HOMOGENEOUS ANISOTROPIC MODELS WITH SHEAR IN NEWTONIAN COSMOLOGY

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The problem of Newtonian analogies for homogeneous anisotropic models with shear in general relativity is considered. As shown by Milne and McCrea, for isotropic Friedmann relativistic models there exists a close analog in Newtonian cosmology. It is shown in the present paper that within the framework of Newtonian hydrodynamics with a Newtonian gravitational potential and under certain assumptions regarding the shear tensor structure, solutions with homogeneous deformation can be obtained which are Newtonian analogs of relativistic anisotropic models with shear of Bianchi type I and also of Bianchi type V in the case when transitivity hypersurfaces are orthogonal to the 4-velocity. Newtonian solutions for dust-like matter and for matter with an ultra-relativistic state equation are considered, as well as the Newtonian analog of vacuum solutions and model with a magnetic field.

In relativistic cosmology, within the framework of general relativity theory (GRT), there are known homogeneous anisotropic models characterized in the general case by the presence of isotropic extension, shear, and rotation in the matter. In this paper, we consider homogeneous models with expansion and displacement without rotation, with metrics admitting groups of motion of Bianchi type I, and also of Bianchi type V[1-3], acting on transitivity surfaces that are orthogonal to the four-velocity.

Relativistic homogeneous isotropic Friedmann models have an analog, as shown by Milne and McCrea[4-8], in Newtonian cosmology (Newtonian hydrodynamics with gravitation).

For anisotropic models, the general analysis of the co-moving space in the quasi-Newtonian approximation is given in[9,10]. It is shown in[10,11], that a number of GRT solutions (in particular, the Heckmann-Schucking and the Kasner solutions) remain valid in the Newtonian approximation.

In this paper we formulate the assumptions under which the Newtonian approximation for the aforementioned general-relativistic Bianchi models of types I and V, which consider their essential properties, can be derived within the framework of Newtonian hydrodynamics with gravitation. The aforementioned assumptions with respect to the structure of the shear tensor in Newtonian theory are made ad hoc, as necessary supplementary relations[9] from the general relativistic solutions. In GRT these relations, which are considered in Sec. 1, are an automatic consequence of the gravitation equations for a metric with definite structure.

In Secs. 2–4 we consider the Newtonian solutions for matter with an equation of state $p = 0$ and $e = 3p$, the Newtonian analog of the vacuum solutions, and the Newtonian analog of the axially symmetrical (with group G_4) model with magnetic field.

The Newtonian approximations are considered here for general-relativistic homogeneous anisotropic models with groups of motions acting on the transitivity hypersurfaces $\tau = \text{const}$ (with zero and negative constant curvature), which are orthogonal to the four-velocity. For such models, the synchronous reference frame with metric (we use the notation of Landau and Lifshitz[12])

$$ds^2 = (cd\tau)^2 - \gamma_{\alpha\beta}dx^\alpha dx^\beta, \quad x^0 = c\tau, \quad \alpha, \beta = 1, 2, 3, \quad i, k = 0, 1, 2, 3 \quad (1)$$

is also co-moving ($u^0 = 1$). The tensor $u_{i;k} - u_{k;i}$ vanishes (there is no rotation), and the invariant breakdown of $u_{i;k}$ into an expansion tensor (Θ—expansion scalar) and a shear tensor q_{ik} with zero trace ($q_i^i = 0$) takes the form[1]

$$cu_{i;k} = c(u_{i;k} + u_{k;i})/2 = \Theta(g_{ik} - u_i u_k) + q_{ik}, \qquad \Theta = cu_{i;i}/3. \quad (2)$$

1. NECESSARY RELATIONS IN HOMOGENEOUS ANISOTROPIC BIANCHI MODELS OF TYPES I AND V IN THE GRT

In the model that admits of an Abelian group of motions G_3 (type I after Bianchi[2,3]), considered first by Heckmann and Schucking[13], the metric (1) has the following form (the three-space x^α is Euclidean

$$ds^2 = (cd\tau)^2 - \{[R_1(\tau)dx^1]^2 + [R_2(\tau)dx^2]^2 + [R_3(\tau)dx^3]^2\}. \quad (1.1)$$

The nonzero components of $u_{i;k}$ are equal to the Hubble constants $h_\alpha(\tau)$:

$$cu_{;1}^{1} = h_1, \quad cu_{;2}^{2} = h_2, \quad cu_{;3}^{3} = h_3,$$
$$3\Theta = h_1 + h_2 + h_3, \quad h_\alpha = d\ln R_\alpha/d\tau.$$

The Einstein equations without the cosmological term)

$$R_i^k - {}^1/_2 R\delta_i^k = (8\pi k/c^4)T_i^k \quad (1.2)$$

with an ideal-gas energy-momentum tensor $T_{ik} = (e + p)u_i u_k - pg_{ik}$ (p—pressure, e—density of internal energy) give expressions for $R_\alpha(\tau)$ in (1.1) and e for dustlike matter ($p = 0$) and for matter with an equation of state $e = 3p$ in the following form[1,14]:

1) Pressure $p = 0$:

$$R_1^3(\tau) = L_1 c^2(\tau - \tau_1)^{1+\alpha_1}(\tau - \tau_2)^{1-\alpha_1},$$
$$R_2^3(\tau) = L_2 c^2(\tau - \tau_1)^{1+\alpha_2}(\tau - \tau_2)^{1-\alpha_2}, \quad (1.1a)$$
$$R_3^3(\tau) = L_3 c^2(\tau - \tau_1)^{1+\alpha_3}(\tau - \tau_2)^{1-\alpha_3},$$
$$e = c^2/6\pi k(\tau - \tau_1)(\tau - \tau_2).$$

2) Ultrarelativistic equation of state e = 3p:

$$R_1 = L_1(\operatorname{sh}\lambda)^{1+\alpha_1}(\operatorname{ch}\lambda)^{1-\alpha_1}, \quad R_2 = L_2(\operatorname{sh}\lambda)^{1+\alpha_2}(\operatorname{ch}\lambda)^{1-\alpha_2},$$
$$R_3 = L_3(\operatorname{sh}\lambda)^{1+\alpha_3}(\operatorname{ch}\lambda)^{1-\alpha_3}, \quad c\tau = L_0(\operatorname{sh}4\lambda - 4\lambda); \tag{1.1b}$$
$$e = 3p = 3c^4/128\pi k L_0 \operatorname{sh}^4 2\lambda$$

(L_1, L_2, L_3, L_0—scale constants).

The constants α_1, α_2, α_3 in (1.1a) and (1.1b) are connected by two relations:

$$\alpha_1+\alpha_2+\alpha_3=0, \quad \alpha_1+\alpha_2+\alpha_3^2=6, \tag{1.3}$$

and can be represented in the form

$$\alpha_1 = 2\sin\gamma, \quad \alpha_2 = 2\sin(\gamma+2\pi/3), \quad \alpha_3 = 2\sin(\gamma+4\pi/3),$$
$$-\pi/6 \leqslant \gamma \leqslant \pi/6. \tag{1.3a}$$

The calculation of the components of the shear tensor q_{ik}, in accordance with (2) for (1.1a) and (1.1b) (and in general at e = e(p)), in mixed indices, yields

$$q_0^0 = q_\alpha^0 = 0, \quad q_\beta^\alpha = Q(\tau)\begin{pmatrix}\alpha_1 & 0 & 0\\ 0 & \alpha_2 & 0\\ 0 & 0 & \alpha_3\end{pmatrix}, \tag{1.4}$$

where

$$Q(\tau) = A/R_1R_2R_3 = A(-g)^{-1/2}, \quad A = \text{const}. \tag{1.5}$$

In the model (1.1), the component (1.2) with i = k = 0 is written in the form

$$h_1h_2+h_1h_3+h_2h_3 = (8\pi k/c^2)e. \tag{1.6}$$

Relation (1.6) can be treated as the quasi-Euclidean condition: the energy density is equal to the "critical" value.

In the case of vacuum $R_\alpha(\tau)$ in (1.1) is given by the Kasner solution[2,12,14]:

$$R_1^3 = \text{const}\cdot\tau^{1+\alpha_1}, \quad R_2^3 = \text{const}\cdot\tau^{1+\alpha_2}, \quad R_3^3 = \text{const}\cdot\tau^{1+\alpha_3} \tag{1.7}$$

with constants α_1, α_2, and α_3 from (1.3) and (1.3a), and Eqs. (1.4), (1.5), and (1.6) with e = 0 are retained for the Kasner solution.

There are known Bianchi models of type I, constructed under the assumption that there is a homogeneous magnetic field in space besides the ideal liquid, ("magnetic models of the universe")[15-20]. In this case we have for matter with p = 0, assuming $R_1 = R_3$ (the magnetic field is directed along the axis x^2),

$$R_1 = R_3 = A(B^2+\eta^2)/2, \quad R_2 = DA(-3B^4+6B^2\eta^2+\eta^4 + 12E\eta)/12R_1,$$
$$c\tau = A(3B^2\eta+\eta^3)/6, \tag{1.8}$$

where A, D, E, and B are constants. The densities of the energy of the matter (e) and of the magnetic field (W) are equal to

$$e = c^4D/4\pi kR_1R_2R_3, \quad W = H^2/8\pi = c^4A^2B^2/8\pi kR_1^4. \tag{1.8a}$$

Calculation of q_{ik} for (1.8) leads to (1.4) with the values

$$\alpha_1 = \alpha_3 = -1, \quad \alpha_2 = 2; \tag{1.4a}$$

the function $Q(\tau)$, however, now turns out to be

$$Q(\eta) = cAD(-2B^2\eta^3+6\eta B^4-9E\eta^2+3EB^2)/9(B^2+\eta^2)(-g)^{1/2}.$$

To solve (1.8) we have, in place of (1.6), the relation[15-18]

$$h_1 = h_3, \quad h_1(h_1+2h_2) = (8\pi k/c^2)(e+W). \tag{1.6a}$$

Formulas (1.4) and (1.6) (as well as (1.10)) will be used essentially in Secs. 2—4.

