INDIRECT PARALLEL PUMPING AND BIRESONANT FREQUENCY DOUBLING IN ANTIFERROMAGNETS

V. I. OZHOGIN

Submitted December 18, 1969

Zh. Eksp. Teor. Fiz. 58, 2079–2089 (June, 1970)

"Three-magnon" nonlinear phenomena in antiferromagnets with an "easy plane" of anisotropy (AFEP) are calculated by classical methods for the case of parallel orientation of the ac and dc magnetic fields applied in the easy plane of the crystal. It is shown that for arbitrary pumping frequencies, parametric transformation of electromagnetic energy into spin-wave energy of the lower branch occurs via oscillations in the other branch of the spectrum. Because of this, the threshold field for appearance of instability in the first branch may be described by an expression that is valid for any pumping frequency and is of a resonance nature. Frequency doubling is also calculated for an AFEP located in a dc magnetic field of such magnitude that the first frequency ω_{10} of the AFMR is exactly equal to half the second ω_{20}. It turns out that in this case the efficiency of transformation of an electromagnetic field of frequency ω_{10} into a field of frequency ω_{20} may be much higher than that attained with a ferrite, especially at high frequencies.

1. INTRODUCTION

AMONG uniaxial antiferromagnets (AF) there is a broad class of substances in which there is an entire plane of easy directions of magnetization of the sublattices. The spectrum ω_{rk} of the electronic spin waves of these substances (see figure) consists of two distinctly separate branches (for values of the external dc magnetic field $\mathbf{H}_0$ that are not too large)—"quasiphonon" (r = 1) and "quasioptic" (r = 2). Borovik-Romanov[1] first called attention to the several physically interesting thermodynamic consequences of this kind of spectrum.

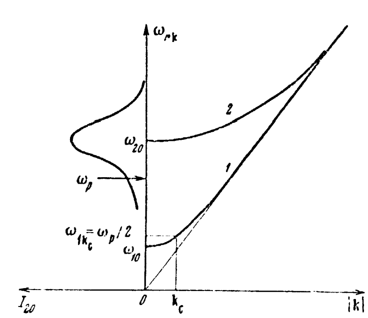
It was also found[2] that antiferromagnetics with easy-plane anisotropy (AFEP) are interesting from still another point of view—that of the nonlinear dynamical phenomena that can be observed at high power levels of a microwave field h applied to the sample. Many of these phenomena are due to the high magnitude of the nonlinear interaction between the branches. One of the types of this interaction is characteristic first of all for AFEP. This is the so-called three-magnon process, in which two magnons of branch r = 1 are converted into one magnon of branch r = 2, or the reverse: one magnon $\omega_{2\mathbf{k}_2}$ disappears, giving rise to two magnons $\omega_{1\mathbf{k}_1}$ and $\omega_{1\mathbf{k}'_1}$.[3] When this happens, of course, along with the conservation law for quasimomentum $\hbar\mathbf{k}_2 = \hbar\mathbf{k}_1 + \hbar\mathbf{k}'_1$, the energy conservation law must be satisfied:

$$\hbar\omega_{2\mathbf{k}_2} = \hbar\omega_{1\mathbf{k}_1} + \hbar\omega_{1\mathbf{k}_1'}, \tag{1}$$

which for small k is possible only if the frequencies for antiferromagnetic resonance (AFMR) ω_{20} and ω_{10} differ significantly (by a factor of two or more). This situation exists primarily in AFEP at moderate fields $H_0 \lesssim \frac{1}{2}H_{AE}$, as well as in antiferromagnets with easy-axis anisotropy (AFEA) close to those values of H_0 for which one of the AFMR frequencies vanishes.

In AFEP, as calculations show, these processes can lead to a marked broadening of the AFMR line at ω_{20} and, beginning at a certain temperature, also of the AFMR line at ω_{10},[4] as well as to "premature" saturation" of the AFMR line at ω_{20}.[2] The magnitude of the "three-phonon" interaction is in the first approximation proportional to the external magnetic field H_0,[2,4] as a consequence of which, in particular, the magnitude of the threshold field for "premature saturation" of the ω_{20} AFMR line due to three-phonon processes is inversely proportional to H_0 (see Eq. (22) in[2]).

Recently, Seavey[5,6] and Prozorova and Borovik-Romanov[7] discovered in monocrystals of the AFEP $CsMnF_3$ a phenomenon of threshold absorption of microwave energy with so-called "parallel pumping" ($\mathbf{h} \parallel \mathbf{H}_0$) at a frequency $\omega_p \ll \omega_{20}$, which they explained as a parametric excitation of pairs of magnons $\omega_{1\mathbf{k}_c}$ and $\omega_{1,-\mathbf{k}_c}$ (see the figure). Seavey[6] and Hinderks and Richards[8] derived semiquantum mechanically that $h_{thr} \sim H_0^{-1}$. This circumstance, and particularly the fact that the ω_{20} AFMR is also excited by a microwave field $\mathbf{h} \parallel \mathbf{H}_0$ (see below), suggests that a single description of the three-magnon "premature saturation"



The spectrum of spin waves of an antiferromagnet with anisotropy of the "easy plane" type with $\mathbf{H}_0 \| EP$: curve 1—r = 1 ($h_\perp \to \mu_\perp, \lambda_\|$); curve 2—r = 2 ($h_\| \to \mu_\|, \lambda_\perp$). The left part of the figure shows schematically the dependence of the intensity of the uniform quasiantiferromagnetic mode (k = 0, r = 2) on the pump frequency ω_p.

of the high-frequency AFMR ($\omega_p = \omega_{20}$) and the "parallel pumping" at $\omega_p \ll \omega_{20}$ is possible. The calculations presented below confirm this.

