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GAUGE INVARIANCE AND REGULARIZATION

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It is pointed out that the Pauli-Villars regularization which is necessary for maintaining gauge invariance at all steps of computations according to perturbation theory, has some unusual features. It leads to the vanishing of integrals of positive definite functions, both for convergent and divergent integrals, and also leads to discontinuities in the regular parts of divergent diagrams. The physical significance of the Pauli-Villars procedure is discussed, as well as the possibility of extending it in such a manner that the divergent parts of all the diagrams are made to vanish.

1. INTRODUCTION

GREAT attention has been paid recently to the problem of definition of currents and their commutators. In doing this, different authors claim different results for the same quantities, so that there arises the impression that certain redefinitions become necessary. Any such redefinition consists in detailing the regularization method employed. We are of the opinion that regularizations should not be introduced anew for each special case under consideration, but rather that one should make use of the regularization which has already been used in carrying out the renormalizations in the S-matrix.

In principle, any regularization in an intermediate stage of the calculation may violate some of the physical conditions imposed on the theory: unitarity, causality, positive definiteness of the metric, relativistic and gauge invariance, etc. Since all these conditions have to be satisfied only for the final expressions[1)], their (inevitable) violation in the intermediate stages of computations is not dangerous. It is however preferable to deal with a regularization which violates a "minimal" set of physical requirements, namely those which are not too hard to verify on the final expressions. From this point of view the Pauli-Villars regularization[1] is attractive for theories having a gauge group, since in distinction from other types of regularizations, it maintains gauge invariance at all stages of the computation.

In the present paper we investigate in detail on the example of quantum electrodynamics those peculiarities of the Pauli-Villars regularization which distinguish it from other regularization methods. We shall be interested most of all in the application of regularization to spectral representations. The reason for this is that the known formal contradictions of vector theories are usually demonstrated most clearly in the language of spectral representations. At the same time the properties of the Pauli-Villars regularization have never been discussed in the literature in these terms, in spite of the fact that it is exactly the presence of these properties which ensure the freedom from any contradictions. In particular, we show that the Pauli-Villars regularization in the form used for effecting renormalizations in the S-matrix are completely sufficient for the computation of the equal-time commutators.

2. THE SPECTRAL REPRESENTATIONS FOR ELECTRODYNAMICS

It is well-known that the spectral representation of the vacuum expectation value of the commutator of vector currents has the form

$$f_{\mu\nu}(x-y)=-i\langle[j_\mu(x),j_\nu(y)]\rangle_0=\int_0^\infty ds I(s)\left(g_{\mu\nu}+\frac{\partial_\mu\partial_\nu}{s}\right)D_s(x-y). \tag{1}$$

In order to obtain the representation for the Green-like functions one can make use of the "equations of motion"[2] for the current-like operators

$$\frac{\delta j_\mu(x)}{\delta a_\nu(y)}=i\theta(y_0-x_0)[j_\mu(x),j_\nu(y)]+\Lambda_{\mu\nu}^{(2)}(x,y). \tag{2}$$

Then

$$f_{\mu\nu}^{a}(x-y)=f_{\mu\nu}^{a(D)}(x-y)+\lambda_{\mu\nu}^{(2)}(x-y). \tag{3}$$

where each term in (3) is the vacuum expectation value of the appropriate term in (2), taken with the opposite sign:

$$\begin{aligned} f_{\mu\nu}^{a(D)}(x-y)=&(-K_{\mu\lambda})\int_0^\infty ds I(s)s^{-2}D_s^a(x-y)(-K_{\lambda\nu})\\ &+c_{1D}(-K_{\mu\nu})\delta(x-y)+c_{0D}(g_{\mu\nu}-\delta_{\mu0}\delta_{\nu0})\delta(x-y) \end{aligned} \tag{4}$$

$$\lambda_{\mu\nu}^{(2)}(x-y)=\lambda_1(-K_{\mu\nu})\delta(x-y)+\lambda_0(g_{\mu\nu}-\delta_{\mu0}\delta_{\nu0})\delta(x-y),$$