In models that admit of the Bianchi group of type V[1-3,21], the metric reduces to the form

$$ds^2 = (c\,d\tau)^2 - \{R_1^2(\tau)(dx^1)^2 + e^{-2x^1}[R_2^2(\tau)(dx^2)^2 + R_3^2(\tau)(dx^3)^2]\}. \tag{1.9}$$

The three-space x^α is in this space a constant with negative curvature. The component (1.2) with i = k = 0 yields

$$h_1h_2+h_1h_3+h_2h_3 = \frac{8\pi k}{c^2}e + \frac{3c^2}{R_1^2(\tau)}, \quad h_\alpha = \frac{d\ln R_\alpha}{d\tau} \tag{1.10}$$

(the energy density is lower than "critical").

From Einstein's equations for (1.9) there follows $T_{10} = (c^3/8\pi k)(-2h_1+h_2+h_3)$, so that in the general case (1.9) the matter has a velocity component $u^1 \neq 0$, and the shear tensor is not diagonal. The corresponding formulas in GRT (and their Newtonian analog) will not be considered here.

There is, however, a special case of the metric (1.9) (at $2h_1 = h_2+h_3$), when $T_{10} = 0$ identically, the synchronous system (1.9) is co-moving, and the transitivity hypersurfaces τ = const are orthogonal to the four-velocity. In this case we have in (1.9)[1]

$$R_1 = R(\tau), \quad R_2 = R(\tau)S^a(\tau), \quad R_3 = R(\tau)S^{-a}(\tau), \quad a = \sqrt{3};$$
$$\Theta = \dot{R}/R. \tag{1.11}$$

The solution of Einstein's gravitational equation with the energy-momentum tensor of an ideal gas leads in this case to the following formula[1]:

1) pressure p = 0:

$$\dot{R}^2 = \frac{c^2R^4+2kMR^3+A^2}{R^4}, \quad S(\tau) = \text{const}\cdot\exp\int\frac{A}{R^3(\tau)}d\tau,$$
$$e = \mu c^2 = 3Mc^2/4\pi R^3, \quad M = \text{const}, \quad A = \text{const}; \tag{1.11a}$$

2) ultrarelativistic equation of state e = 3p:

$$\dot{R}^2 = \left(c^2R^4 + \frac{8\pi kK}{3c^2}R^2 + A^2\right)\frac{1}{R^4}, \quad S(\tau) = \text{const}\cdot\exp\int\frac{A}{R^3(\tau)}d\tau, \tag{1.11b}$$
$$e = 3p = K/R^4, \quad K = \text{const} > 0, \quad A = \text{const}.$$

The shear tensor for the solutions (1.11a) and (1.11b) is expressed by formula (1.4) with

$$\alpha_1 = 0, \quad \alpha_2 = -\alpha_3 = \sqrt{3} \tag{1.12}$$

and by the function $Q(\tau)$ from (1.5).

The Bianchi type-V model exists also for vacuum[2]. In this case (1.11) and (1.11a) are valid with M = 0, as are (1.4), and (1.5) with the values of the parameters in (1.12).

2. HOMOGENEOUS MODELS WITH SHEAR IN NEWTONIAN COSMOLOGY FOR DUSTLIKE MATTER

The fundamental equations in Newtonian cosmology are the equations of hydrodynamics with a Newtonian gravitational potential, having in the case of zero pressure (p = 0) the form

$$\partial\rho/\partial t + \partial(\rho v^\alpha)/\partial x^\alpha = 0, \tag{2.1}$$

$$\partial v_\alpha/\partial t + (v^\beta\partial v_\alpha/\partial x^\beta) = -\partial\varphi/\partial x^\alpha, \tag{2.2}$$

and the Poisson equation for the gravitational potential

$$\partial^2\varphi/\partial x^\alpha\partial x_\alpha = 4\pi k\rho \tag{2.3}$$

(here and below we use rectangular Cartesian coordinates x^α in three-dimensional Euclidean space; the upper and lower indices are of equal value; the Greek indices run through the values from 1 to 3).

For motions with homogeneous deformation, the dependence of the velocity v^α on x^α and on the time t in "Euler" coordinates is given by

$$v^\alpha = h^\alpha{}_\beta(t)\, x^\beta \tag{2.4}$$

with the Hubble-constant tensor $h^\alpha_\beta(t)$.

For motions with expansion and shear without rotation, the tensor $h_{\alpha\beta}$ is symmetrical and reduces to a diagonal form. We denote the eigenvalues of $h_{\alpha\beta}$ by h_1, h_2, and h_3, and introduce $R_\alpha(t)$ in accordance with

$$h_\alpha = d\ln R_\alpha/dt, \quad \alpha = 1, 2, 3. \tag{2.5}$$

The Hubble-constant tensor breaks up into a spherical tensor (Θ is the expansion scalar) and a shear tensor with zero trace:

$$h^\alpha_\beta = \Theta\delta^\alpha_\beta + q^\alpha_\beta, \quad \Theta = {}^1/_3 h^\alpha_\alpha, \quad q^\alpha_\alpha = 0. \tag{2.6}$$

Elimination of φ from (2.2) and (2.3) leads, with allowance for (2.4) and (2.6), to the following relation (the dot denotes differentiation with respect to t)

$$3(\dot\Theta + \Theta^2) + q_{\alpha\beta}q^{\beta\alpha} + 4\pi k\rho = 0. \tag{2.7}$$

Expression (2.7) is the Newtonian analog of the well known Raychaudhuri identity in GRT[1,22].