In addition, we consider the "reverse" phenomenon-frequency doubling in AFEP, i.e., the excitation of homogeneous oscillations of the second branch (at frequency ω_{20}) by means of an external microwave field of half the frequency $\omega_p = \omega_{20}/2$ with such a choice of the external field H_0 that $\omega_{20} = 2\omega_{10}$. In this case, the conditions for the three-magnon process with $\mathbf{k}_1 = \mathbf{k}_1 = \mathbf{k}_2 = 0$ are fulfilled, and the conversion efficiency can be extremely high.

2. INITIAL EQUATIONS

Unlike[2] we shall use a purely classical method, which is more pictorial in a treatment of non-resonant oscillations, as well as more convenient for a qualitative analysis of possible nonlinear phenomena in arbitrary AF structures.

We shall consider a two-sublattice model of an AFEP. In the system of two Landau-Lifshitz equations the dissipative terms can be introduced in any one of the well-known ways, but we shall do this in such a way that physically possible "cross" dissipation is taken into account (see the terms with κ'), the system is invariant with respect to a change in numeration of the sublattices, and at all moments of time the condition $|\mathbf{M}_1| = |\mathbf{M}_2| = M_0$, the natural one for $T \ll T_N$, is exactly fulfilled:

$$\dot{\mathbf{M}}_1 = -\gamma[\mathbf{M}_1\mathbf{H}_1^{\text{eff}}] + \frac{\varkappa}{M_0}[\mathbf{M}_1\dot{\mathbf{M}}_1] + \frac{\varkappa'}{M_0}[\mathbf{M}_1\dot{\mathbf{M}}_2],$$
$$\dot{\mathbf{M}}_2 = -\gamma[\mathbf{M}_2\mathbf{H}_2^{\text{eff}}] + \frac{\varkappa}{M_0}[\mathbf{M}_2\dot{\mathbf{M}}_2] + \frac{\varkappa'}{M_0}[\mathbf{M}_2\dot{\mathbf{M}}_1]. \qquad (2)^*$$

Here M_ν is the magnetization of the sublattices ($\nu = 1, 2$); $H_\nu^{\text{eff}} \equiv -\delta\mathscr{H}/\delta M$; $\mathscr{H}$ is the Hamiltonian (energy) of the system; κ, κ' are the dissipative coefficients; γ is the magnetomechanical ratio ($\gamma > 0$); anisotropy of the g factor and the field of the spin waves are neglected. Converting to variables $\mathbf{m} \equiv \mathbf{M}_1 + \mathbf{M}_2$ and $\mathbf{l} \equiv \mathbf{M}_1 - \mathbf{M}_2$, we have

$$\dot{\mathbf{m}} = -\gamma[\mathbf{m}\mathbf{H}_m] - \gamma[\mathbf{l}\mathbf{H}_l] + \frac{\eta}{2M_0}[\mathbf{m}\dot{\mathbf{m}}] + \frac{\zeta}{2M_0}[\mathbf{l}\dot{\mathbf{l}}],$$
$$\dot{\mathbf{l}} = -\gamma[\mathbf{m}\mathbf{H}_l] - \gamma[\mathbf{l}\mathbf{H}_m] + \frac{\eta}{2M_0}[\mathbf{l}\dot{\mathbf{m}}] + \frac{\zeta}{2M_0}[\mathbf{m}\dot{\mathbf{l}}], \qquad (3)$$

where

$$\mathbf{H}_m \equiv -\delta\mathscr{H}/\delta\mathbf{m}, \quad \mathbf{H}_l \equiv -\delta\mathscr{H}/\delta\mathbf{l}, \quad \eta \equiv \varkappa + \varkappa', \quad \zeta \equiv \varkappa - \varkappa'.$$

It is easy to show that the power absorbed in the sample in this case is given by

$$P_a = -\frac{\overline{d\mathscr{H}}}{dt} = \frac{1}{2M_0\gamma}\int(\eta\overline{\dot{\mathbf{m}}^2} + \zeta\overline{\dot{\mathbf{l}}^2})\,dV. \qquad (4)$$

For the sake of generality, we take into account also the Dzyaloshinskiĭ interaction (DI) in writing down the energy of the AFEP magnetic system, having in mind not only $CsMnF_3$, but also $MnCO_3$, $CoCO_3$, α-Fe_2O_3, etc.:

$$\mathscr{H} = \int dV\left\{\frac{B}{2}\mathbf{m}^2 + \frac{b}{2}(\mathbf{nm})^2 + \frac{a}{2}(\mathbf{nl})^2 - \beta\mathbf{n}[\mathbf{ml}] - \mathbf{mH} + \frac{\alpha}{2}\left(\frac{\partial\mathbf{l}}{\partial x_j}\right)^2 + \frac{\rho}{2}\left(\frac{\partial\mathbf{m}}{\partial x_j}\right)^2\right\}. \qquad (5)$$

*$[\mathbf{M}_1\mathbf{M}_1^{\text{eff}}] \equiv \mathbf{M}_1 \times \mathbf{M}_1^{\text{eff}}$.

Here B is the exchange interaction constant ($2BM_0 \equiv 2H_E \equiv \omega_E/\gamma$); a, b are anisotropy constants ($a > 0$, $2aM_0 \equiv H_A$); β is the Dzyaloshinskiĭ constant ($2\beta M_0 \equiv H_D$); H_E, H_A, H_D are the exchange, anisotropy, and Dzyaloshinskiĭ fields; α, ρ are phenomenological constants of the inhomogeneous exchange interaction; n is a unit vector in the direction of the hard axis.

