

*ASYMPTOTIC ANALYSIS OF OSCILLATORY MODE OF APPROACH TO A SINGULARITY IN
HOMOGENEOUS COSMOLOGICAL MODELS*

E. M. LIFSHITZ, I. M. LIFSHITZ, and I. M. KHALATNIKOV

Institute of Physics Problems, USSR Academy of Sciences, L. D. Landau Institute of Theoretical
Physics, USSR Academy of Sciences

Submitted February 25, 1970

Zh. Eksp. Teor. Fiz. 59, 322-336 (July, 1970)

An analysis of the evolution of the metric in the oscillatory mode of approach to a singularity in homogeneous cosmological models is carried out in the asymptotic region of arbitrarily small times. It is shown that in this region the successive interchange of the "Kasner epochs" depends in each case on a single "perturbation" (in other words, the case when two types of perturbations are operative never arises). This permits one to carry out analytic and statistical investigations of the evolution of the model with a high degree of completeness. Recurrent formulas are derived for the oscillation periods and the amplitudes during a single era (series of Kasner epochs). Formulas relating successive eras are also derived. The statistical properties of alternation of successive eras are analyzed and statistical distribution functions are obtained for the quantities characterizing the process. The law governing the probability of increase of density of matter on approach to a singularity is found.

In preceding communications by two of us and V. A. Belinskiĭ, we discovered an oscillatory mode of approach to a singularity in time in the cosmological solutions of Einstein's equations^[1-3]. The particular cases of homogeneous Bianchi models of type IX and VIII were considered (the former was considered also by Misner^[4]), and arguments were advanced indicating that this should be precisely the character of the singularity in the general solution of the gravitational equations.

The present communication is devoted to a further study of the homogeneous models. We shall show that in the asymptotic limit of times t that are arbitrarily close to the singularity it is possible to carry out analytic and statistical investigation of the evolutions of a model with appreciable density.

We leave aside here the question of the possible connection between the parameters of this evolution and the time scale of a real world. It may turn out that the asymptotic limit lies in the region of such high densities of matter that the use of the existing gravitational theory may not be realistic. In this connection, however, it is appropriate to recall that although the physical applicability of Einstein's equations in their present form can be clarified under the indicated "singular" conditions only by a future synthesis of physical theories, the existing gravitational theory itself does not lose its logical cohesion (it does not lead to internal contradictions) at any density of matter. In other words, this theory is not limited by any conditions that follow from the theory itself and can make its application at very high densities logically incorrect and contradictory; the limitations can arise only as a result of factors that are extraneous with respect to the gravitational theory itself. This circumstance makes it valid, at any rate formally, and in our opinion necessary, to consider the question of the singularities in cosmological models even within the framework of the existing theory.

1. GENERAL CHARACTER OF THE EVOLUTION OF THE MODEL

Let us recall the main properties of the considered solution of the gravitational equations in the form needed for the subsequent investigation.

The spatial metric of the homogeneous model is written in the form

$$dl^2 = (a^2 l_\alpha l_\beta + b^2 m_\alpha m_\beta + c^2 n_\alpha n_\beta) dx^\alpha dx^\beta, \quad (1.1)$$

where \mathbf{l} , \mathbf{m} , and \mathbf{n} are three-dimensional reference vectors that are definite functions of the spatial coordinates. The concrete form of these functions is immaterial. It is only important that for metrics of types IX and VIII the quantities

$$\lambda = \frac{1}{v} (\mathbf{l} \text{ rot } \mathbf{l}), \quad \mu = \frac{1}{v} (\mathbf{m} \text{ rot } \mathbf{m}), \quad \nu = \frac{1}{v} (\mathbf{n} \text{ rot } \mathbf{n}) \quad (1.2)$$

(where $v = \mathbf{l} \cdot \mathbf{m} \times \mathbf{n}$) are constant, and all the remaining products of the type $\mathbf{l} \text{ curl } \mathbf{m}$, $\mathbf{l} \text{ curl } \mathbf{n}$, ... are equal to zero. The numbers λ , μ , and ν are none other than the structural constants of the group of motions of space. For a space of type IX, all three coordinates have the same sign and we can put $\lambda = \mu = \nu = 1$. On the other hand, for a space of type VIII, one of the constants has a sign opposite to that of the two others; we can put $\lambda = -1$ and $\mu = \nu = 1$. The entire analysis that follows pertains to an equal degree to both models.

The quantities a , b , and c in (1.1) are functions of the world synchronous time t , and determine the scales of the spatial distances in the directions \mathbf{l} , \mathbf{m} , and \mathbf{n} . The temporal evolution of the model is described in terms of these functions.

The key to the understanding of the character of the evolution of the metric on approaching the singular point is the interchange of the "Kasner epochs," during the course of which the functions a , b , and c vary like

$$a \sim t^{p_1}, \quad b \sim t^{p_2}, \quad c \sim t^{p_3}, \quad (1.3)$$

where the three numbers p_l , p_m , and p_n coincide with the numbers of any one of the triads p_1 , p_2 , p_3 satisfying the condition¹⁾

$$p_1 + p_2 + p_3 = p_1^2 + p_2^2 + p_3^2 = 1. \quad (1.4)$$

These triads, arranged once and for all in a definite sequence $p_1 < p_2 < p_3$, can be parametrized in the form

$$p_1(u) = \frac{-u}{1+u+u^2}, \quad p_2(u) = \frac{u+1}{1+u+u^2}, \quad p_3(u) = \frac{u(u+1)}{1+u+u^2} \quad (1.5)$$

where the parameter u runs through values in the region $u \geq 1$. On the other hand, values $u < 1$ can be reduced to the same region in accordance with the formulas

$$p_1\left(\frac{1}{u}\right) = p_1(u), \quad p_2\left(\frac{1}{u}\right) = p_3(u), \quad p_3\left(\frac{1}{u}\right) = p_2(u). \quad (1.6)$$

The function $p_1(u)$ is always negative, and $p_2(u)$ and $p_3(u)$ are positive; $|p_1(u)|$ and $p_2(u)$ decrease monotonically, while $p_3(u)$ increases monotonically as u increases from 1 to ∞ .