From the continuity equation (2.1) we have for $\rho(t)$ a relation analogous to the general-relativistic one, expressing the mass conservation:

$$\rho = \frac{3M}{4\pi}\exp\int 3\Theta\, dt = \frac{3M}{4\pi R^3}, \quad M = \mathrm{const}, \tag{2.8}$$

where R(t) is expressed in terms of the expansion scalar Θ in the form

$$\Theta = \dot R/R. \tag{2.6a}$$

The equations of Newtonian hydrodynamics (2.2) together with the Poisson equation (2.3) do not suffice for a complete determination of $h^\alpha_\beta(t)$. The system (2.2) and (2.3) leads only to the Raychaudhuri identity (2.7).

Thus, within the framework of Newtonian hydrodynamics, the relation (2.4) in itself does not define the motion; there is still a complete leeway in the choice of the expression for the shear tensor[7].

As the supplementary relations that fix the tensor q^α_β ($q^\alpha_\alpha = 0$), and consequently the entire motion, we shall use for the models in question the following two relations.

1. In accordance with (1.4), for the shear tensor in the general relativistic Bianchi models of type I and also for the metric (1.9) and (1.11) of the Bianchi type V, we shall consider the Newtonian problem with the shear tensor q in the form

$$q^\alpha_\beta \equiv \begin{pmatrix} h_1-\Theta & 0 & 0 \\ 0 & h_2-\Theta & 0 \\ 0 & 0 & h_3-\Theta \end{pmatrix} = Q(t)\begin{pmatrix} \alpha_1 & 0 & 0 \\ 0 & \alpha_2 & 0 \\ 0 & 0 & \alpha_3 \end{pmatrix}, \tag{2.9}$$

where the constants α_1, α_2, and α_3, which satisfy as a result of $q^\alpha_\alpha = 0$ the condition $\alpha_1 + \alpha_2 + \alpha_3 = 0$, can be normalized, without loss of generality, also by the condition $\alpha_1^2 + \alpha_2^2 + \alpha_3^2 = 6$, so that Eqs. (1.3) and (1.3a) hold for them.

2. Besides (2.9), we shall assume, in accordance with the general-relativistic relations (1.6) and (1.10), that the following relation holds true in the Newtonian problem for dustlike matter

$$h_1h_2 + h_1h_3 + h_2h_3 = 8\pi k\rho \mp [3\alpha^2c^2/R^2(t)]. \tag{2.10}$$

The constant α^2 in (2.10) can assume values $\alpha^2 = 0$ and $\alpha^2 = 1$. Formula (2.10) with allowance for (2.6), (2.9), and (2.8) reduces to

$$Q^2 = \Theta^2 - (2kM/R^3) \pm (\alpha^2c^2/R^2), \tag{2.10a}$$

which fixes the function Q(t) in (2.9).

The specification of (2.4), (2.9), and (2.10) determines the parameters of motion in accordance with (2.8) and (2.7), and determines φ in accordance with (2.2). The resultant relations are the Newtonian approximation for the solution in the GRT considered in Sec. 1.

Assumptions other than (2.9) and (2.10) with respect to the shear tensor (or, equivalently, with respect to φ) lead to other Newtonian solutions with a deformation in the form (2.4), which in final analysis is connected with the specific character of the problem in infinite space.

Relation (2.9) means that the eigenvectors of the shear tensor do not change their orientation with time, and the eigenvalues at each instant are proportional to the constants α_1, α_2, and α_3.

Let us define the function S(t) with the aid of the relation

$$Q = \dot S/S. \tag{2.9a}$$

We introduce "Lagrangian" coordinates ξ^α and t such that $v^\alpha = \partial x^\alpha(\xi^\beta, t)/\partial t$. Then we obtain from (2.4) and (2.5) the law of motion of the particles in the form

$$x^1 = R_1(t)\xi^1, \quad x^2 = R_2(t)\xi^2, \quad x^3 = R_3(t)\xi^3. \tag{2.11}$$

The dependence $R_\alpha(t)$ in (2.11) has in accordance with (2.6), (2.6a), and (2.9), (2.9a) the form

$$R_1 = RS^{\alpha_1}, \quad R_2 = RS^{\alpha_2}, \quad R_3 = RS^{\alpha_3}; \quad R_1R_2R_3 = R^3. \tag{2.12}$$

Relation (2.10) generalizes (1.6) and (1.10). The value $\alpha^2 = 0$ corresponds to the solution in the GRT with Euclidean three-space; the corresponding solution in Newtonian cosmology will be called the flat model. The solution with $\alpha^2 = 1$ and the lower sign in (2.10) in Newtonian cosmology will be called the open model; it corresponds in GRT to a solution with a three-space of constant negative curvature. (For completeness, we consider also a Newtonian solution with $\alpha^2 = 1$ and the upper sign of (2.10), corresponding to a closed model.)

From (2.7) with allowance for (2.6a), (2.8), (2.9), and (2.10a) we have

$$\ddot R R^2 + 2\dot R^2 R - 3kM \pm 2\alpha^2c^2R = 0.$$

After a single integration we get from this

$$\dot R^2 = (\mp\alpha^2c^2R^4 + 2kMR^3 + A^2)/R^4, \quad A = \mathrm{const} \tag{2.13}$$

(the integration constant is denoted A^2 in the sense of (2.10a)).