For effective fields we find

$$\mathbf{H}_m = \mathbf{H} - B\mathbf{m} - b\mathbf{n}(\mathbf{nm}) - \beta[\mathbf{nl}] + \rho\partial^2\mathbf{m}/\partial x_j^2,$$
$$\mathbf{H}_l = -a\mathbf{n}(\mathbf{nm}) + \beta[\mathbf{nm}] + \alpha\partial^2\mathbf{l}/\partial x_j^2. \qquad (6)$$

Let the dc magnetic field $\mathbf{H}_0$ be applied in the easy plane of the crystal. We take its direction along x, and the z axis along n; $\mathbf{n} = (0, 0, 1)$, $x_j \to (x, y, z)$, $\mathbf{H} = (H_0 + h_x, h_y, h_z)$, h is the ac magnetic field ($h \ll H_0$). The state of equilibrium is described by the known expressions: $\mathbf{m}_0 = (m_0, 0, 0)$, $\mathbf{l}_0 = (0, l_0, 0)$, $m_0 \approx (H_0 + \beta l_0)/B$, $l_0 = [(2M_0)^2 - m_0^2]^{1/2} \approx 2M_0$. Here and everywhere below we make use of the relations a, b, β, $H_0/2M_0 \ll B$, $\alpha, \rho \sim Ba_0^2$ (a_0 is the lattice constant), $a_0k \ll 1$, $\eta, \zeta \ll 1$. In this case terms containing b and ρ, as it turns out, can generally be neglected.

Solution of the system (3) in the linear approximation ($\mathbf{m} = \mathbf{m}_0 + \boldsymbol{\mu}$, $\mathbf{l} = \mathbf{l}_0 + \boldsymbol{\lambda}$, $\mu \ll m_0$, $\lambda \ll l_0$) leads to the following results (they differ from known results only in the means of introduction of the dissipative terms, which, of course, does not pretend to any physical uniqueness).

1) Homogeneous (inhomogeneous) oscillations of the variables μ_y, μ_z, λ_x can be excited by a homogeneous (inhomogeneous) ac field of polarization y and (or) z, which is briefly notated thus: $h_\perp \to \mu_\perp, \lambda_\parallel$ (see the figure). Natural and relaxation frequencies of the oscillations of these variables are given by the expressions (valid only for long waves with $k \ll a_0^{-1}$)

$$\omega_{1k} \approx \gamma[H_0(H_0 + \beta l_0) + \alpha B l_0^2 k^2]^{1/2}, \qquad (7a)$$

$$\delta_{1k} \approx [\eta(\gamma^2 B^2 m_0^2 + \omega_{1k}^2) + \zeta\omega_E^2]/2\omega_E. \qquad (7b)$$

This means that μ_y, μ_z, λ_x vary as $\exp(-\delta_{1k}t - i\omega_{1k}t + i\mathbf{k}\cdot\mathbf{r})$ when $h = 0$.

2) The field h_x excites oscillations in the variables, μ_x, λ_y, λ_z, i.e., $h_\parallel \to \mu_\parallel, \lambda_\perp$. The natural and relaxation frequencies for these are

$$\omega_{2k} \approx \gamma[aBl_0^2 + \beta l_0(H_0 + \beta l_0) + \alpha B l_0^2 k^2]^{1/2}, \qquad (8a)$$

$$\delta_{2k} \approx (\eta\omega_{2k}^2 + \zeta\omega_E^2)/2\omega_E. \qquad (8b)$$

Classifying these two modes not by their relative magnitude of natural frequencies but by the type of excitation by the external electromagnetic field, we shall call the first quasiferromagnetic and the second quasiantiferromagnetic.

3. QUALITATIVE ANALYSIS OF THE SIMPLEST NONLINEAR PROCESSES

The explicit form of the equations of motion immediately permits a number of qualitative conclusions. For simplicity, we neglect for the moment the DI and attenuation and consider only homogeneous oscillations; we take the external action into account in the linear approximation (i.e., we neglect products of μ_j and λ_j with h_j). Here and everywhere below (except in the final expressions) we shall for brevity use a system of units in which $2M_0 = 1$ and $\gamma = 1$. Considering that at equilibrium $Bm_0 = H_0$, we obtain from (3), (5), and (6):

a) quasiferromagnetic mode ($r = 1$):

$$\begin{aligned} \dot{\mu}_y &= -H_0\mu_z + m_0 h_z - a\lambda_z\lambda_x, \\ \dot{\mu}_z &= H_0\mu_y - m_0 h_y, \\ \dot{\lambda}_x &= Bl_0\mu_z - l_0 h_z + B\lambda_y\mu_z - (B-a)\lambda_z\mu_y; \end{aligned} \tag{9}$$

b) quasiantiferromagnetic mode ($r = 2$):

$$\begin{aligned} \dot{\mu}_x &= al_0\lambda_z + a\lambda_y\lambda_z, \\ \dot{\lambda}_y &= -am_0\lambda_z + (B-a)\mu_x\lambda_z - B\mu_z\lambda_x, \\ \dot{\lambda}_z &= -Bl_0\mu_x + l_0 h_x + B\mu_y\lambda_x - B\mu_x\lambda_y. \end{aligned} \tag{10}$$

The nonlinear terms $a\lambda_z\lambda_x$, $B\lambda_y\mu_z$, $(B - a)\lambda_z\mu_y$, $B\mu_z\lambda_x$, $B\mu_y\lambda_x$ provide a nonlinear coupling between the modes.

"Parallel pumping" $h = (\tilde{h}_x \cos\omega_p t, 0, 0)$ with $\omega_p < \omega_{20}$ excites the second mode (variables μ_x, λ_y, λ_z) in the wings of the resonance line (see figure), i.e., generally speaking, the weaker, the greater ω_{20}, i.e., the greater constants a and B. These very same variables, by virtue of the nonlinear terms in (9) act parametrically on the oscillations of the first mode, and the stronger, the greater constants a and B. Hence when the condition $\omega_p \approx 2\omega_{10} = 2H_0$ is fulfilled excitation of parametric resonance of the first mode is possible. A similar effect with $\omega_p = 2\omega_{1k}$ ("indirect parallel pumping") has indeed been observed, evidently, in $CsMnF_3$.[5,7] An analogous phenomenon (nonresonant magnetic excitation of phonons in an AF) was discussed in[9].

