gamma_0'/\tau_p}.$$

Since usually $\tau_p \sim 10^{-3}$–10^{-5} sec, it follows that $\omega_k \gg \tau_p^{-1}$, so that we neglected the corresponding terms in the derivation of (29).

The approximate solution of (29) has the form

$$\pi(t) \approx \mp a_k \sin(\omega_k t + \varphi_k), \quad a_k = m_\epsilon \gamma_0' / \omega_k. \qquad (30)$$

As it follows from (30), the amplitude a_k of the kinematic modulation of the single-mode generation of moving bodies is determined by the value of m_ϵ on the one hand, and by the ratio of γ_0' to ω_k on the other. The values of m_ϵ are computed from (14): for ruby $m_\epsilon \approx 1/2$, and for glass $m_\epsilon \approx 1/3$, so that strictly speaking the solution of (28) yields in the linear approximation only a qualitative picture of the phenomenon. This should be emphasized because the ratio γ_0'/ω_k also often exceeds unity appreciably. We can therefore expect both a large modulation amplitude and the appearance of modulation components $2\omega_k$, $3\omega_k$, etc., since the exact solution of (29) contains harmonics with the frequencies $2\omega_k$, $3\omega_k$, etc. We also add that if the kinematic modulation frequency ω_k is close to one of the resonance frequencies of laser intensity $\omega_r/2$, ω_r, $2\omega_r \ldots$, the occurrence of both stable and unstable resonance oscillations at the frequency ω_r is possible[7].

According to (14) the amplitudes of "traveling medium" laser modes are in phase, so that in the case of multimode generation we should consider the possibility of mode locking. Then, however, to describe the stimulated emission the kinematic balance equations such as (26) should be supplemented with terms that take amplitude interference into account.

3. KINEMATIC MODULATION AND DOPPLER EFFECT

As noted above, the Doppler effect is the foundation of the kinematic modulation phenomenon. Namely, modes interacting with the moving body are no longer represented as a linear combination of opposed plane waves that are degenerate in frequency, but as a linear combination of waves whose frequencies differ by $2|\Delta\omega_D| = \omega_k$. However, this is yet insufficient to bring about kinematic modulation of the field. In fact if $\epsilon = 1$, then $m_\epsilon = m = 0$ and there is no modulation.

The discontinuity in dielectric permittivity of a moving body relative to the medium filling the resonator causes the beats of the Doppler mode components outside and inside the body to occur at different frequencies. The beats outside the body, acting like a driving force, impose a time-varying amplitude upon the beats inside the body. In turn, the amplitude of beats outside the body within the resonator also varies in accord with the law of conservation of energy.

On the other hand, according to (27), even for $\epsilon = 1$ kinematic modulation of field amplitude occurs at a given point z of the active medium with the frequency $\omega_k^\epsilon \approx 2K_\epsilon v$. However the average (with respect to z) field amplitude is the same at any time, i.e., this modulation merely means that each active center periodically varies its position relative to the spatial distribution of the field; the narrowing of generation spectrum determined in[5] is a result of this.

Consequently the Doppler effect in stimulated emission of moving bodies is manifested in two ways:

1. In averaging of the mode field relative to a given active center; at a sufficiently large velocity of motion (corresponding to sufficiently high frequency of kinematic modulation) this converts the "traveling medium" laser into a system analogous to the traveling-wave laser in terms of the emission spectrum[2,5];
2. In a kinematic modulation of the gain and of the mode-loss coefficients, leading to a corresponding modulation of the laser emission intensity.

4. CONCLUSION

The experimental results concerning kinematic modulation of laser emission presented in[4] confirmed the universal validity of the phenomenon discussed above.

A detailed spectral analysis of kinematic modulation of stimulated emission shows also that under certain conditions it can be the cause of mode locking (broad continuous generation spectra[4,8]). Under other conditions the generation becomes single-mode, although kinematic modulation remains. This will be discussed in greater detail in a special paper on the spectral properties of emission from "traveling medium" ruby lasers.

The experiment also confirmed the fact that kinematic modulation of stimulated emission vanishes at a Brewster orientation of the end faces of the active rod. Furthermore, kinematic modulation is not observed even when the angle between normal to the rod face and the resonator axis is of the order of one degree. This is obviously due to the fact that at an inclined-position of the active layer the phase of kinematic molulation varies continuously in the transverse direction so that the modulation effect is smeared out if the diaphragm inside the resonator is sufficiently large.

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