Substitution of (2.13) in (2.10a) yields Q(t) in the form

$$Q = A/R^3. \tag{2.14}$$

For S(t) defined by (2.9a), we get from (2.14)

$$S(t) = \mathrm{const}\cdot\exp\int\frac{A}{R^3(t)}\,dt. \tag{2.14a}$$

Relations (2.11), (2.12), (2.13), (2.14a) and (1.3), (1.3a) with $A \neq 0$ are the Newtonian formulas for a homogeneous anisotropic model with expansion and shear for dustlike matter. When $A = 0$ we have $S(t) = \text{const}$, and these formulas give the Newtonian analog of the flat, open and closed isotropic Friedmann models, respectively.

The expansion scalar Θ and $Q(t)$ in (2.9) are determined by expressions (2.13) and (2.14), (2.14a), which coincide with the corresponding general-relativistic formulas for the Bianchi model I (formulas (1.1a), (1.5)) and for the Bianchi model V with transitivity hypersurfaces orthogonal to the four-velocity (formulas (1.11a), (1.5); in (2.13) we have $\alpha^2 = 1$ and the lower sign; the constants α_1, α_2, and α_3 in (2.9) and (2.12) are defined in (1.12)).

According to (2.13) and (2.14), in the plane and open models, as $R \to \infty$, the value of Q at $M \neq 0$ tends to zero more rapidly than Θ, so that the asymptotic solution becomes isotropic at a rate that increases with decreasing A^2.

Near the value $R = 0$, which corresponds to the singular state with $\rho = \infty$, the last term of the right side of (2.13) predominates, and for all the models (in the isotropic case with $A \neq 0$) we have

$$R \approx (3At)^{1/3} \quad Q \approx 1/3t, \quad S \sim t^{1/3}, \quad R_1^3 \sim t^{1+\alpha_1}, \quad R_2^3 \sim t^{1+\alpha_2}, R_3^3 \sim t^{1+\alpha_3}.$$

These formulas, which remain in force in the GRT[12], coincide with the corresponding vacuum solution of Kasner (1.7), demonstrating that the presence of matter does not influence the asymptotic form near the singularity in the principal term[23] (unlike the isotropic models with $A = 0$, in which this asymptotic form is determined by the equation of state of the matter).

When $A \neq 0$, near the singularity, $R \sim (t - t_0)^{1/3}$, and for dustlike matter, in accord with (2.8), the density ρ (in GRT—the energy density e) reverses sign on going through $t = t_0$, this being essentially connected with the equation of state $p = 0$. For matter with $e = 3p$, in view of the different $e(R)$ dependence (formula (3.9)), the transition through the singularity is not accompanied by a change in the sign of e.

In the <u>flat</u> model, according to (2.13) with $\alpha = 0$ and (2.14a),

$$R^3 = {}^9/_2 kM(t - t_1)(t - t_2), \quad t_1 = \text{const}, \quad t_2 = t_1 - 2A / 3kM,$$

$$S^3 = \text{const}\cdot(t - t_1)/(t - t_2). \tag{2.15}$$

Expressions (2.12), (2.15), and (2.8) lead to formulas (1.1a) for $R_\alpha(t)$ and $\rho(= e/c^2)$.

In the <u>open</u> model, introducing the dimensionless quantities $\overline{R} = Rc^2/kM$ and $\overline{A} = Ac^3/(kM)^2$, we obtain from (2.13) ($\alpha^2 = 1$):

$$(\dot{R}/c)^2 = (\overline{R}^4 + 2\overline{R}^3 + \overline{A}^2) / \overline{R}^4. \tag{2.16}$$

When $\overline{A}^2 > 27/16$ (the right side of (2.16) is larger than zero for all $\overline{R}$), the value of R changes from $-\infty$ to $+\infty$ and passes through $R = 0$, at which ρ reverses sign. When $0 < \overline{A}^2 < 27/16$, the numerator in the right side of (2.16) has two roots: $\overline{R}_1(<0)$ and $\overline{R}_2(<0)$ and is not negative at $-\infty < \overline{R} \leq \overline{R}_1 < 0$ and $\overline{R}_2 \leq \overline{R} < +\infty$. In this case, two regimes are possible. In one of them $\overline{R}$ changes from $-\infty$ to $\overline{R}_1(<0)$ and $\rho < 0$ (since $\overline{R} < 0$), which is physically unsatisfactory. The other regime corresponds, with increasing t, to a decrease of $\overline{R}$ from $+\infty$ to $\overline{R}_2(<0)$ with transition at $t = t_0$ through the singularity $R = 0$ and subsequent growth from $\overline{R}_2$ to $+\infty$ with a second transition at $t = t_1$ through $R = 0$. When $t < t_0$ and $t > t_1$ we have for the density $\rho > 0$, and when $t_0 < t < t_1$ we have $\rho < 0$. This regime is analogous to the behavior of R in the flat model.