But if, on the other hand, in the linear approximation oscillations of the first mode (μ_y, μ_z, λ_x) are strongly excited, e.g., by a field $\tilde{h}_y \cos\omega_{10}t$, the nonlinear terms $-B\mu_z\lambda_x$ and $B\mu_y\lambda_x$ in (10) will contain harmonics with frequency $2\omega_{10}$, which will excite oscillations of the second mode (μ_x, λ_y, λ_z). Their intensity will be particularly great if the choice of external field H_0 (and in the real cases of $CsMnF_3$, $MnCO_3$, α-Fe_2O_3, the choice also of the temperature or deformation directions) satisfies the equality $2\omega_{10} = \omega_{20}$. And, since the magnetic moment of the sample $\int m_x dV$ also oscillates with the frequency ω_{20}, excitation of electromagnetic oscillations in the external electromagnetic circuit is possible, i.e., there will be frequency doubling. In contrast to the known phenomenon of frequency doubling with a ferrite,[10] in this case the conversion efficiency can be extremely high, since at the frequency $\omega_{20} = 2\omega_{10}$ the oscillations are excited, as also at the frequency ω_{10}, in resonant fashion (hence the name "biresonant frequency doubling").

In AFEP with the aid of a dc magnetic field H_0 of a certain direction and magnitude, it is possible to markedly decrease one of the AFMR frequencies (e.g., close to the flipping field of the sublattices in AF of the MnF_2 type or close to the field for AF to AWF transition in an AFEA with DI[11,12]). In these cases, when $H_0 > H_0^{cr}$ the frequency ω_{20} of the quasiantiferromagnetic mode (in which the variables $\mu_{\parallel}$, $\lambda_{\perp}$ oscillate and which is excited by an HF field $h_{\parallel}$) lies, as a rule, below the frequency ω_{10} of the quasiferromagnetic mode (in which $h_{\perp}$ excites $\mu_{\perp}$, $\lambda_{\parallel}$). Indirect parallel pumping, nonetheless, is theoretically possible, since the condition necessary for this $\omega_p > 2\omega_{10}$ can be fulfilled for certain values of ω_p and H_0. The threshold amplitude can be calculated on the basis of scheme presented below for AFEP.

4. INDIRECT PARALLEL PUMPING IN AFEP WITH DI

For the calculation of the threshold field for "indirect parallel pumping" in AFEP we return to Eq. (3). Consider the effect of a homogeneous ac field $\mathbf{h} = (h\cos\omega_p t, 0, 0)$. We seek a solution in the form $\mathbf{m} = \mathbf{m}_0 + \boldsymbol{\mu}(t)$, $\mathbf{l} = \mathbf{l}_0 + \boldsymbol{\lambda}(t)$. Only the quasiantiferromagnetic mode with $\mathbf{k} = 0$ is excited:

$$\mu_{x0}(t) = -(l_0/m_0)\dot{\lambda}_{y0}(t) = (\chi_1\cos\omega_p t + \chi_2\sin\omega_p t)\,l_0 h, \tag{11}$$

$$\lambda_{z0}(t) = (-\xi_1\sin\omega_p t + \xi_2\cos\omega_p t)\,l_0 h;$$

here

$$\begin{aligned} \chi_1 &\approx \omega_{20}^2(\omega_{20}^2 - \omega_p^2)/Bl_0\Delta_{20}, \\ \chi_2 &\approx \omega_p(\eta\omega_{20}^4 + \zeta\omega_p^2\omega_E^2)/Bl_0\omega_E\Delta_{20}, \\ \xi_1 &\approx \omega_p(\omega_{20}^2 - \omega_p^2)/\Delta_{20}, \quad \xi_2 \approx 2\omega_p^2\delta_{20}/\Delta_{20}, \\ \Delta_{20} &= (\omega_{20}^2 - \omega_p^2)^2 + 4\omega_p^2\delta_{20}^2. \end{aligned} \tag{12}$$

Substituting (11) into the equation for the quasiferromagnetic mode (μ_{yk}, μ_{zk}, $\lambda_{xk} \sim e^{i\mathbf{k}\cdot\mathbf{r}}$) with no account taken of attenuation, we obtain

$$\begin{aligned} \dot{\mu}_{yk} &= \beta\lambda_{z0}(t)\mu_{yk} - [H_0 + \beta l_0 + \beta\lambda_{y0}(t) + h\cos\omega_p t]\mu_{zk} - (a - \alpha k^2)\lambda_{z0}(t)\lambda_{xk}, \\ \dot{\mu}_{zk} &= (H_0 + h\cos\omega_p t)\mu_{yk} - \alpha k^2[l_0 + \lambda_{y0}(t)]\lambda_{xk}, \\ \dot{\lambda}_{xk} &= +B\lambda_{z0}(t)\mu_{yk} + [Bl_0 + B\lambda_{y0}(t) + \beta\mu_{x0}(t)]\mu_{zk} - \beta\lambda_{z0}(t)\lambda_{xk}. \end{aligned} \tag{13}$$

This is a system of linear equations with coefficients that have a periodic time dependence, which, as is well known, can have for $\omega_p \approx 2\omega_{1k}$ a solution with an exponentially increasing amplitude (parametric resonance).