$$K_{\mu\nu}=g_{\mu\nu}\partial_\lambda^2-\partial_\mu\partial_\nu,$$

so that in the renormalized theory

$$c_{1D}+\lambda_1=0,\quad c_{0D}+\lambda_0=0; \tag{6}$$

$$c_{1D}=-\int\frac{I(s)ds}{s^2},\quad c_{0D}=\int\frac{I(s)ds}{s}. \tag{7}$$

It is convenient to write down the spectral representation for the Fourier transform of the advanced current Green's function, $f_{\mu\nu}^a(k)$:

$$\begin{aligned} f_{\mu\nu}^{a}(k)=&(g_{\mu\nu}k^2-k_\mu k_\nu)k^2\int\frac{I(s)ds}{s^2(s-k^2-i\varepsilon k_0)}\\ &+(g_{\mu\nu}k^2-k_\mu k_\nu)\left[\int\frac{I(s)ds}{s^2}-\lambda_1\right]+(g_{\mu\nu}-\delta_{\mu0}\delta_{\nu0})\left[\int\frac{I(s)ds}{s}+\lambda_0\right]. \end{aligned} \tag{8}$$

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1)Of course, at all stages of the computation the regularization should be the same, otherwise contradictions might (and will) appear. To their number we refer the so-called Schwinger paradox.

By the definition of c_{0D} as the subtraction constant in the spectral representation (8), it should be related to the mass renormalization constant of the photon: $c_{0D} = \delta\mu^2/Z_3$. On the other hand, the spectral integral c_{0D} gives an expression for the equal-time current commutator

$$\delta(x_0-y_0)f_{\mu\nu}(x-y)=(\delta_{\mu0}\delta_{\nu k}+\delta_{\mu k}\delta_{\nu 0})\int\frac{I(s)\,ds}{s}\partial_k\delta(x-y), \quad (9)$$

the consideration of which leads to the Schwinger paradox.

The physical photon mass must vanish, from considerations of gauge invariance. For a vanishing unrenormalized photon mass this requirement will be satisfied owing to the stability condition (cf. the second of the conditions (6)), which can be enforced in two ways[2].

If the integral c_{0D} vanishes, then $\lambda_0 = 0$, i,e., in the Lagrangian formalism there is no need to introduce a mass renormalization counterterm for the photon. In this case gauge invariance holds at all stages of the calculation.

Conversely, c_{0D} may be nonzero. Then $\lambda_0 \neq 0$ in order to compensate c_{0D}, so that the physical photon mass remains equal to zero. However, now the interaction Hamiltonian must contain gauge non-invariant and noncovariant counterterms of the type

$$\lambda_0(g_{\mu\nu}-\delta_{\mu0}\delta_{\nu0})\,\delta(x-y):a_\mu(x)a_\nu(y):.$$

It is clear that in this case gauge invariance is satisfied only in the final stage of the calculation.

The main disadvantage of the second possibility consists in the fact that there is no continuous transition to the case of finite renormalizations, where gauge-noninvariant counterterms would be completely inadmissible as a physical part of the interaction. One would therefore like to give unconditional preference to the first possibility. However, the requirement for the integral c_{0D} to vanish is quite nontrivial. After all, c_{0D} is (albeit divergent in a realistic theory) an integral over a positive definite function. Therefore, in order to achieve the first possibility, we require such a regularization procedure which will not only make an integral over a "positive definite" function vanish, but will also maintain the zero result after the regularization is removed. In addition, in order to achieve the desired continuous transition to the case of finite renormalizations, the procedure we are searching for must do the same to some convergent integrals over positive definite functions.

It turns out that the procedure introduced already more than 20 years ago by Pauli and Villars does indeed exhibit all these "miraculous" properties. In a narrow sense, this procedure consists in replacing the contribution of each fermion loop by a sum of such contributions from loops with new fermions of masses M_i, weighted with coefficients c_i in such a manner that

$$\sum c_i=0, \quad \sum c_i M_i^2=0,$$

and such that $c_1 = 1$, $M_1 = m$, and as the regularization is removed by letting $M_i \to \infty$ $(i > 1)$ the coefficients c_i remain uniformly bounded. For the spectral density this procedure is tantamount to the substitution

$$I(s)=I(s,m^2)\to I^{reg}(s)=\sum c_i I(s,M_i^2),$$

and the regularization is to be removed after the spectral integrals are computed.