When t decreases, one of the functions a , b , or c increases, and the two other decrease. Let, for example,

$$p_l = p_1(u), \quad p_m = p_2(u), \quad p_n = p_3(u), \quad (1.7a)$$

so that the function $a(t)$ increases and the function $b(t)$ decreases with the smaller of the two positive exponents. This process leads to a replacement of the Kasner regime with indices (1.7a) by a regime with exponents

$$p_l' = p_2(u-1), \quad p_m' = p_1(u-1), \quad p_n' = p_3(u-1). \quad (1.7b)$$

The function $a(t)$ acquires a positive exponent and begins to decrease, the function $b(t)$ acquires a negative exponent and begins to decrease, while the functions $c(t)$ continue to decrease.

The transition region between the interchange of regimes is described by the formulas

$$a^2 = \frac{2|p_1|\Lambda}{\text{ch}(2|p_1|\Lambda\tau)}, \quad b^2 = \epsilon_0^2 e^{2\Lambda(p_2-1)p_1\tau} \text{ch}(2|p_1|\Lambda\tau), \quad (1.8)$$

$$c^2 = c_0^2 e^{2\Lambda(p_3-1)p_1\tau} \text{ch}(2|p_1|\Lambda\tau),$$

where τ is a variable connected with t by the equation $dt = abcd\tau$, and the point $\tau = 0$ is arbitrarily chosen to coincide with the instant of the maximum of the function $a(\tau)$. The asymptotic forms of these expressions as $\tau \rightarrow +\infty$ and $\tau \rightarrow -\infty$ correspond to the initial and final Kasner regimes with exponents (1.7a) and (1.7b) respectively. In the former

$$abc = \Lambda t, \quad \tau = \Lambda^{-1} \ln t + \text{const}, \quad (1.9a)$$

and in the latter

$$abc = \Lambda' t, \quad \tau = \Lambda'^{-1} \ln t + \text{const}, \quad \Lambda' = \Lambda(1-2|p_1|). \quad (1.9b)$$

The maximum value of the function $a(\tau)$ is

$$a_{\max} = \sqrt{2|p_1|\Lambda}, \quad (1.10)$$

and it is assumed that this value is large compared with b_0 and c_0 (more accurately, we should have $a^2 \gg b^2, c^2$).

¹⁾The law (1.3)–(1.4) is a solution of Einstein's equations for a homogeneous (but anisotropic) Euclidean space (the Kasner solution—see [5], Sec. 103), and is exact for all time.

Further evolution with increasing function $b(t)$ leads in analogous fashion to the next alternation of Kasner epoch, etc. The successive alternation in accordance with the rule (1.7), with exchange of the negative exponent between the functions a and b (i.e., between the directions l and m), continues until the integer part of the initial value of u is exhausted and we get $u < 1$. The value $u < 1$ is transformed into $u > 1$ in accordance with (1.6), and p_n becomes the smaller of two positive numbers ($p_n = p_2$). The next series of interchanges will now transfer the negative exponent from c to a or from c to b . At an arbitrary (irrational) initial value of u , the process of interchanges continues without limit.

The process of evolution of the metric on approaching the singular point consists, consequently, of a successive periods (which we shall call *eras*), during each of which the distance scales oscillate along two spatial axes, and decrease monotonically along the third axis. On going from one era to another, the direction along which the monotonic decrease of the distances takes place is transferred from one axis to another.

To each (s -th) era there corresponds a series of values of the parameter u , starting with a certain largest one, $u_{\max}^{(s)}$, and reaching the smallest one, $u_{\min} < 1$, via the values $u_{\max}^{(s)} - 1, u_{\max}^{(s)} - 2, \dots$ we put

$$u_{\max}^{(s)} = k^{(s)} + x^{(s)}, \quad u_{\min}^{(s)} = x^{(s)}, \quad (1.11)$$

i.e., $k^{(s)} = [u_{\max}^{(s)}]$ (the square brackets denote the integer part of a number). The number $k^{(s)}$ determines the "length" of the era, measured in terms of the number of Kasner epochs it contains. For the next era

$$u_{\max}^{(s+1)} = 1/x^{(s)}, \quad k^{(s+1)} = [1/x^{(s)}]. \quad (1.12)$$

In the exact solution of the equations, the exponents p_l , p_m , and p_n lose, of course, their literal meaning. We note that a certain "fuzziness" introduced by this circumstance in the definition of these numbers (and with them also of the parameter u), albeit small, makes it meaningless to consider some selected (say, rational) values of u . This is precisely why the only laws with any real meaning are those pertinent to the general case of arbitrary (irrational) values of u .

In an infinite sequence of the series of numbers u , made up in accordance with the rules (1.11)–(1.12), there will be observed arbitrarily small (but never vanishing) values $x^{(s)}$ and accordingly arbitrary large lengths $k^{(s+1)}$. Large values of the parameter u correspond to Kasner exponents

$$p_l \approx -\frac{1}{u}, \quad p_2 \approx \frac{1}{u}, \quad p_3 \approx 1 - \frac{1}{u^2}, \quad (1.13)$$

close to the values $(0, 0, 1)$. Two exponents that are close to zero are by the same token close to each other, and therefore the laws governing the variation of two of functions a , b , or c are also close. If at the start of such a "long" era these functions at the instant of interchange of two Kasner epochs turn out to be close to each other also in absolute magnitude (or if such are arbitrarily specified in accordance with the initial conditions), then they will continue to stay close also during the greater part of the duration of the era; the evolution of the metric requires in this case a

special analysis, which was carried out in^[2], Sec. 4. We shall see, however, that during the process of spontaneous evolution in the asymptotic region of arbitrarily small times t , such cases cease to appear: even in "long" eras, both oscillating functions remain so different in magnitude during the instants of interchange, that the interchanges themselves will be described, as before, by the described rules.