Following the general scheme for calculating the threshold field in parametric phenomena,[13] we seek a solution of the system (13) in the form

$$\begin{aligned} (\mu_{yk}, \mu_{zk}, \lambda_{xk}) &= (c_1, c_5, c_3)[\cos(\omega_{1k}t - \mathbf{kr}) + \cos(\omega_{1k}t + \mathbf{kr})] \\ &+ (c_4, c_2, c_6)[\sin(\omega_{1k}t - \mathbf{kr}) + \sin(\omega_{1k}t + \mathbf{kr})], \end{aligned} \tag{14}$$

where $c_n(t)$ is a slowly varying function of time ($c_n \sim e^{st}$, $|s| \ll \omega_{1k}$, $n = 1, \ldots, 6$), and $\omega_p = \omega_{1k} + \omega_{1,-k} = 2\omega_{1k}$ (ω_{1k} given by Eq. (7a)). Substituting (14) into (13), we express the products of harmonics in terms of harmonics of the sums and differences of the arguments, neglect harmonics of triple frequency, collect terms with the same harmonics, and obtain a system of homogeneous equations for $\bar{c} \equiv \{c_n\}$:

$$\dot{\bar{c}} = (\hat{A} + {}^1/_2 h\hat{B})\bar{c}. \tag{15}$$

The matrices $\hat{A}$ and $\hat{B}$ have the form

$$\hat{A} \equiv \left\| \begin{matrix} 0 & 0 & 0 & -\omega_{1k} & -(H_0 + \beta l_0) & 0 \\ 0 & 0 & 0 & H_0 & \omega_{1k} & -\alpha l_0 k^2 \\ 0 & 0 & 0 & 0 & Bl_0 & -\omega_{1k} \\ \omega_{1k} & -(H_0 + \beta l_0) & 0 & 0 & 0 & 0 \\ H_0 & -\omega_{1k} & -\alpha l_0 k^2 & 0 & 0 & 0 \\ 0 & Bl_0 & \omega_{1k} & 0 & 0 & 0 \end{matrix} \right\|, \tag{16}$$

$$\hat{B} \equiv \left\| \begin{matrix} \beta l_0\xi_2 & \beta m_0\chi_2 & -(a - \alpha k^2)l_0\xi_2 & -\beta l_0\xi_1 & -1 + \beta m_0\chi_1 & (a - \alpha k^2)l_0\xi_1 \\ 0 & 0 & \alpha k^2 m_0\chi_2 & -1 & 0 & -\alpha k^2 m_0\chi_1 \\ -Bl_0\xi_2 & -H_0\chi_2 & -\beta l_0\xi_2 & Bl_0\xi_1 & -H_0\chi_1 & \beta l_0\xi_1 \\ -\beta l_0\xi_1 & 1 - \beta m_0\chi_1 & (a - \alpha k^2)l_0\xi_1 & -\beta l_0\xi_2 & \beta m_0\chi_2 & (a - \alpha k^2)l_0\xi_2 \\ 1 & 0 & \alpha k^2 m_0\chi_1 & 0 & 0 & \alpha k^2 m_0\chi_2 \\ Bl_0\xi_1 & H_0\chi_1 & \beta l_0\xi_1 & Bl_0\xi_2 & -H_0\chi_2 & \beta l_0\xi_2 \end{matrix} \right\| \tag{17}$$

For $h = 0$ the characteristic equation of the system (15) is

$$\Delta_6 = s^6 + 5\omega_{1k}^2 s^4 + 4\omega_{1k}^4 s^2 = 0.$$

Its roots are $s_{1,2} = 0$, $s_{3,4} = \pm i\omega_{1k}$, $s_{5,6} = \pm 2i\omega_{1k}$. For small $h \neq 0$ these roots change slightly, but in view of our method of calculation, we are interested only in values of s with small absolute value $|s| \ll \omega_{1k}$. Hence, in writing down the characteristic equation of system (15) with small $h \neq 0$, it suffices to keep only terms proportional to s^2, s^1, and s^0 and to calculate their coefficients in the first nonzero approximation with respect to $h/2$. The result is

$$\Delta_6 \approx Q_0 s^2 - R_2 (h/2)^2 = 0, \tag{18}$$

where $Q_0 = 4\omega_{1k}^4$, and

$$R_2 = [\beta l_0 \omega_{1k} + H_0 \omega_{20}^2 \xi_1 + \beta B^{-1}(2\omega_{10}^2 - \omega_{1k}^2)\omega_{1k}\chi_1]^2 + [H_0 \omega_{20}^2 \xi_2 + \beta B^{-1}(2\omega_{10}^2 - \omega_{1k}^2)\omega_{1k}\chi_2]^2. \tag{18a}$$

From (18) we have for the growth increment $s_+ \approx (R_2/Q_0)^{1/2} h/2$. When attenuation of the oscillations of the quasiferromagnetic mode are taken into account there should appear in the expressions for $c_n(t)$ a factor $\exp(-\delta_{1k} t)$, where δ_{1k} is the relaxation frequency of the spin waves ω_{1k} and $\omega_{1,-k}$ (see Eq. (7b)). An exponential growth of amplitude (parametric instability) becomes possible only for $s_+ - \delta_{1k} > 0$. From this we obtain for the threshold amplitude

$$h_{\text{thr}} = 2\delta_{1k}(Q_0/R_2)^{1/2}. \tag{19}$$

Using Eq. (12) for $\chi_{1,2}$ and $\xi_{1,2}$, we can show that under the conditions of our calculation the contribution of χ_1 and χ_2 to R_2 is practically for all pump frequencies much smaller than that of ξ_1 and ξ_2. Thus, considering that $\delta_{1k} = \Delta\omega_{1k}/2$, $\delta_{20} = \Delta\omega_{20}/2$, where $\Delta\omega_{rk}$ are the total line widths of the corresponding oscillations, we find from (19) and (18a) a final expression for the threshold field for indirect parallel pumping at frequency ω_p (in the usual system of units):

$$h_x^{\text{thr}} = \frac{\Delta\omega_{1k_c}}{\gamma} 2\omega_{1k_c}^2 [(\gamma H_D \omega_{1k_c} + \gamma H_0 \omega_{20}^2 \xi_1)^2 + (\gamma H_0 \omega_{20}^2 \xi_2)^2]^{-1/2}, \tag{20}$$