3. TWO-DIMENSIONAL ELECTRODYNAMICS

The simplest example on which one can illustrate the realization of our general reasoning is spinor electrodynamics in a two-dimensional space-time. In this case, among the strongly connected diagrams, the only divergent one is the second-order photon self-energy. Indeed, writing the polarization operator according to the Feynman rules (this operator corresponds to the current Green's function):

$$\Pi_{\mu\nu}(k)=-\frac{ie^2}{(2\pi)^2}\int d^2p\,\frac{\mathrm{Sp}\,\{\gamma_\mu(\hat p+m)\gamma_\nu(\hat p-\hat k+m)\}}{[p^2-m^2+i\varepsilon]\,[(p-k)^2-m^2+i\varepsilon]} \quad (10)$$

one obtains an expression which formally has a logarithmic divergence. We note however, that if one contracts this operator with respect to the indices:

$$\Pi(k^2)=g_{\mu\nu}\Pi_{\mu\nu}(k),$$

and takes the trace, the highest powers of p in the numerator cancel mutually, so that $\Pi(k^2)$ turns out to be convergent

$$\begin{aligned}\Pi(k^2)&=-ie^2(2\pi)^{-2}4\int d^2p[p^2-m^2+i\varepsilon]^{-1}[(p-k)^2-m^2+i\varepsilon]^{-1}\\&=-\frac{e^2}{2\pi}\frac{4m^2}{k^2}\left(1-\frac{4m^2}{k^2}\right)^{-1/2}\cdot\left\{\ln\left|\frac{1+(1-4m^2/k^2)^{1/2}}{1-(1-4m^2/k^2)^{1/2}}\right|-i\theta(k^2-4m^2)\pi\right\}\\&=\int\frac{I(s)\,ds}{s-k^2-i\varepsilon}=k^2\int\frac{I(s)\,ds}{s(s-k^2-i\varepsilon)}+\int\frac{I(s)\,ds}{s};\\I(s)&=\frac{e^2}{2\pi}\frac{4m^2}{s}\left(1-\frac{4m^2}{s}\right)^{-1/2}\theta(s-4m^2).\end{aligned} \quad (11)$$

Such a result for $\Pi(k^2)$ corresponds to a non-transverse $\Pi_{\mu\nu}(k)$, and can be obtained by contracting the gauge-noninvariant spectral representation

$$\Pi_{\mu\nu}'(k)=(g_{\mu\nu}k^2-k_\mu k_\nu)\int\frac{I(s)\,ds}{s(s-k^2-i\varepsilon)}+(g_{\mu\nu}-\delta_{\mu0}\delta_{\nu0})\int\frac{I(s)\,ds}{s}, \quad (12)$$

in which all integrals are convergent; in second order of e the spectral density I(s) corresponds to the contribution from a closed fermion loop with $S^{(-)}$-lines, and behaves like s^{-1} for $s \to \infty$.

The restoration of gauge invariance (transversality of $\Pi_{\mu\nu}$) can be achieved by applying the Pauli-Villars procedure. For the two-dimensional electrodynamics it is sufficient to use one auxiliary mass and to subject the spectral density to the substitution

$$I(s)=I(s,m^2)\to I^{reg}(s)=I(s,m^2)-I(s,M^2).$$

Gauge invariance will be restored if one can make the spectral integral $\int ds\, s^{-1}I(s)$ vanish in the representation (12). It is easy to see that in using the Pauli-Villars regularization

$$\int\frac{I(s)\,ds}{s}\to\int\frac{I^{reg}(s)\,ds}{s}=\int\frac{I(s,m^2)\,ds}{s}-\int\frac{I(s,M^2)\,ds}{s}=\frac{e^2}{2\pi}(2-2)=0,$$

and this result remains valid after the regularization has been removed.

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[2] In principle there is a third conceivable possibility: when the stability condition is not satisfied and a nonvanishing integral c_{0D} compensates a nonvanishing unrenormalized photon mass in the absence of a mass renormalization counterterm; but such a possibility is not conveniently discussed within our formalism (cf. [3] Secs. 2.2 and 2.4.1).