The presence of matter does not influence the evolution of the metric of space near the singularity^[6]. In other words, the matter can be "written in" into the specified metric, and its reaction on the metric can be neglected. During each of the Kasner epochs, the density of matter ϵ varies like

$$\epsilon \sim t^{-2(1-\mu)}, \quad (1.14)$$

where p_3 , by agreement, is the larger of the numbers p_1 , p_2 , and p_3 (see^[6], Sec. 3, or^[7]). The density of matter increases monotonically during the entire evolution to the singular point.

The law (1.14) corresponds to matter "written in" with an arbitrary initial velocity distribution. On the other hand, if matter is written-in into the considered model, understood as an exact solution of the Einstein equations, then the resultant picture of the evolution of matter would have no general character at all, and would be unique only for the high symmetry possessed by this model. Mathematically this unique character is connected with the fact that, for the considered homogeneous spatial geometry, the components R_0^α of the Ricci tensor are identically equal to zero, and therefore Einstein's equations would not admit of motion of matter (which would lead to the appearance of nonzero components T_0^α of the energy-momentum tensor). In other words, the synchronous reference system should be also co-moving with respect to the matter. We would then obtain for the variation of the density of matter the formula $\epsilon \sim t^{4/3}$.

2. EVOLUTION OF THE MODEL DURING ONE ERA

For further analysis it is convenient to introduce in place of the functions a , b , and c the logarithms α , β , and γ :

$$a = e^\alpha, \quad b = e^\beta, \quad c = e^\gamma. \quad (2.1)$$

During each Kasner epoch we have in accordance with (1.9) $\alpha + \beta + \gamma = \ln \Lambda + \ln t$. Changing over from one epoch to another, the constant $\ln \Lambda$ changes (in accordance with (1.9)) by an amount of the order of 1. But in the asymptotic region of arbitrarily large values of $|\ln t|$, we can neglect not only these changes, but also the entire constant $\ln \Lambda$. In other words, the employed approximation corresponds to neglecting all the quantities whose ratio to $|\ln t|$ tends to zero as $t \rightarrow 0$. We then have

$$\alpha + \beta + \gamma = -\Omega, \quad (2.2)$$

where Ω denotes the "logarithmic time":

$$\Omega = -\ln t. \quad (2.3)$$

In the same approximation, we can consider the processes of interchanges of epochs as instantaneous. We can also neglect the constant in the right side of

the condition (1.10), $\alpha_{\max} = \frac{1}{2}n(2|p_1| \Lambda)$, which determines the instants of the interchanges, i.e., to assume as this condition the equality $\alpha = 0$ (or similar equalities for β or γ , if the initial negative exponent pertains to the functions b or c). We thus put

$$\alpha_{\max} = 0, \quad \beta_{\max} = 0, \quad \gamma_{\max} = 0, \quad (2.4)$$

so that the quantities α , β , and γ run only through negative values that are connected with one another at each instant of time by the relation (2.2).

Regarding the interchange of epochs as instantaneous, we neglect the widths of the transition regions between epochs compared with the durations of the epochs themselves; this condition is actually satisfied (see footnote 5 below). On the other hand, replacement of (1.10) by (2.4) requires that the quantity $|\ln(|p_1| \Lambda)|$ be small compared with the amplitudes of the oscillations of the corresponding functions α , β , and γ . But on going from one era to the next there can appear, as was noted in Sec. 1, very small values of $|p_1|$, and these values and the probability of their appearance are not connected in any way with the value of the oscillation amplitudes reached by that instant of time. One can therefore not exclude, in principle, also the appearance of such small values of $|p_1|$, for which the required condition would be violated²⁾. Such a strong lowering of α_{\max} can lead to different specific situations, in which the joining together of the Kasner epoch in accordance with the rule (1.7) becomes incorrect (for example in a situation in which two of the functions α , β , and γ are close during the entire era). These "dangerous" cases call for a special analysis, and at any rate would violate the laws employed for the statistical analysis in Sec. 4 below. It can be shown, however, that the probability of such violations tends asymptotically to zero; we shall return to this question at the end of Sec. 5.

Let us consider an era that contains k Kasner epoch corresponding to the parameter u running through the values

$$u_n = k + x - 1 - n, \quad n = 0, 1, \dots, k-1 \quad (2.5)$$

and let the oscillating functions during that era be α and β (see Fig. 1)³⁾.

During the instants of the start of the Kasner epoch with parameters u_n by Ω_n . At each of these instants, one of the quantities, α or β , is equal to zero, and the other has a minimum. The values of α or β at the succeeding minima, i.e., at the instants Ω_n , will be denoted by

$$\alpha_n = -\delta_n \Omega_n \quad (2.6)$$

(without distinguishing between the minima of α and β). The quantities δ_n , which measure these minima in units corresponding to Ω_n , can have values between 0 and 1.

²⁾This danger was pointed out to us by A. G. Doroshkevich and I. D. Novikov.

³⁾The definition of the limits of the era in accordance with (2.5) is natural in that sense that it combines all the epochs during which the third function $\gamma(t)$ decreases monotonically. Were we to define the era in accordance with the sequence of values of u from $k+x$ to $1+x$, then the monotonic decrease of $\gamma(t)$ would continue also during the first epoch of the following era.

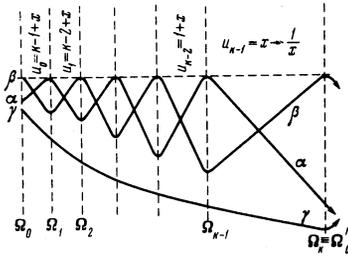


FIG. 1. Schematic representation of the course of variation of α , β , and γ as functions of the logarithmic time Ω during one era. The vertical dash lines note the instants of interchanges of the Kasner epoch, corresponding to the linear segments of the curves between the dashed lines. On the top are indicated the values of the parameter u which determine the Kasner exponents. The last epoch has a longer duration if x is small. In the first epoch of the next era, γ begins to increase and α becomes a monotonically decreasing function.