where

$$\xi_1 = \frac{\omega_p(\omega_{20}^2 - \omega_p^2)}{(\omega_{20}^2 - \omega_p^2)^2 + \omega_p^2(\Delta\omega_{20})^2}, \quad \xi_2 = \frac{\omega_p^2 \Delta\omega_{20}}{(\omega_{20}^2 - \omega_p^2)^2 + \omega_p^2(\Delta\omega_{20})^2}$$

$$\omega_{1k}^2 = \gamma^2 H_0(H_0 + H_D) + (\Theta_c a_0 k/\hbar)^2,$$

$$\omega_{2k}^2 = \gamma^2[2H_A H_E + H_D(H_0 + H_D)] + (\Theta_c a_0 k/\hbar)^2,$$

$$\Theta_c = 2M_0(\alpha B)^{1/2}\hbar\gamma / a_0,$$

and the value of the modulus of the wave vector k_c is given by $\omega_p = 2\omega_{1k_c}$.

5. ANALYSIS OF SPECIAL CASES

A. Equation (20) can describe phenomena which previously had been considered separately, such as: "direct" parallel pumping of the quasiferromagnetic mode,[2] indirect parallel pumping of the Seavey-Prozorova-Borovik-Romanov type,[5,7] "three-magnon saturation" of the quasiantiferromagnetic mode.[2] In fact, we have, for pump frequencies $\omega_p \ll \omega_{20}$,

$$\xi_1 \approx \omega_p/\omega_{20}^2, \quad \xi_2 \approx 2\omega_p^2\delta_{20}/\omega_{20}^4 \ll \xi_1,$$

i.e.,

$$h_x^{\text{thr}}(\omega_p \ll \omega_{20}) = \omega_p \Delta\omega_{1k_c}/\gamma^2(2H_0 + H_D). \tag{21a}$$

For $H_D \gg H_0$, this leads to Eq. (27) of[2] ("direct" parallel pumping), and for $H_D = 0$ (the case of $CsMnF_3$) to an expression that differs from the one derived in[5,6] by a factor ½ (see below).

And if $\omega_p = \omega_{20}$, then $\xi_1 = 0$, $\xi_2 = (2\delta_{20})^{-1}$ and

$$h_x^{\text{thr}}(\omega_p = \omega_{20}) = \Delta\omega_{1k_c}\Delta\omega_{20}/2\gamma^2 H_0, \tag{21b}$$

i.e., we obtain an expression that is the same as the one obtained in[2] for the threshold field of "three-magnon premature saturation" of the quasiantiferromagnetic mode.

Note that the frequency ω_{20} does not appear in Eqs. (21a) and (21b).

B. It is useful to have a general expression for the threshold field of indirect parallel pumping ($\mathbf{h} \parallel \mathbf{H}_0 \parallel x$) for $H_D = 0$ and for given values of H_0 and pump frequency ω_p:

$$h_x^{\text{thr}} = \frac{\Delta\omega_{1k_c}}{\gamma} \frac{\omega_p}{2\gamma H_0} \frac{[(\omega_{20}^2 - \omega_p^2)^2 + \omega_p^2(\Delta\omega_{20})^2]^{1/2}}{\omega_{20}^2}. \tag{22}$$

It is seen that the function $h_x^{\text{thr}}(\omega_p)$ has a sharp resonance. Equation (22) can be applied to the case of parametric excitation of electronic spin waves in $CsMnF_3$, if the assumption is made that the hyperfine interaction in this substance affects h_x^{thr} only via its effect on the AFMR frequencies ω_{10} and ω_{20}.

C. The difference between (21a) for $H_D = 0$ and the analogous formula in[5,6] is evidently not purely arithmetical. In fact, the expression for h_x^{thr} in $CsMnF_3$ is derived by the method of Hinderks and Richards under the assumption of direct coupling between the magnetic field of the pump and the spin waves of the quasiferromagnetic mode. However, as follows from[2], in the absence of DI there is no such coupling in AFEP. In the notation of[8] this can be shown in the following way.

For the description of three-magnon processes of direct conversion of one photon into two spin waves it is necessary to consider this part of the total Hamiltonian:

$$\Delta\mathcal{H} = \gamma\hbar(H_E\cos 2\theta + H\sin\theta)\sum_k (a_k^+ a_k + b_k^+ b_k),$$

where $H = H_0 + h_\parallel \cos\omega_p t$. In AF of the $CsMnF_3$ type, where ω_{20} is quite large (L. A. Prozorova, private communication), when $\omega_p \ll \omega_{20}$ the equilibrium directions of the sublattice moments are obviously able to follow the changes in the external field; hence, $\sin\theta = (H_0 + h_\parallel \cos\omega_p t)/2H_E$. This leads to

$$\Delta\mathcal{H} = \gamma\hbar H_E \sum_k (a_k^+ a_k + b_k^+ b_k),$$

i.e., in this way the parts of $\Delta\mathcal{H}$ that depend explicitly on time compensate each other. The quantum-mechanical calculation of the threshold amplitude of parallel pumping must obviously take into account terms like $d_0 c_k^+ c_{-k}^+$ and the time dependence $d_0(t)$ induced by the external ac field.