The fact of the vanishing of a convergent integral over a positive definite function is explained by the peculiarities of the Pauli-Villars regularization. Indeed, this regularization is not equivalent to the introduction of some regularizing multiplier, which becomes equal to one when the regularization is removed. On the contrary, it consists in subtracting from the integrand to be regularized a function (which in our case is negative definite), such that its contribution to the integral $\int ds\, s^{-1} I^{reg}(s)$ does not depend on the values of the parameter M, although for $M \to \infty$ the region of s values where this function is nonzero goes off to infinity.

The characteristics of the Pauli-Villars regularization for the polarization operator as a whole can be illustrated by means of the graph of the function

$$\Pi^{reg}(k^2) = k^2 \int \frac{I^{reg}(s)\, ds}{s(s - k^2 - i\varepsilon)} \tag{13}$$

for large but finite values of M. As is usual in two-dimensional theories the function $\Pi^{reg}(k^2)$ suffers a root-type discontinuity at the thresholds $k^2 = 4m^2$, $k^2 = 4M^2$. In order not to complicate the picture one can smooth out the behavior of $\Pi^{reg}(k^2)$ near $k^2 \sim 4M^2$, by means of an appropriate smearing out in M. Then the graph will have the form shown in the figure. It has the following characteristic regions: I: $4m^2 \ll |k^2| \ll 4M^2$, and II: $|k^2| \gg 4M^2$. The region I corresponds to carrying out the limiting process in the order

$$\lim_{k^2 \to \infty} \lim_{M \to \infty} \Pi^{reg}(k^2) = -2,$$

and the region II corresponds to the order of the limits:

$$\lim_{M \to \infty} \lim_{k^2 \to \infty} \Pi^{reg}(k^2) = 0.$$

As is evident, the two limits are different.

In other words, although for the spectral density we have

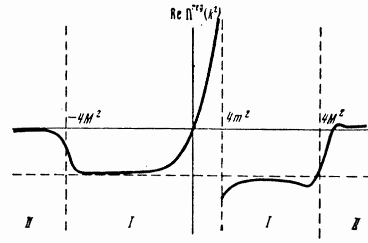
$$\lim_{M \to \infty} I^{reg}(s) = I(s),$$

we shall have for the spectral integrals

$$\lim_{M \to \infty} \int I^{reg}(s) f(s, k^2)\, ds \neq \int \lim I^{reg}(s) f(s, k^2)\, ds = \int I(s) f(s, k^2)\, ds.$$

Therefore, after the regularization is removed, the contracted polarization operator $\Gamma(k^2)$ is discontinuous: the limits of $\Pi(k^2)$ for $|k^2| \to \infty$ are -2, whereas the value when k^2 is "exactly infinity" is zero. The appearance of the discontinuity is the price one has to pay for the attractive features of the Pauli-Villars regularization we have discussed[3)]. As we shall see below, for the same price one achieves also the gauge invariance in a realistic four-dimensional electrodynamics, but there the discontinuities turn out to be infinite.

It would seem that the difficulties with the discon-



tinuity at $k^2 = \infty$ is inessential, since this point could be completely eliminated from consideration, for example by redefining $\Pi(k^2)$ by continuity. However this point is physically distinguished: it corresponds to equal times in the coordinate representation. Therefore it is related to such important concepts as the equal-time commutation relations which are at the foundation of the Hamiltonian formalism, and also the initial conditions. If one eliminates from consideration, completely exactly equal times, it becomes unclear how far one can proceed with the construction of the Hamiltonian formalism of the theory. Indeed, considering only approximately equal, but not strictly equal, times one is forced to remember one of the versions of nonlocal field theory with its inherent difficulties.

The answer for the contracted polarization operator comes out unique for any method used, owing to the sufficiently rapid decrease of the spectral density. At the same time the longitudinal part of the operator $\Pi_{\mu\nu}(k)$, more precisely the expression $k_\mu \Pi_{\mu\nu}(k)$, which should vanish owing to the requirement of gauge invariance, will in fact depend on the method of computation. Here the formal divergence of $\Pi_{\mu\nu}(k)$ in perturbation theory manifests itself in full measure.