On the other hand, the function γ decreases monotonically during the course of the given era; according to (2.2), its value at the instant Ω_n is

$$\gamma_n = -\Omega_n(1 - \delta_n). \quad (2.7)$$

During the course of the epoch that begins at the instant Ω_n and ends at the instant Ω_{n+1} , one of the functions, α or β , increases from $-\delta_n\Omega_n$ to zero, and the other decreases from zero to $-\delta_{n+1}\Omega_{n+1}$, in accordance with the linear laws $\text{const} + |p_1(u_n)|\Omega$ and $\text{const} - p_2(u_n)\Omega$, respectively. From this we obtain a recurrence relation

$$\delta_{n+1}\Omega_{n+1} = \frac{1 + u_n}{u_n} \delta_n\Omega_n = \frac{1 + u_0}{u_n} \delta_0\Omega_0, \quad (2.8)$$

and for the logarithmic duration of the epoch we get

$$\Delta_{n+1} \equiv \Omega_{n+1} - \Omega_n = \frac{f(u_n)}{u_n} \delta_n\Omega_n = \frac{f(u_n)(1 + u_{n-1})}{f(u_{n-1})u_n} \Delta_n, \quad (2.9)$$

where we put for brevity $f(u) = 1 + u + u^2$. For the total duration n of the epoch we can obtain the formula

$$\Omega_n - \Omega_0 = \left[n(n-1) + \frac{nf(u_{n-1})}{u_{n-1}} \right] \delta_0\Omega_0. \quad (2.10)$$

It is seen from (2.8) that $|\alpha_{n+1}| > |\alpha_n|$, i.e., the swing of the oscillations of the functions α and β increases during the entire era (whereas the coefficients δ_n may be also small). If the depth of the minimum at the start of the error was large, then it will no longer be small in the succeeding minima; in other words, the difference $|\alpha - \beta|$ at the instants of the interchange of the Kasner epochs remains large. We emphasize that this statement does not depend on the length of the era k , so that for long eras the interchanges of the epochs will be determined by the usual rule (1.7)⁴.

The amplitude of the last oscillation of the function α or β in the given era is connected with the amplitude of the first oscillation by the relation $|\alpha_{k-1}| = |\alpha_0|(k+x)/(1+x)$. Even at lengths k that amount to only several units it is already possible to neglect x compared with k , so that the increase of the amplitude

of the oscillations of the functions α and β will be proportional to the length of the era. For the functions $a = e^\alpha$ and $b = e^\beta$ this means that if the amplitude of their oscillations at the start of the era with A_0 , at the end of the era it will be $A_0^{k/(1+x)}$.

In the approximation in question, the maxima of the oscillating functions α and β remain at a constant level (condition (2.4)). Actually these maxima decrease somewhat in the successive oscillations during the course of the era. Consequently, we get from the condition (10)

$$(a_{\max}^2)_{n+1}/(a_{\max})_n^2 = |p_1(u_{n+1})|\Lambda_{n+1}/|p_1(u_n)|\Lambda_n,$$

According to (1.9), $\Lambda_{n+1}/\Lambda_n = 1 - 2|p_1(u_n)|$. From this we get

$$\frac{(a_{\max})_{n+1}}{(a_{\max})_n} = \sqrt{1 - \frac{1}{u_n}}. \quad (2.11)$$

This lowering of the maxima, of course, is small compared with the lowering of the minima; thus, at $u_n \gg 1$ we have from (2.11)

$$|(a_{\max})_{n+1} - (a_{\max})_n| \approx 1/2u_n,$$

whereas the lowering of the minima amounts to, according to (2.8),

$$|(\alpha_{\min})_{n+1} - (\alpha_{\min})_n| \approx |(\alpha_{\min})_n|/u_n,$$

where $|(\alpha_{\min})_n|$ is already assumed to be large.

During the era, an increase takes place also in the duration (in logarithmic time) of the successive Kasner epochs; it is easy to conclude from (2.9) that $\Delta_{n+1} > \Delta_n$ ⁵. The total duration of the era is

$$\Omega'_0 - \Omega_0 \equiv \Omega_k - \Omega_0 = \left(k + x + \frac{1}{x} \right) \delta_0\Omega_0 \quad (2.12)$$

(the term with $1/x$ is the result of the last k -th epoch, which is large at small values of x —see the figure). The instant Ω_k of the termination of the k -th epoch of a given era is at the same time the instant Ω'_0 at the start of the next era.

In the first Kasner epoch of this new era, the function γ begins first to grow from the minimal value $\gamma_k = -\Omega_k(1 - \delta_k)$ attained by it in the preceding era; this value will play the role of the initial amplitude $\delta'_0\Omega'_0$ of the new series of oscillations. It is easy to obtain for it

$$\delta'_0\Omega'_0 = \left(\frac{1}{\delta_0} + k^2 + kx - 1 \right) \delta_0\Omega_0. \quad (2.13)$$

Obviously $\delta'_0\Omega'_0 > \delta_0\Omega_0$. Even at not very large lengths k , the growth of the amplitude is quite appreciable: the function $c = e^\gamma$ begins to oscillate from the amplitude $A'_0 \sim A_0^{k^2}$. (We leave aside the "dangerous" cases, mentioned in Sec. 1, of very strong lowering of the lower limit of the oscillations.)

According to (1.14), the growth of the density of matter during each of the first $(k-1)$ -st epochs is given by the formula

$$\ln(\varepsilon_{n+1}/\varepsilon_n) = 2[1 - p_3(u_n)]\Delta_{n+1}.$$

⁴As already indicated, the investigation reported in [2], Sec. 4, pertains to the inverse case of a long era during the entire length of which $|\alpha - \beta| \ll 1$.

⁵We note also that these durations are large compared with the widths of the transition regions between the epochs. According to (1.8), these widths are large at small $|p_1|$ (i.e., large u) and amount to $\sim 1/|p_1| \sim u$. But even in this case $\Delta_n \sim u_n|\alpha_n| \gg u_n$.