D. In the case of AFEA in fields parallel to the easy axis (EA) and greater than the flipping field H_c (e.g., $RbMnF_3$, cf.[8]), the AFMR frequency of the quasiantiferromagnetic mode ω_{20} may be close to zero; hence the question of the validity of the model of "direct" parallel pumping with parametric excitation of two electronic spin waves in the framework of a simplified quantum-mechanical scheme remains open. More complex methods are obviously required-see, for example,[14]. It is more probable, however, that in uniaxial AFEA with "flipped" sublattices parallel pumping is generally impossible without taking into account the field of the spin waves or the magnetoelastic interaction.[15]

Classically, AFEA with $H_0 \parallel$ EA and $H_0 > H_c$ can be described by the addition of the term $(-\frac{1}{2}Al_x^2)$ to (5), with $A > 0$. Then the x axis will be the easy axis ($H_c^2 \approx ABl^2$), and the term $\frac{1}{2}al_z^2$ describes the anisotropy in the plane perpendicular to the easy axis. The spin-wave spectrum takes on the following form (for $H_D = 0$ and $k \ll a_0^{-1}$):

$$\omega_{nk}^2 = \omega_{n0}^2 + \gamma^2 aBl_0^2k^2 \quad (n = 1, 2),$$
$$\omega_{10}^2 = \gamma^2(H_0^2 - H_c^2), \quad \omega_{20}^2 = \gamma^2 aBl_0^2.$$

Parallel pumping is possible if $\omega_p > 2\omega_{10}$, and the threshold amplitude, as calculation shows, is described by Eq. (22). Thus, spin-wave instability of the type considered in the presence of an easy axis is possible for $H_0 \parallel$ EA and $H_0 > H_c$ only in biaxial AF (since the condition $\omega_{20} \neq 0$ is required). In particular, when $\omega_p \gg \omega_{20}$ we have

$$h_x^{\text{thr}}\,(\omega_p \gg \omega_{20}) \approx \frac{\Delta\omega_{1k_c}}{\gamma}\,\frac{\omega_p^3}{2\gamma H_0\omega_{20}^2},$$

i.e., the magnitude is relatively high.

From the experimental point of view, the biaxial AF, $CuCl_2 \cdot 2H_2O$, is of interest. Here, when $H_0 \parallel$ EA and $H_0 > H_c$, ω_{20}/γ is evidently about 10 kOe.[16] This permits working with $\omega_p \sim \omega_{20}$, which, in accordance with (21b), gives a very low threshold for the excitation of instability in parallel pumping.

6. BIRESONANT FREQUENCY DOUBLING

In AFEP, by changing the magnitude of the external magnetic field H_0 perpendicular to the hard axis, it is easy to reach the situation where $2\omega_{10} = \omega_{20}$ (see Eqs. (7a) and (8a) for $k = 0$). In this case a linear (by fields h_y and h_z) excitation of oscillations in the quantities μ_y, μ_z, λ_x at the frequency ω_{10} leads, on account of nonlinear terms in the equations of motion, to a resonant excitation of oscillations in the quantities μ_x, λ_y, λ_z at frequency ω_{20}. Let us calculate the magnitude of the effect, i.e., the dependence of the amplitudes of the oscillations of the quasiantiferromagnetic mode on the amplitudes $\tilde{h}_y$ and $\tilde{h}_z$.

In the general case resonant excitation of the quasiferromagnetic mode can be accomplished by a uniform ac field $h = (0, h_y, h_z)$, where $h_y = \tilde{h}_y \cos(\omega_{10}t + \varphi)$, $h_z = \tilde{h}_z \cos\omega_{10}t$. Then the solution of the initial system of equations (3) in the first approximation with respect to δ_{10}/ω_{10} gives

$$\mu_y \approx -\frac{m_0}{l_0}\lambda_x \approx \frac{m_0\tilde{h}_z}{2\delta_{10}}\cos\omega_{10}t + \frac{m_0\tilde{h}_y}{2\delta_{10}\varepsilon}\sin(\omega_{10}t + \varphi),$$
$$\mu_z \approx \frac{\varepsilon m_0\tilde{h}_z}{2\delta_{10}}\sin\omega_{10}t - \frac{m_0\tilde{h}_y}{2\delta_{10}}\cos(\omega_{10}t + \varphi), \quad (23)$$

where $\epsilon \equiv [H_0/(H_0 + H_D)]^{1/2}$, and ω_{10}, δ_{10} are given by Eqs. (7a) and (7b) for $k = 0$.

We now substitute (23) into the troika of equations for the quasiantiferromagnetic mode (μ_x, λ_y, $\lambda_z \sim e^{-i\omega t}$)—into those nonlinear terms which contain the variables of the quasiferromagnetic mode. We express the products of harmonics as harmonics of the zero and doubled frequencies and keep only the latter. Considering the variable μ_x at $\omega = \omega_{20} = 2\omega_{10}$ and using the conditions $\delta_{10} \ll a$, β and $H_0 \ll B$, we obtain

$$\mu_x \approx \frac{H_0\omega_{20}}{16B\delta_{10}^2\delta_{20}}\Big[-\tilde{h}_z^2\sin\omega_{20}t + \frac{1}{\varepsilon^2}\tilde{h}_y^2\sin(\omega_{20}t + 2\varphi) + \frac{2}{\varepsilon}\tilde{h}_y\tilde{h}_z\cos(\omega_{20}t + \varphi)\Big]. \quad (24)$$

If we introduce $\tilde{h}_z/\tilde{h}_y \equiv \tan\theta$ and $h \equiv (\tilde{h}_y^2 + \tilde{h}_z^2)^{1/2}$, then the maximum of the average value of μ_x^2 with h = const is attained when $\varphi = \pm\pi/2$ and $\theta = \pm\arctan\epsilon$, i.e., when

$$h_y = \mp\frac{h}{\sqrt{1+\varepsilon^2}}\sin\omega_{10}t, \quad h_z = \pm\frac{\varepsilon h}{\sqrt{1+\varepsilon^2}}\cos\omega_{10}t.$$

This is natural, since oscillations of the quasiferromagnetic mode are best excited by an electromagnetic field with elliptical polarization of a certain sign.