Let us try, for instance, to verify the transversality of $\Pi_{\mu\nu}(k)$. Owing to the formal divergence of the integral (10), the expression $k_\mu \Pi_{\mu\nu}(k)$ should be interpreted as the improper integral

$$k_\mu \Pi_{\mu\nu}(k) = -\frac{ie^2}{4\pi^2} \lim_{A_i B_i \to \infty} \int_{-A_0}^{B_0} dp_0 \int_{-A_1}^{B_1} dp_1 \frac{p_\nu(2kp - k^2) - k_\nu(p^2 - m^2)}{[p^2 - m^2 + i\varepsilon][(p - k)^2 - m^2 + i\varepsilon]}.$$

Before passing to the limit one can add and subtract in the numerator the combination $p_\nu(p^2 - m^2)$ and reducing similar terms and simplifying with the denominator, we obtain

$$k_\mu \Pi_{\mu\nu} = -\frac{ie^2}{4\pi^2} \lim_{A_i B_i \to \infty} \int_{-A_0}^{B_0} dp_0 \int_{-A_1}^{B_1} dp_1 \left(\frac{p_\nu - k_\nu}{(p - k)^2 - m^2 + i\varepsilon} - \frac{p_\nu}{p^2 - m^2 - i\varepsilon} \right).$$

By a change of variables one can achieve that the second term in the integrand will be equal, with opposite sign, to the first, but so that the limits of integration for the two terms will be different. Therefore, depending on the relations between the quantities A_i and B_i, one obtains different results after going to the limit: if, e.g., $A_1, B_1 \gg A_0, B_0$, then

$$\mathrm{Re}\, k_\mu \Pi_{\mu\nu}(k) = \frac{e^2}{2\pi} \frac{k_\nu}{2} \delta_{\nu 0},$$

and if $A_1, B_1 \ll A_0, B_0$, then

$$\mathrm{Re}\, k_\mu \Pi_{\mu\nu}(k) = \frac{e^2}{2\pi} \frac{k_\nu}{2} \delta_{\nu 1} \text{ etc.}$$

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3)We note that if one considers the successive operations as a formal intermediate regularization

$$I(s) \to I^{reg}(s) \to \lim_{M \to \infty} I^{reg}(s) = I(s)$$

which should have no influence on the physical answers, such an approach is not legitimate in our case. The only possibility to interpret the Pauli-Villars procedure is to postulate that the "usual theory" is by definition the limit of a theory with finite M. For the two-dimensional electrodynamics before taking the limit, in addition to the physical fermion there are two auxiliary ones with mass M, one of which has indefinite metric.

It is clear that the different methods of passing to the limit, corresponding to different methods of regularization may violate not only the gauge invariance, but also the covariance of the result for $k_\mu \Pi_{\mu\nu}(k)$. It is easy to verify that the regularization according to Pauli-Villars leads to a unique gauge-invariant (and covariant) reply: $k_\mu \Pi_{\mu\nu}(k) = 0$. In this case one obtains in perturbation theory for the total polarization operator a transverse expression, which agrees with the gauge-invariant spectral representation:

$$\Pi_{\mu\nu}(k) = \lim_{M\to\infty} \Pi_{\mu\nu}^{reg}(k) = -\frac{e^2}{\pi}\left(g_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\right)\left\{1 + \frac{2m^2/k^2}{\sqrt{1-4m^2/k^2}}\right.$$
$$\left.\times\left[\ln\left|\frac{1+\sqrt{1-4m^2/k^2}}{1-\sqrt{1-4m^2/k^2}}\right| - i\pi\theta(k^2-4m^2)\right]\right\}$$
$$= (g_{\mu\nu}k^2 - k_\mu k_\nu)\int \frac{I(s)\,ds}{s(s-k^2-i\varepsilon)},$$

and the integral $c_0D = \int ds\, s^{-1} I(s)$ does not participate at all in this representation, since $\int ds\, s^{-1} I^{reg}(s) = 0$. It is clear in this connection that there is no need to include in the current-like operator terms proportional to $g_{\mu\nu}\delta(x-y)$ and $\delta_{\mu 0}\delta_{\nu 0}\delta(x-y)$, i.e., if this method is used the photon mass is not subject to renormalization.