For the last, k -th epoch of a given era, on the other hand, it is necessary to recognize that when $u = x < 1$ the largest exponent is $p_2(x)$ (and not $p_3(x)$). As a result we obtain for the growth of the density during the entire era

$$\ln(\epsilon_k/\epsilon_0) \equiv \ln(\epsilon_0'/\epsilon_0) = 2(k-1+x)\delta_0\Omega_0. \quad (2.14)$$

Already at not very large values of k we have, consequently, $\epsilon_0'/\epsilon_0 \sim A_0^{2k}$. During the next era (with length k'), the increase of the density will be even faster, by virtue of the increase of the initial amplitude A_0' : $\epsilon_0''/\epsilon_0' \sim A_0'^{2k'} \sim A_0^{2kk'}$, etc. These formulas illustrate the vigorous character of the growth of the density of matter.

In concluding this section, let us make one more remark of methodological character. Misner has proposed, for the description of the considered solution of Einstein's equations, a mechanical model in which a "particle" moves in the field of a time-dependent potential^[4]. It seems to us that this model can reflect only some of the most general properties of the solution, and is not convenient for a more detailed investigation. The very introduction of the potential is connected with difficulty-estimated approximations (for example, the quantity Λ introduced in^[4] should be regarded in one case as a constant and in one case as a variable); it is in any case not suitable in regions near the corners of the diagram. By way of a condition determining the boundaries of the motion of the "particle," Misner proposes the equality of the potential of the kinetic energy of the free motion, which does not depend on its direction: by the same token, for example, effects connected with the dependence of a_{\max} on p_1 in (1.10) drop out. Yet, as seen from the foregoing, an analytic consideration based on the exact Einstein equations for the functions a , b , and c makes it possible to trace the evolution of the model in a rather simple and lucid manner (and this analysis can be carried out without particular difficulty, and with greater accuracy than in the present paper).

3. STATISTICAL PROPERTIES OF THE NUMERICAL SEQUENCE OF VALUES OF THE PARAMETER u

The sequence of the lengths $k^{(s)}$ of the successive eras (measured in terms of the number of the Kasner epochs contained in them) acquires asymptotically the character of a random process. The same pertains also to the sequence of the interchanges of the pairs of oscillating functions on going over from one era to the next one (it depends on whether the numbers $k^{(s)}$ are even or odd).

A source of this statistical behavior is the rule (1.12), according to which the transition from one era to the next is determined in an infinite numerical sequence of the values of the parameter u . This rule states, in other words, that if the entire infinite sequence begins with a certain initial value $u^{(0)} = k^{(0)} + x^{(0)}$, then the lengths of the series $k^{(0)}, k^{(1)}, \dots$, are the numbers in the continuous-fraction expansion

$$k^{(0)} + x^{(0)} = k^{(0)} + \frac{1}{k^{(1)} + \frac{1}{k^{(2)} + \dots}}. \quad (3.1)$$

It is possible to change over to a probabilistic de-

scription of such a sequence by considering not a definite initial value $x^{(0)}$ but the values $x^{(0)} = x$ distributed in the interval from 0 to 1 in accordance with a certain specified law $w_0(x)$. Then the values of $x^{(s)}$ terminating each series will also have distributions that follow certain laws. Let $w_s(x)dx$ be the probability that the s -th series terminates with the value $u_{\min}^{(s)} = x$ lying in a specified interval dx .

The value $x^{(s)} = x$, which terminates the s -th series, can result from initial (for this series) values $u_{\max}^{(s)} = x + k$, where $k = 1, 2, \dots$; these values of $u_{\max}^{(s)}$ correspond to the values $x^{(s-1)} = 1/(k+x)$ for the preceding series. Noting this, we can write the following recurrence relation, which expresses the distribution of the probabilities $w_s(x)$ in terms of the distribution $w_{s-1}(x)$:

$$w_s(x)dx = \sum_{k=1}^{\infty} w_{s-1}\left(\frac{1}{k+x}\right) \left|d\frac{1}{k+x}\right|$$

or

$$w_s(x) = \sum_{k=1}^{\infty} \frac{1}{(k+x)^2} w_{s-1}\left(\frac{1}{k+x}\right). \quad (3.2)$$

If the distribution $w_s(x)$ tends with increasing s to a stationary (independent of s) limiting distribution $w(x)$, then the latter should satisfy an equation which we obtain from (3.2) by dropping the indices of the functions w_{s-1} and w_s . This equation has indeed a solution

$$w(x) = 1/(1+x) \ln 2 \quad (3.3)$$

(normalized to unity), as can be readily verified directly⁶⁾.

In order for the s -th series to have a length k , the preceding series must terminate with a number x in the interval between $1/(k+1)$ and $1/k$. Therefore the probability that the series will have a length k is equal to (in the stationary limit)

$$W(k) = \int_{1/(k+1)}^{1/k} w(x)dx = \frac{1}{\ln 2} \ln \frac{(k+1)^2}{k(k+2)}. \quad (3.4)$$

At large values of k

$$W(k) \approx 1/k^2 \ln 2. \quad (3.5)$$

An idea of the rate at which the stationary distribution sets in is obtained from the following example. Let the initial values $x^{(0)}$ be distributed in a narrow interval of width $\delta x^{(0)}$ about some definite number. From the recurrence relation (3.2) (or directly from the expansion (3.1)) it is easy to conclude that the widths of the distributions $w_s(x)$ (about other definite numbers) will then be equal to

$$\delta x^{(s)} \approx \delta x^{(0)} \cdot k^{(1)/2} k^{(2)/2} \dots k^{(s)/2} \quad (3.6)$$

(this expression is valid only so long as it defines quantities $\delta x^{(s)} \ll 1$).

The mean value of k calculated from the distribution (3.4) diverges logarithmically. For a sequence cut

⁶⁾Formula (3.3) was known already to Gauss, and an equation of type (3.2) was considered in this connection by R. O. Kuz'min (see, example, [8]). Without claiming originally, we have deemed it advantageous to present here a simple derivation in a form appropriate to the problem formulation of interest to us.

off at a very large but finite number N of these numbers we would obtain $\bar{K} \sim \ln N$. However, the meaning of a mean value in this case is very limited in view of its instability: the slowness of the decrease of $W(k)$ causes the fluctuations of the number k to diverge even more rapidly than its mean value. A more adequate characteristic of the properties of the considered sequence is the probability that a number selected from it at random turns out to belong to a series with length $k \leq K$, where K is large. This probability is equal to $\ln K / \ln N$. It is small if $1 \ll K \ll N$. In this sense one can say that a number randomly selected from the sequence turns out to belong, with large probability, to a long series.