The gravitational potential φ for the considered Newtonian models is determined from (2.2), and reduces, when account is taken of (2.4), (2.6), (2.7), and (2.9), to the form

$$\varphi = -{}^1/_2[-{}^4/_3\pi k\rho\delta_{\alpha\beta} + (\dot{Q} + 2Q\Theta)A_{\alpha\beta} + Q^2(t)B_{\alpha\beta}]x^\alpha x^\beta + \text{const}, \tag{2.17}$$

where the matrices $A_{\alpha\beta}$ and $B_{\alpha\beta}$ (with zero trace) are given by

$$A_{\alpha\beta} = \begin{pmatrix} \alpha_1 & 0 & 0 \\ 0 & \alpha_2 & 0 \\ 0 & 0 & \alpha_3 \end{pmatrix}, \quad B_{\alpha\beta} = \begin{pmatrix} \alpha_1^2 - 2 & 0 & 0 \\ 0 & \alpha_2^2 - 2 & 0 \\ 0 & 0 & \alpha_3^2 - 2 \end{pmatrix} \tag{2.18}$$

(the components of $B_{\alpha\beta}$ also satisfy relations of the type (1.3)).

In expression (2.17) for φ, the sum of the terms with $A_{\alpha\beta}$ and $B_{\alpha\beta}$ forms a tensor with zero trace. The isotropic part of (2.17) forms the first term, which is represented also in the form $(kM/2R^3)(x^2 + y^2 + z^2)$. Its existence is connected with the presence of matter. In the case of a completely isotropic model with matter ($A = 0$, $Q = 0$), only this term remains in (2.17). As is well known, in this case the gravitational force $-\text{grad}\,\varphi$ at a given point is equal to the force produced by a sphere with the mass $M[(\xi^1)^2 + (\xi^2)^2 + (\xi^3)^2]^{3/2}$ of the matter inside the sphere passing to this point. This leads to a corresponding physical interpretation of the isotropic Newtonian cosmological model[4,5,6,8,19]. It should be noted in this connection that in the anisotropic case (2.17) yields for $-\text{grad}\,\varphi$ an expression that differs from the force produced at the given point by the mass contained within the ellipsoid passing through this point[24]. This difference is manifest in significant manner in the case of empty space ($M = 0$). Equation (2.17) leads in this case to a nonzero expression for $-\text{grad}\,\varphi$, in accordance with the fact that vacuum anisotropic solutions exist in the GRT.

Let us consider the Newtonian analog of the general relativistic vacuum anisotropic solutions with shear. In accordance with the general relativistic relations for the vacuum solutions (1.4) and (1.6) with $e = 0$, it is necessary to assume for the Newtonian analog relations (2.9) and (2.10) with $\rho = 0$. The Newtonian analog of the vacuum solutions presupposes the existence of a gravitational field in infinite space without sources (with zero in the right side of (2.3)) in the presence of trial particles (the presence of which does not distort the gravitational field), which move in accordance with (2.2) and with relations (2.9) and (2.10) with $\rho = 0$. Expressions (2.13) and (1.3) with the functions R and S satisfying (2.14a) remain in force in this case, as does relation (2.13) with $M = 0$, which takes the form

$$\dot{R}^2 = (\mp\alpha^2c^2R^4 + A^2) / R^4. \tag{2.13a}$$

In the "flat" model (at $\alpha = 0$) we get from (2.13a)

$$R = (3At)^{1/3}, \quad Q = \Theta = 1/3t,$$

which leads then to (1.7). This solution is the Newtonian analog of Kasner's vacuum solution.

In the "open" model ($\alpha^2 = 1$ and the lower sign in (2.13a)) the value of R^2 changes from $+\infty$ to 0 and from 0 to $+\infty$. At the values (1.12) of the constants in (2.12), this solution is the Newtonian analog of the Bianchi type V vacuum solution in the GRT.

The gravitational potential φ for the "vacuum" Newtonian solutions has no isotropic part, and in accordance with (2.17) it is expressed, with allowance for (2.14), in the form

$$\varphi = -{}^1/_2 Q[(-\Theta)A_{\alpha\beta} + QB_{\alpha\beta}]x^\alpha x^\beta + \text{const}$$

with matrices $A_{\alpha\beta}$ and $B_{\alpha\beta}$ from (2.18). For the Newtonian analog of the Kasner vacuum solution, this expression simplifies (since $Q = \Theta$)

$$\varphi = (-1/18t^2)D_{\alpha\beta}x^\alpha x^\beta + \text{const};$$

Here the matrix is $D_{\alpha\beta} = B_{\alpha\beta} - A_{\alpha\beta}$.

The "vacuum" Newtonian solutions with total isotropy ($A = 0$) in (2.13a)) can occur only in the open model. Then $R^2 = (ct)^2$, $v = r/t$, and φ = const in accordance with (2.17). This solution, which describes the inertial expansion, is the Newtonian analog of the Milne model[19].

3. NEWTONIAN ANISOTROPIC MODELS WITH SHEAR FOR MATTER WITH AN EQUATION OF STATE $e = e(p)$

As before, we start from the Newtonian momentum equations with a gravitational potential φ in the form (2.2) (the pressure p depends only on the time t and its 3-gradient is equal to zero).