We also remark that in computing $\Pi_{\mu\nu}$ to second order by using the α-representation, then after taking the trace under the integral sign with respect to α_i, the divergent terms cancel out, but the finite result obtained in this manner (covariant, but gauge-noninvariant) is in disagreement with all spectral representations.

The listed facts show that in the presence of formal divergences a regularization is always necessary. Even if we are not encountering any really divergent integrals in a selected method of computation, some regularization is "at work" in fact, but we do not realize exactly which it is. The transformation group with respect to which the theory must be invariant is here a criterion which allows one to select a "natural" regularization, and thus to obtain definite, unique answers.

4. FOUR-DIMENSIONAL ELECTRODYNAMICS

The next example is four-dimensional quantum electrodynamics. The spectral density $I(s)$ to second order in e, has the form:

$$I(s) = I(s, m^2) = \frac{e^2}{12\pi^2} s\left(1 + \frac{2m^2}{s}\right)\sqrt{1 - \frac{4m^2}{s}}\,\theta(s-4m^2).$$

In this case the Pauli-Villars procedure requires the introduction of two auxiliary masses, and is equivalent to the substitution

$$I(s, m^2) \to I^{reg}(s) = \sum_{i=1}^{3} c_i I(s, M_i^2).$$

We first investigate how this regularization affects the quadratically divergent spectral integral $c_0D = \int ds\, s^{-1} I(s)$, corresponding to the gauge-noninvariant photon mass renormalization counterterm. Now this integral is to be interpreted as the limit

$$c_{0D} = \lim_{M_{2,3}\to\infty} \int \frac{I^{reg}(s)}{s}\, ds.$$

For its computation it is technically convenient to take the sum $\sum_i$ outside the integral sign. Taking into account that the integral converges absolutely for finite M_i, we can represent it in the form

$$c_{0D} = \lim_{M_{2,3}\to\infty} \lim_{\Lambda\to\infty} \int_0^\Lambda ds\, s^{-1} I^{reg}(s).$$

Then, after going to the limit the integral of each term of the sum will be absolutely convergent, and we are entitled to permute summation and integration:

$$c_{0D} = \lim_{M_{2,3}\to\infty} \lim_{\Lambda\to\infty} \sum_i c_i \int_0^\Lambda ds\, s^{-1} I(s, M_i^2).$$

After the change of variables $y_i = (1 - 4M_i^2/s)^{1/2}$ in each term of the sum we have

$$c_{0D} = \frac{e^2}{12\pi^2} \lim_{M_{2,3}\to\infty} \lim_{\Lambda\to\infty} \sum_i c_i \Lambda^2 \left(1 - \frac{4M_i^2}{\Lambda^2}\right)^{3/2}$$
$$= \frac{e^2}{12\pi^2} \lim_{M_{2,3}\to\infty} \lim_{\Lambda\to\infty} \sum_i c_i \left(\Lambda^2 - 6M_i^2 + O\left(\frac{1}{\Lambda^2}\right)\right). \tag{14}$$

In the same manner as for the two-dimensional case, the possibility that c_0D vanishes is explained by the fact that the regularized spectral density turns out not to be positive definite: one of the coefficients c_i is negative. However, in itself this circumstance does not guarantee that after the regularization is removed the divergences do not show up again. It is also important that there are no logarithmic terms of the form

$$\sum c_i \Lambda^2 \ln \frac{M_i^2}{m^2} \quad \text{or} \quad \sum c_i M_i^2 \ln \frac{M_i^2}{m^2}$$

in the limiting expression (14). Therefore, in the limit $\Lambda \to \infty$ the integral $\int ds\, s^{-1} I^{reg}(s)$ does indeed vanish, and the result remains true when the regularization is removed, i.e., in the limit $M_{2,3} \to \infty$.