In the next section we shall find it convenient to average expressions that depend simultaneously on $k^{(s)}$ and $x^{(s)}$. Since both these quantities are derived from the same quantity $x^{(s-1)}$ (which terminates the preceding series), in accordance with the formula $k^{(s)} + x^{(s)} = 1/x^{(s-1)}$, their statistical distributions cannot be regarded as independent. The joint distribution $W_S(k, x)dx$ of both quantities can be obtained from the distribution $w_{S-1}(x)dx$ by making in the latter the substitution $x \rightarrow 1/(x+k)$. In other words, the function $W_S(k, x)$ is given by the very expression under the summation sign in the right side of the equation (3.2). In the stationary limit, taking w from (3.3), we obtain

$$W(k, x) = 1 / (k + x)(k + x + 1) \ln 2. \tag{3.7}$$

Summation of this distribution over k brings us back to (3.3), and integration with respect to dx to (3.4).

4. STATISTICAL ANALYSIS OF THE EVOLUTION OF THE MODEL UPON APPROACHING THE SINGULAR POINT

Proceeding to investigate the statistical properties of the evolution of a model, we write out again the initial recurrence formulas that determine the rules for the transition from one era to the next. The index s will now number successive eras (and not Kasner epochs in one era!), starting from a certain era ($s = 0$), taken to be the initial one. $\Omega^{(s)}$ and $\epsilon^{(s)}$ denote respectively the initial instant of time and the initial density of matter in the s -th era; $\delta^{(s)}\Omega^{(s)}$ is the initial amplitude of the oscillations of that pair from among the functions α, β , and γ , which experiences oscillations in the given era; $k^{(s)}$ is the length (number of Kasner epochs) of the s -th era, and the quantity $x^{(s)}$ determines the length of the next era, in accordance with $k^{(s+1)} = [1/x^{(s)}]$. According to (2.12)–(2.14) we have

$$\frac{\Omega^{(s+1)}}{\Omega^{(s)}} = 1 + \delta^{(s)}k^{(s)} \left(k^{(s)} + x^{(s)} + \frac{1}{x^{(s)}} \right) \equiv \exp \xi^{(s)}, \tag{4.1}$$

$$\delta^{(s+1)} = 1 - \frac{(k^{(s)}/x^{(s)} + 1)\delta^{(s)}}{1 + \delta^{(s)}k^{(s)}(1 + x^{(s)} + 1/x^{(s)})} \equiv f(k^{(s)}, x^{(s)}, \delta^{(s)}), \tag{4.2}$$

$$\ln(\epsilon^{(s+1)} / \epsilon^{(s)}) = 2(k^{(s)} + x^{(s)} - 1)\delta^{(s)}\Omega^{(s)} \tag{4.3}$$

(in (4.1)–(4.2) we introduce, for future use, the symbols $\xi^{(s)}$ and f).

In the probabilistic approach described in Sec. 3, the quantities $\delta^{(s)}$ (which run through values between

0 and 1) also have their own statistical distributions that tend with increasing s to a definite stationary (independent of s) distribution; we denote it by $P(\delta)$. It satisfies the integral equation

$$P(z) = \sum_{k=1}^{\infty} \int_0^1 \int_0^1 \delta [f(k, x, y) - z] W(k, x) P(y) dx dy, \tag{4.4}$$

which expresses the fact that the quantities $\delta^{(s)} \equiv y$ and $\delta^{(s+1)} \equiv z$, connected with relation (4.2), have the same distribution; $W(k, x)$ is the distribution function (3.7); the δ function in the integrand is eliminated by integration with respect to dy .⁷⁾ In view of the absence of any singularities in (4.2), the distribution defined by (4.4) has a perfectly stable character, namely, the mean values of δ raised to any power, as calculated by means of this formula, will be definite finite numbers. Figure 2 shows a plot of the function $P(\delta)$, obtained by numerical integration of Eq. (4.4) (using several iterations) with the aid of an electronic computer⁸⁾. The mean value of δ turns out to be $\bar{\delta} = 0.5$.

Let us examine the statistical connection between the large time intervals Ω and the number s of the eras that have become interchanged during that time.

A second application of formula (4.1) yields

$$\Omega^{(s)}/\Omega^{(0)} = \exp \left(\sum_{p=0}^{s-1} \xi^{(p)} \right). \tag{4.5}$$

Direct averaging of this equation, however, would be meaningless, for by virtue of the slow decrease of the function $W(k)$ (3.5), the average values of the quantity $\exp \xi^{(s)}$ are unstable in the sense indicated in Sec. 3—the fluctuations increase even more rapidly than the mean value itself with increasing region of averaging. This instability is eliminated by taking the logarithm: the ‘‘doubly-logarithmic’’ time interval

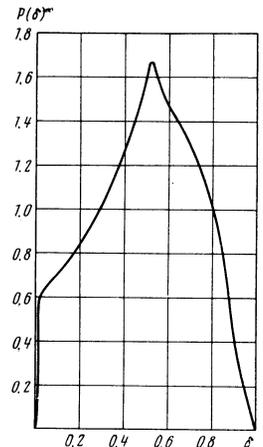


FIG. 2. Plot of the distribution function $P(\delta)$.

⁷⁾Strictly speaking, this equation is approximate. The approximation corresponds to termination of the correlation chain between the values of the succeeding $x^{(s)}$ and $k^{(s)}$, making it possible to assume that the statistical distribution of $\delta^{(s)}$ is independent of the distributions of $x^{(s)}$ and $k^{(s)}$.