To obtain the corresponding Newtonian equation for the gravitational potential φ (similar to the Poisson equation (2.3)) in the considered case of matter with an equation of state $e = e(p)$, it is necessary to take into consideration the fact that this equation can be obtained[12] as the limit of Einstein's equation

$$R_0^0 = (8\pi k/c^4)(T_0^0 - T/2) \tag{3.1}$$

(with an energy-momentum tensor of the ideal gas, $T_0^0 = z$, and $T = e - 3p$). This leads to the relation

$$\frac{\partial^2\varphi}{\partial x^\alpha\partial x_\alpha} = \frac{8\pi k}{c^2}\left(e - \frac{e-3p}{2}\right) = \frac{4\pi k}{c^2}(e+3p). \tag{3.2}$$

The dependence of v^α on x^α and t is given as before by (2.4), (2.6), and (2.6a).

In the present case of matter with an equation of state $e = e(p)$, just as in the preceding case with $p = 0$, we shall use the relation (2.9) and a relation of the type (1.6) and (1.10), which generalizes (2.10):

$$h_1h_2 + h_1h_3 + h_2h_3 = (8\pi k/c^2)e \mp 3\alpha^2c^2/R^2(t). \tag{3.3}$$

Taking (2.4), (2.6), (2.6a), and (2.9) into account, the momentum equations (2.2) and (3.2) lead to the expression

$$3(\Theta^2 + \dot\Theta) + 6Q^2 + (4\pi k/c^2)(e+3p) = 0, \tag{3.4}$$

which plays the role of the Raychaudhuri identity for the given case, and relation (3.3) yields

$$3(\Theta^2 - Q^2) = (8\pi k/c^2)e \mp 3\alpha^2c^2/R^2(t). \tag{3.5}$$

From (3.4) and (3.5) we get

$$\frac{\ddot R}{R} + 2\left(\frac{\dot R}{R}\right)^2 - \frac{4\pi k}{c^2}(e-p) \pm \frac{2\alpha^2c^2}{R^2} = 0. \tag{3.6}$$

To obtain the dependence of e on R it is necessary to use a relation whose role in the case of dustlike matter is played by the mass continuity equation (2.1). In this case it is necessary to use the scalar equation

$$u^i(T_i^k)_{;k} = 0$$

of relativistic hydrodynamics for an ideal gas with an equation of state $e = e(p)$ (in the co-moving system it is the law of conservation of energy). This equation is written in the form[25]

$$\left[\frac{(e+p)u^k}{\mathscr{B}(p)}\right]_{;k} = 0, \quad \mathscr{B}(p) = \exp\int\frac{dp}{e+p} \tag{3.7}$$

and in the considered Newtonian approximation it reduces to

$$\frac{\partial}{\partial t}\left[\frac{e+p}{\mathscr{B}(p)}\right] + \frac{\partial}{\partial x^\alpha}\left[\frac{e+p}{\mathscr{B}(p)}v^\alpha\right] = 0, \tag{3.7a}$$

where $\mathscr{B}$ is given by (3.7).

Inasmuch as e and p depend on t (in accord with the momentum equation), we get from (3.7a), as a result of (2.4), (2.6), and (2.6a)

$$(e+p)/\mathscr{B}(p) = \text{const}/R^3. \tag{3.8}$$

Equation (3.8) holds also in the corresponding GRT solutions.

Relations (3.6) and (3.8) determine $R(t)$ for the equation of state $e = e(p)$.

In the case of the nonrelativistic equation of state $e = 3p$ we obtain from (3.8) and (3.7)

$$e = 3p = K/R^4, \quad K = \text{const}, \quad K > 0. \tag{3.9}$$

According to (3.9), e and p are always positive.

Single integration of (3.6) with allowance for (3.9) yields

$$ct = L_0(\operatorname{sh}4\lambda - 4\lambda), \quad S = \text{const}\cdot\operatorname{th}\lambda, \quad L_0 \equiv c^2L^2(3/128\pi kK)^{1/2}, \tag{3.10}$$

Substitution of (3.10) in (3.5) leads to the previous expressions (2.14) and (2.14a) for the quantity $Q(t)$ in the shear tensor (2.9).

Just as in Sec. 2, (2.11) and (2.12) remain in force in this case, with the sole exception that at $e = 3p$ we have Eq. (3.10) for $R(t)$.

In the case of the flat model, introducing the parameter λ in accordance with

$$\dot R^2 = \mp\alpha^2c^2 + 8\pi kK/3c^2R^2 + A^2/R^4, \quad A^2 = \text{const}.$$

we obtain after integrating (3.10) (with $\alpha^2 = 0$)

$$R = L\operatorname{sh}2\lambda, \quad L \equiv Ac(3/8\pi kK)^{1/2},$$

after which (2.12) leads to formulas (1.1b).

In the open model, (3.10) (with $\alpha^2 = 1$ and with the lower sign) coincides with the general-relativistic relation (1.11b) and takes the following form in terms of the dimensionless quantities $\bar R = Rc^2/(8\pi kK/3)^{1/2}$, $\bar A = Ac/(8\pi kK/3)^{1/2}$

$$(\dot R/c)^2 = (\bar R^4 + \bar R^2 + \bar A^2)/\bar R^4.$$

The quantity R^2 decreases monotonically from $+\infty$ to zero, and then increases.

When $A \neq 0$, the asymptotic form near the singularity $R = 0$ is $R \sim t^{1/3}$ for all models, just as in (1.7).