The presence of logarithmic terms would lead to the appearance of divergences when the regularization is removed. The absence of such terms is a characteristic of vector theories (more precisely of Yukawa type interactions and vector bosons. It is easy to foresee this feature, if one expands the spectral density $I(s)$ in series with respect to the powers of $1/s$: $I(s)s^{-1} = 1 - 6(m^2/s^2)^2 - 8(m^2/s^2)^3 - \ldots$. The terms proportional to s^k with $k < -1$ give a vanishing contribution to the spectral integral, owing to the Pauli-Villars condition. The contribution of terms proportional to s^k with $k > -1$ vanishes in the limit $\Lambda \to \infty$. The only dangerous term, namely the one proportional to s^{-1} (which would give a logarithmic contribution to the spectral integral) is absent from the expansion of the function $s^{-1}I(s)$ for vector theories. At the same time in other theories (with scalar, pseudoscalar or axial vector mesons) $s^{-1}I(s)$ contains such a term, and after the regularization is removed there remains a logarithmic divergence in the spectral integral c_0D. We note that in the second divergent spectral integral, c_1D the logarithmic divergence reappears for this method of procedure in any theory, even the vector one.

Since the spectral integral c_0D vanishes, the total polarization operator has the form

$$\Pi_{\mu\nu}(k) = (g_{\mu\nu}k^2 - k_\mu k_\nu)\int \frac{I(s)\,ds}{s(s-k^2-i\varepsilon)}. \tag{15}$$

Therefore it suffices to talk in the sequel only about the contracted operator $\Pi(k^2) = g_{\mu\nu}\Pi_{\mu\nu}$, as in the two-dimensional case. It is convenient to exhibit the characteristic of the Pauli-Villars regularization for this quantity on the example of the "dipole" regularization, which corresponds to auxiliary masses situated close to each other: $M_2 = M$; $M_3 = M(1+\epsilon)$. Then

$$c_1 = 1, \quad c_2 = -1-(1-m^2/M^2)/\varepsilon, \quad c_3 = (1-m^2/M^2)/\varepsilon,$$

and the removal of the regularization corresponds to the successive limits $\lim_{M\to\infty}\lim_{\epsilon\to 0}$. The first limit may already be taken in the spectral density, so that the regularized expressions will depend only on the one parameter M. In the "dipole" case

$$I^{reg}(s) = I(s, m^2) - I(s, M^2) + (M^2 - m^2)\frac{\partial}{\partial M^2} I(s, M^2).$$

Accordingly, we have

$$\Pi^{reg}(k^2) = k^2 \int \frac{I^{reg}(s)\,ds}{s(s-k^2-i\varepsilon)} = \frac{e^2}{4\pi^2}\Big\{2(M^2-m^2) + k^2 \ln\frac{M^2}{m^2} - k^2\Big(1+\frac{2m^2}{k^2}\Big)X(k^2, m^2) + \frac{k^4 - 2k^2M^2 - 12m^2M^2 + 4M^4}{k^2-4M^2} X(k^2, M^2)\Big\}, \qquad (16)$$

where

$$X(a, b) = \frac{1}{2}\sqrt{1-\frac{4b}{a}}\Big[\ln\frac{1+\sqrt{1-4b/a}}{|1-\sqrt{1-4b/a}|} - i\pi\theta(a-4b)\Big].$$

Here, as before, two different sequences of taking the limit are interesting:

$$\lim_{k^2\to\infty}\lim_{M^2\to\infty} \Pi^{reg}(k^2) = \frac{e^2}{4\pi^2}k^2\Big(\ln\frac{M^2}{k^2}+\frac{2}{3}\Big)$$

and

$$\lim_{M^2\to\infty}\lim_{k^2\to\infty} \Pi^{reg}(k^2) = 0.$$

The second of these results is quite understandable and corresponds to a transition to the integral $-c_0D = -\int ds\, s^{-1} I(s)$ in the spectral representation (16); since without regularization c_0D simply diverges, it will make sense only with such an order of taking the limits. The first result should be interpreted in the sense that the asymptotic behavior of $\Pi^{reg}(k^2)$ "goes off to infinity" when the regularization is removed.

Thus, we have again reached the conclusion that $\Pi(k^2)$ is discontinuous at the point $k^2 = \infty$ after the regularization is removed. In distinction from the two-dimensional case this discontinuity is infinite, which means that the degree of divergence in four-dimensional electrodynamics is higher.