⁸⁾It is seen from (4.2) that $\delta^{(s+1)} \rightarrow 0$ is possible only when $x^{(s)} \rightarrow 0$; therefore the region of integration with respect to dx in (4.4) tends to zero as $z \rightarrow 0$; so that $P(0) = 0$. On the other hand, the value $\delta^{(s+1)} = 1$ is possible only when $\delta^{(s)} = 0$; it therefore follows from $P(0) = 0$ that also $P(1) = 0$.

$$\tau_s = \ln(\Omega^{(s)}/\Omega^{(0)}) = \sum_{p=0}^{s-1} \xi^{(p)} \quad (4.6)$$

is expressed by the sum of the quantities $\xi^{(p)}$ having a stable statistical distribution. The mean values of the quantities $\xi^{(s)}$,

$$\bar{\xi} = \sum_{k=1}^{\infty} \int_0^1 \int_0^1 \ln \left[1 + \left(k + x + \frac{1}{x} \right) k \delta \right] P(\delta) W(k, x) dx d\delta,$$

and also of their powers, are finite. A numerical calculation yields $\bar{\xi} = 2.1$ and $\bar{\xi}^2 = 6.8$.

Averaging (4.6) for a given s , we obtain

$$\bar{\tau}_s = \overline{\ln(\Omega^{(s)}/\Omega^{(0)})} = 2.1s, \quad (4.7)$$

thereby determining the mean doubly-logarithmic time interval necessary for the occurrence of s successive eras.

On the other hand, to calculate the mean square of the fluctuations of this quantity we write

$$\overline{(\tau_s - \bar{\tau}_s)^2} = \sum_{p, q=0}^{s-1} \overline{(\xi^{(p)} \xi^{(q)} - \bar{\xi}^{(p)} \bar{\xi}^{(q)})} = s \sum_{p=0}^{s-1} \overline{(\xi^{(p)} \xi^{(p)} - \bar{\xi}^2)}.$$

In the last equation we have taken into account the fact that in the stationary limit the statistical correlation between $\xi^{(s)}$ and $\xi^{(s')}$ depends only on the difference $|s - s'|$. In view of the existence of a recurrence relation between $x^{(s)}$, $k^{(s)}$, $\delta^{(s)}$ and $x^{(s+1)}$, $k^{(s+1)}$, $\delta^{(s+1)}$, the correlation, strictly speaking, differs from zero. It decreases very rapidly, however, with increasing $|s - s'|$, and already at $|s - s'| = 1$ a numerical calculation yields $\overline{\xi_s \xi_{s+1}} - \bar{\xi}^2 = -0.4$. Retaining the first two terms in the sum over p , we obtain

$$[(\tau_s - \bar{\tau}_s)^2]^{1/2} = 1.4\bar{\xi} s. \quad (4.8)$$

When $s \rightarrow \infty$, the relative fluctuation (i.e., the ratio of the mean-square fluctuation (4.8) to the mean value (4.7)) tends, consequently, to zero like $s^{-1/2}$. In other words, the statistical relation (4.7) becomes almost certain at large s . Of course, this certainty is a consequence of the fact that in accordance with (4.6) τ_s can be represented by a sum of a large number of quasi-independent terms (i.e., it has the same origin as the certainty of the additive thermodynamic quantities of a macroscopic body). It follows from this that the probabilities of different values of τ_s (at a specified s) have a Gaussian distribution:

$$\rho(\tau_s) \propto \exp \{ -(\tau_s - 2.1s)^2 / 4s \}. \quad (4.9)$$

The certainty of relation (4.7) makes it possible also to invert it, i.e., to represent it as the dependence of the average number \bar{s}_τ of the eras that are interchanged in a given interval of the doubly logarithmic time τ :

$$\bar{s}_\tau = 0.47 \cdot \tau \quad (4.10)$$

The corresponding statistical distribution is given by the same Gaussian distribution, in which the random quantity is now s_τ at a specified τ :

$$\rho(s_\tau) \propto \exp \{ -(s_\tau - 0.47\tau)^2 / 0.43\tau \}. \quad (4.11)$$

From this point of view, the source of the statistical behavior is the arbitrariness in the choice of the starting point of the interval τ superimposed on the infinite sequence of the interchanging eras.

Returning to the density of matter, we rewrite (4.3)

with allowance for (4.5) in the form

$$\ln \ln \frac{\epsilon^{(s+1)}}{\epsilon^{(s)}} = \eta^{(s)} + \sum_{p=0}^{s-1} \xi^{(p)}, \quad \eta^{(s)} = \ln [2\delta^{(s)}(k^{(s)} + x^{(s)} - 1)\Omega^{(0)}]$$

and then we have for the total change of energy during s eras:

$$\ln \ln \frac{\epsilon^{(s)}}{\epsilon^{(0)}} = \ln \sum_{p=0}^{s-1} \exp \left\{ \sum_{q=0}^p \xi^{(q)} + \eta^{(p)} \right\}. \quad (4.12)$$

The main contribution to this expression is made by the last term of the sum over p , containing the exponential with the largest exponent. Retaining only this term and averaging (4.12), we obtain in the right side of this equation an expression for $s\bar{\xi}$ coinciding with (4.7); all the remaining terms in the sum (and also the terms $\eta^{(p)}$ in the exponents) lead only to corrections of relative order $1/s$. Thus, we have

$$\overline{\ln \ln (\epsilon^{(s)}/\epsilon^{(0)})} = \overline{\ln (\Omega^{(s)}/\Omega^{(0)})}. \quad (4.13)$$

By virtue of the almost certain character of the connection between τ_s and s , which we have established above, relation (4.13) can be written in one of the forms

$$\overline{\ln \ln (\epsilon_s/\epsilon^{(0)})} = \tau \quad \text{or} \quad \overline{\ln \ln (\epsilon^{(s)}/\epsilon^{(0)})} = 2.1s,$$

in which it determines the double logarithm of the growth of the density, averaged over a specified doubly logarithmic interval of time τ or over a specified number of eras s .