In order to compare the magnitude of this effect with that of the well-known frequency-doubling effect in ferrites,[10] we set for brevity $H_D = 0$ and $h_z = 0$. Then we get for μ_x (going over to the usual system of units)

$$\mu_x^{AF} = \frac{H_0\omega_{20}\tilde{h}_y^2}{16B\delta_{10}^2\delta_{20}} \to \frac{2M_0H_C\omega_{20}}{4H_E\Delta\omega_{20}}\left(\frac{\gamma\tilde{h}_y}{\Delta\omega_{10}}\right)^2 = 2M_0\frac{\gamma H_A}{4\Delta\omega_{20}}\left(\frac{\gamma\tilde{h}_y}{\Delta\omega_{10}}\right)^2, \quad (25)$$

where we have used $2\gamma H_0 = 2\omega_{10} = \omega_{20} = \gamma(2H_AH_E)^{1/2} \equiv \gamma H_{AE}$.

For a thin ferrite plate magnetized in the plane by a field $H_0 \parallel z$, it is known[10] that

$$\mu_z^F = \frac{\omega_M^2}{8\delta_F^2}\left(1 + \frac{\omega_M}{\omega_0}\right)^{1/2}\frac{\gamma\tilde{h}_y^2}{\omega_p},$$

where $\omega_M \equiv 4\pi\gamma M_F$, $\omega_0 \equiv \gamma H_0$, M_F is the saturation magnetization of the ferrite, $\delta_F = \Delta\omega_F/2$, $\Delta\omega_F$ is the total FMR line width ($\Delta\omega_F \sim \gamma\Delta H_F$), $\tilde{h}_y$ is the amplitude of the HF field of frequency $\omega_p = [\omega_0(\omega_0 + \omega_M)]^{1/2}$ and polarization y, parallel to the plane of the plate.

Since we are interested in the doubling of high frequencies, we take $\omega_p = \omega_{20}/2 \gg \omega_M$; then the relative effectiveness of the processes (using the power radiated at frequency ω_{20} as criterion) is characterized by the ratio of the amplitudes squared:

$$\left(\frac{\mu_x^{AF}}{\mu_z^F}\right)^2 \approx \left[\frac{2M_0}{4\pi M_F}\,\frac{\gamma H_A}{\omega_M}\,\frac{\omega_{20}}{4\Delta\omega_{20}}\left(\frac{\Delta\omega_F}{\Delta\omega_{10}}\right)^2\right]^2.$$

If, for example, we take $\omega_{20}/\gamma \sim 5 \times 10^4$ Oe, $2M_0 \sim 2\pi M_F \sim 10^3$ G, $H_A \sim 10^4$ Oe, $\Delta\omega_{20}/\gamma \sim 3 \times 10^2$ Oe, $\Delta\omega_F \sim \Delta\omega_{10}$, then it is easy to see that at high frequencies (microwaves, 1 to 4 mm) the use of AFEP for the generation of harmonics can be many times more effective that the use of a ferrite even with an extremely narrow resonance line.

It is interesting to note that the quadratic dependence between the pump power $P_p \sim h^2$ at frequency ω_{10} and the power emitted at frequency $2\omega_{10}$, $P_e \sim \mu_x^2 \sim h^4$, can change radically in character beginning with some value of P_p (a very high one, of course)—if such a value of the amplitude of μ_x should be attained at which the "reverse" process (discussed in Sec. 4) becomes possible: "three-magnon" parametric excitation of oscillations in the quasiferromagnetic mode. But is is obviously more probable that the quadratic dependence of P_e on P_p will change even earlier (at smaller values of P_p) because of another threshold effect—four-magnon premature saturation of the low-frequency AFMR.

I am deeply grateful to I. K. Kikoin for his steady interest in this work and A. S. Borovik-Romanov and L. A. Prozorova for numerous fruitful discussions.

[1] A. S. Borovik-Romanov, Zh. Eksp. Teor. Fiz. 36, 766 (1959) [Sov. Phys.-JETP 9, 539 (1959)].

[2] V. I. Ozhogin, Zh. Eksp. Teor. Fiz. 48, 1307 (1965) [Sov. Phys.-JETP 21, 874 (1965)].

[3] V. G. Bar'yakhtar and O. V. Kovalev, Tezisy dokladov IX Vsesoyuznogo soveshchaniya po fizike nizkikh temperatur (Abstracts of Reports to the IX-th All-Union Conference on Low-Temperature Physics), Leningrad, 1962.

[4] V. I. Ozhogin, Zh. Eksp. Teor. Fiz. 46, 531 (1964) [Soviet Phys.-JETP 19, 362 (1964)].

[5] M. H. Seavey, J. Appl. Phys. 40, 1597 (1969).

[6] M. H. Seavey, Phys. Rev. Lett. 23, 132 (1969).

[7] L. A. Prozorova and A. S. Borovik-Romanov, ZhETF Pis. Red. 10, 316 (1969) [JETP Lett. 10, 201 (1969)].

[8] L. W. Hinderks and P. M. Richards, Phys. Rev. 183, 575 (1969).

[9] R. M. White, M. Sparks, and I. Ortenburger, Phys. Rev. 139, A450 (1965).

[10] B. Lax and K. Button, High-Frequency Ferrites and Ferromagnets, (Russ. Transl., Mir, 1965).

[11] V. I. Ozhogin and V. G. Shapiro, ZhETF Pis. Red. 6, 467 (1967) [JETP Lett. 6, 7 (1967)].

[12] G. Cinader, Phys. Rev. 155, 453 (1967).

[13] L. D. Landau and E. M. Lifshitz, Mekhanika (Mechanics), Fizmatgiz, 1958. (Engl. Transl., Addison-Wesley, 1960).

[14] L. D. Filatova and V. M. Tsukernik, Zh. Eksp. Teor. Fiz. 57, 498 (1969) [Sov. Phys.-JETP 30, 273 (1970)].

[15] A. Platzker and F. R. Morgenthaler, 15th Annual Conference on Magnetism and Magnetic Materials, Abstracts, 2D-3, 1969.

[16] M. Date and K. Nagata, J. Appl. Phys. 34, 1039 (1963).

Translated by L. M. Matarrese
254