The analog (insofar as the degree of divergence is concerned) of the polarization operator of the two-dimensional case is now the photon-photon scattering diagram in fourth order of e. Its contribution is formally logarithmically divergent. However, if all the traces are computed in the integrand and similar terms are reduced, the total degree of growth diminishes. For vanishing external momenta the photon-photon scattering tensor after removal of the cutoff becomes a finite constant, independent of the masses:

$$T_{\mu\nu\lambda\sigma} \sim {}^4/_3(g_{\mu\nu}g_{\lambda\sigma} + g_{\mu\lambda}g_{\nu\sigma} + g_{\mu\sigma}g_{\nu\lambda}).$$

As a result of the Pauli-Villars regularization (owing to the condition $\Sigma c_i = 0$) this finite constant "turns into" zero.

5. DISCUSSION

We have already praised the advantages of a computational scheme which maintains gauge invariance throughout all steps of the calculation. Such a scheme is guaranteed only by the Pauli-Villars regularization. All other regularization methods can achieve only the gauge invariance of the final expressions. However, the renormalization terms (or the unrenormalized theory) will in this case be gauge-noninvariant.

But, in ensuring full gauge invariance, the Pauli-Villars regularization exhibits one concern causing feature: it turns into zero not simply divergent integrals, but divergent integrals of positive definite functions. It does moreover the same with convergent integrals of positive definite functions. This means that the procedure of introducing and subsequently removing the regularization a la Pauli-Villars is not an innocent computational device, but a "physical" operation.

In order to understand its physical meaning we note that the Pauli-Villars regularization is equivalent to replacing the usual current density $-e{:}\,\bar{\psi}\gamma_\mu\psi{:}$ by the expression[4)]:

$$-e : \bar{\psi}\gamma_\mu\psi + \sqrt{|c_2|}\,(\bar{\psi}_2'\gamma_\mu\psi_2 + \bar{\psi}_2\gamma_\mu\psi_2') + \sqrt{c_3}\bar{\psi}_3\gamma_\mu\psi_3 : .$$

Thus, from a theory with a single fermion we go over to a theory with indefinite metric and with four types of fermions: one physical and three auxiliary ones. The auxiliary fields ψ_2 and ψ_3 with masses M_2 and M_3 satisfy the same commutation relations as the original physical field ψ: $\{\bar{\psi}(x), \psi(y)\}_+ = -iS(x-y)$. The auxiliary field ψ_2' of mass M_2 generates states of indefinite metric: $\{\bar{\psi}_2'(x), \psi_2'(y)\} = iS(x-y)$.

Only in such a theory with finite M_i are dynamical relations among different quantities meaningful. In particular, the T-product is finite for the modified currents. In addition, their commutator at equal times yields directly zero, if one makes the natural assumption that the indeterminacy in each of the four separate commutators should be removed in the same way. Considered in the whole Hilbert space with indefinite metric, the modified theory is unitary and causal.

However, in this theory it is impossible to give a physical interpretation to the auxiliary fields. The genuinely physical quantities which are obtained after taking the limit $M_i \to \infty$ appear as boundary values of quantities belonging to a modified, dynamical theory. It looks as if the physical quantities alone are insufficient for the construction of a dynamical formalism, similar to the fact that in quantum mechanics the probability density $\Phi^*\Phi$ is insufficient and one needs to use the wave function Φ itself, which is devoid of direct physical meaning.

An investigation of the Pauli-Villars regularization has yet another aspect. In its simple form it only makes some divergent integrals vanish. However, it can be generalized (adding some new auxilliary masses, and imposing additional conditions on the coefficients c_i) in such a manner as to make all divergent integral vanish. In fact, such a possibility was considered in the papers by Slavnov[4], who has shown that if one

4)The authors are indebted to D. A. Slavnov, who called their attention to this circumstance.

introduces new auxilliary fields and requires that conditions of the type $\Sigma c_i \ln (M_i^2/m^2) = 0$ be satisfied (which can be achieved maintaining the uniform boundedness of all coefficients c_i), one can make the divergent parts of all diagrams vanish. Then only the regular parts survive for all diagrams (the number of new conditions increases with the order of perturbation theory).

In the light of this remark one can question all the results which are based on the positive definiteness of divergent spectral integrals. Such results are, e.g., the known restrictions on the magnitudes of the renormalization constants.

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