We emphasize once more that stable statistical relations are obtained just for the double logarithmic time intervals and density increments. On the other hand, for quantities such as $\Omega^{(s)}/\Omega^{(0)} = \exp \tau_s$ the relative fluctuation increases exponentially with increasing averaging region, by the same token depriving the mean-value concept of a stable meaning.

The origin of the statistical connection (4.13) can be traced already from the initial law governing the variation of the density during the individual Kasner epochs. According to (1.14), during the entire evolution we have

$$\ln \ln \epsilon(t) = \text{const} + \ln \Omega + \ln 2(1 - p_3(t)),$$

with $1 - p_3(t)$ changing from epoch to epoch, running through values in the interval from 0 to 1. The term $\ln \Omega = \ln \ln (1/t)$ increases monotonically; on the other hand, the term $\ln 2(1 - p_3)$ can assume large values (comparable with $\ln \Omega$) only when values of p_3 very close to unity appear (i.e., very small $|p_1|$). These are precisely the "dangerous" cases mentioned in Sec. 2, which violate the "regular" course of the evolution expressed by the recurrence relations (4.1)–(4.3).

It remains for us to show such cases actually do not arise that in the asymptotic limiting regime. We trace the spontaneous evolution of the model, starting with a certain instant at which definite initial conditions are specified in an arbitrary manner. Accordingly, by "asymptotic" is meant here a regime sufficiently far away from the chosen initial instant.

Dangerous cases are those in which excessively small values of the parameter $u = x$ (and hence also of $|p_1| \approx x$) appear at the end of the era. Let us assume by way of a criterion for the selection of such cases the inequalities

$$x^{(s)} \exp |a^{(s)}| \lesssim 1, \quad (4.14)$$

where $|\alpha^{(s)}|$ is the initial depth of the minima of the functions oscillating in the s -th era (it would be more appropriate to choose the final amplitude, but this only would intensify the selection criterion).

The value of $x^{(0)}$ in the initial era is specified by the initial conditions. The dangerous values are those in the interval $\delta x^{(0)} \sim \exp(-|\alpha^{(0)}|)$, and also in the intervals that lead to a dangerous case in the succeeding eras. In order for $x^{(s)}$ to fall in the dangerous interval $\delta x^{(s)} \sim \exp(-|\alpha^{(s)}|)$, the initial value of $x^{(0)}$ should, according to (3.6), lie in an interval of width $\delta x^{(0)} \sim \delta x^{(s)} / k^{(1)2} k^{(2)2} \dots k^{(s)2}$. Altogether, consequently, from the initial single interval of all the possible values of $x^{(0)}$, a dangerous case will result from values lying in a fraction λ of this interval, with

$$\lambda = \exp(-|\alpha^{(0)}|) + \sum_{s=1}^{\infty} \sum_{(k)} \frac{\exp(-|\alpha^{(s)}|)}{k^{(1)2} k^{(2)2} \dots k^{(s)2}} \quad (4.15)$$

(the internal sum is taken over all the values $k^{(1)}$, $k^{(2)}$, \dots , $k^{(s)}$ from 1 to ∞). It is easy to see that this series converges to a value $\lambda \ll 1$, the order of magnitude of which is determined already by the first term in (4.15).

It is easy to demonstrate this by strongly majoring the series, for which purpose we put $|\alpha^{(s)}| = (s+1)|\alpha^{(0)}|$, regardless of the lengths of the eras $k^{(1)}$, $k^{(2)}$, \dots (Actually, the $|\alpha^{(s)}|$ increase much more rapidly; even in the most unfavorable case $k^{(1)} = k^{(2)} = \dots = 1$ the values of $|\alpha^{(s)}|$ increase more readily like $q^s |\alpha^{(0)}|$, where $q > 1$). Noting that

$$\sum_{(k)} 1/k^{(1)2} k^{(2)2} \dots k^{(s)2} = (\pi^2/6)^s,$$

we then obtain

$$\lambda = \exp(-|\alpha^{(0)}|) \sum_{s=0}^{\infty} \left[\frac{\pi^2}{6} \exp(-|\alpha^{(0)}|) \right]^s \approx \exp(-|\alpha^{(0)}|),$$

as required.

If the initial value $x^{(0)}$ lies outside the dangerous

section λ , then no dangerous cases arise at all. On the other hand, if it lies in this section, then a dangerous case arises, but after emerging from it the model begins a "regular" evolution with a new initial value of $x^{(0)}$, which may only accidentally (with probability λ) turn out to be again in the dangerous interval. Repetitions of such cases can lead to a dangerous situation only with probabilities $\lambda^2, \lambda^3, \dots$, which tend asymptotically to zero. This reasoning proves indeed the statement above.

We are grateful to A. G. Doroshkevich and I. D. Novikov for a useful discussion, and also to V. Mil'man and G. Kargopolova for help with the numerical calculations.

¹I. M. Khalatnikov and E. M. Lifshitz, Phys. Rev. Lett. 24, 76 (1970).

²V. A. Belinskiĭ and I. M. Khalatnikov, Zh. Eksp. Teor. Fiz. 56, 1700 (1969) [Sov. Phys.-JETP 29, 911 (1969)].

³E. M. Lifshitz and I. M. Khalatnikov, ZhETF Pis. Red. 11, 200 (1970) [JETP Lett. 11, 123 (1970)].

⁴Ch. W. Misner, Phys. Rev. Lett. 22, 1071 (1969).

⁵L. D. Landau and E. M. Lifshitz, Teoriya polya, 5th Nauka, 1967 [Classical Theory of Fields, Addison-Wesley, 1969].

⁶E. M. Lifshitz and I. M. Khalatnikov, Usp. Fiz. Nauk 80, 391 (1963) [Sov. Phys.-Usp. 6, 495 (1964)]; Advances in Physics 12, 185 (1963).

⁷E. M. Lifshitz and I. M. Khalatnikov, Zh. Eksp. Teor. Fiz. 39, 800 (1960) [Sov. Phys.-JETP 12, 558 (1961)].

⁸A. Ya. Khinchin, Tsepnye drobi (Continuous Fractions), Moscow, 1961.

Translated by J. G. Adashko

37