4. NEWTONIAN ANALOG OF THE COSMOLOGICAL MODEL WITH MAGNETIC FIELD

To obtain the Newtonian analog of the cosmological model with dust ($p = 0$) in a homogeneous magnetic field, it is necessary to use as the fundamental equations, besides (2.1), also the equation for the gravitational potential φ, obtained as the limit of Einstein's equation (3.1) with $T_0^0 = e + W$ and $T = e$ (W—energy density of the magnetic field), in the form (cf. (3.2))

$$\partial^2\varphi / \partial x^\alpha \partial x_\alpha = 4\pi k(\rho + 2W / c^2), \qquad (4.1)$$

and the equations of magnetohydrodynamics in the nonrelativistic approximation[26]*

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\,\mathrm{grad})\mathbf{v} = -\mathrm{grad}\,\varphi + \frac{1}{\rho c}[\mathbf{jH}],$$

$$\mathrm{rot}\,\mathbf{E} = -\frac{\partial \mathbf{H}}{c\partial t}, \quad \mathrm{div}\,\mathbf{H} = 0, \quad \mathbf{j} = \frac{c}{4\pi}\mathrm{rot}\,\mathbf{H}.$$

In the Newtonian approximation, the magnetic field H is not transformed on going over to another inertial system. According to the formulation of the problem, in the proper reference frame, the field H^* depends only on t, and the electric field is $E^* = 0$. In the reference frame under consideration, when the matter has a velocity v, there arises, in accordance with (2.4), an induced electric field

$$\mathbf{E} = -[\mathbf{vH}] / c. \qquad (4.2)$$

Since $H = H^*(t)$ and $j = 0$, the Lorentz force is equal to zero, and the momentum equation has in this case the form (2.2). It is necessary to add to the system (2.1), (2.2), and (4.1) the induction equation

$$\mathrm{rot}[\mathbf{vH}] = \partial \mathbf{H} / \partial t. \qquad (4.3)$$

Choosing the y axis along H ($H_y = H$), we obtain from (4.2), (4.3), and (2.4):

$$E_x = Hh_3 z / c,\ E_y = 0,\ E_z = -Hh_1 x / c,\ (h_1 + h_3)H = -dH / dt. \qquad (4.4)$$

Let us consider the case with $R_1 = R_3$ and $h_1 = h_3$ (isotropy in a plane perpendicular to the direction of the magnetic field H). Equation (4.4), together with (2.5), yields expressions for the intensity H and the energy density W of the magnetic field:

$$H = K_1/R_1^2(t), \quad W = H^2/8\pi = K_1^2/8\pi R_1^4, \quad K_1 = \mathrm{const}, \qquad (4.5)$$

which coincide with the general relativistic equations (1.8a). As before, we use in the given Newtonian problem (in accordance with the general-relativistic solution) the relations (1.4) and (1.4a) for the shear tensor and the relation (1.6a) for the Hubble constants (we consider the case of a flat model).

The Raychaudhuri identity, which follows from (4.1), (2.2), (2.6), (2.9), (2.8), and (4.5), has in this case the form

$$3(\Theta^2 + \dot\Theta) + 6Q^2 + 3Mk/R^3 + K_1^2 k/c^2 R_1^4 = 0. \qquad (4.6)$$

Relation (1.6a) leads, with allowance for (2.8) and (4.5), to

$$3(\Theta^2 - Q^2) = 6kM/R^3 + K_1^2 k/c^2 R_1^4. \qquad (4.7)$$

As a result of (1.4a), we obtain in accord with (2.12)

$$R_1 = R_3 = RS^{-1}, \quad R_2 = RS^2. \qquad (4.8)$$

*$[\mathbf{jH}] \equiv \mathbf{j} \times \mathbf{H}$.

To obtain the final formulas for $a(t) \equiv R_1 = R_3$ and $b(t) \equiv R_2$ from (4.6) and (4.7), it is necessary to carry out an additional transformation (in accordance with the form of the GRT equations for the given case[18]). Namely, we take into consideration the fact that according to (4.8) we have $\dot a/a = \Theta - Q$ and $\dot b/b = \Theta + 2Q$, and setting up $2\ddot a/a + (\dot a/a)^2$ we obtain, after differentiating (4.7) and taking (4.6) and (4.7) into account, the following equation (which coincides with the component $i = k = 2$ of Eqs. (1.2)):

$$2a\ddot a + \dot a^2 - (B_1^2 c^2/a^2) = 0, \quad B_1^2 = K_1^2 k/c^4 = \mathrm{const}.$$

Integration leads to the general relativistic formulas (1.8) for a(t), and then also to the relation (1.8) for b.

The gravitational potential φ is expressed in this case (in analogy with (2.17) and (2.18) with allowance for (4.6)) in the form (the matrices $A_{\alpha\beta}$ and $B_{\alpha\beta}$ are given by (2.18) with the values of (1.4a), and turn out to be equal)

$$\varphi = \frac{1}{2}\left[\left(\frac{kM}{R^3} + \frac{K_1^2 k}{3c^2 R_1^4}\right)\delta_{\alpha\beta} - A_{\alpha\beta}(\dot Q + 2\Theta Q + Q^2)\right] x^\alpha x^\beta + \mathrm{const}.$$

It is possible to construct analogously the Newtonian analog of the model with magnetic field at $R_1 = R_3$ for matter with $e = 3p$[18].

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22