EXACT SOLUTIONS OF THE EQUATIONS OF NONLINEAR GEOMETRIC OPTICS

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The nonlinear dynamics of a medium containing high-intensity plane beams of electromagnetic waves is investigated within the framework of geometric optics. A broad class of exact analytic solutions of the equations of nonlinear geometric optics is indicated. It is shown that, as the solution develops, singularities appear in the intensity and angular distributions of rays in the beam, and the situation is similar to the development of a simple wave in the hydrodynamics of an ideal compressible fluid. A classification of the singularities is given and three basic types are indicated.

1. INTRODUCTION

THE field of a high-intensity electromagnetic wave may give rise to very considerable disturbances in a medium. In particular, it may change the permittivity. This, in turn, leads to a change in the propagation conditions for the waves producing the disturbance, so that the wave propagation process becomes nonlinear. The corresponding nonlinear phenomena are described by the Maxwell equation and the constitutive equations for the medium. In general, these constitute a very complicated system and have been investigated by a number of authors.[1-6]

We shall restrict our attention to effects arising during the propagation of a weakly inhomogeneous wave beam, for which the characteristic distance R over which there is a substantial change in the amplitude of the field E is much greater than the wavelength, i.e.,

$$kR \gg 1$$

where k is the wave vector. This ensures that we can use the geometric optics approximation. The wave field can then be represented by a beam of rays having a definite direction of k and definite intensity I at each point. The most interesting case is that of narrow beams for which it is possible to isolate a mean direction of beam propagation (z axis). The deviation of the different rays from this mean direction is small and is characterized only by the vectors $\mathbf{k}_\perp$ lying in the plane orthogonal to the z axis. If, moreover, we confine ourselves to plane beams, i.e., beams whose intensity in the y direction is constant, then the equation of nonlinear geometric optics can be written in the form

$$\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial x}+w\frac{\partial u}{\partial x}=0$$
$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}-\frac{\partial w}{\partial x}=0;$$
$$t=z\beta^{1/2},\quad \beta=\varepsilon_2 I_0/2\varepsilon_0. \qquad (1)$$

In these equations $w(x, t) = I/I_0$ is the dimensionless intensity, $u(x, t) = \beta^{-1/2}k_x/k$ is a function which is a measure of the extent to which the direction of a given ray deviates from the mean direction of propagation of the beam. The variables x and t will also be considered as dimensionless, i.e., $x \to x/a$, $t \to t/a$, where a is a characteristic transverse size of the beam, which is determined by the boundary conditions.

The derivation of Eq. (1) was based on the assumption that the permittivity was

$$\varepsilon=\varepsilon_0+\varepsilon_2 I,$$

where ϵ_0 is the permittivity of the undisturbed medium and $\epsilon_2 I$ is the perturbation proportional to the wave intensity. The second term takes into account the effect of the wave field on the medium. The above expression for ϵ is valid for isotropic nonabsorbing media, provided the wave intensity is not too high, so that $|\epsilon_2 I| \ll \epsilon_0$, i.e., $\beta \ll 1$.[1)] The form of the medium and the nature of the interaction with the waves affect only the coefficients ϵ_0 and ϵ_2. These two coefficients, in fact, disappear from the equations after transformation to the dimensionless functions u and w (only the signs of ϵ_0 and ϵ_2 are important; we shall consider transparent ($\epsilon_0 > 0$) and focusing ($\epsilon_2 > 0$) media). We note also that if we omit the last term $\partial w/\partial x$ from Eq. (1), which represents the effect of the waves on the medium, we obtain the usual equations of linear geometric optics (for a narrow plane beam propagating in a homogeneous medium). A detailed derivation of Eq. (1) is given in reviews[3-5].

The boundary conditions for Eq. (1) are specified on the t = 0 plane:

$$w(x,0)=w_0(x),\quad u(x,0)=u_0(x). \qquad (2)$$

In other words, the problem is formulated as follows: suppose that the intensity and angular distributions of the rays, $w_0(x)$ and $u_0(x)$, are specified for the beam propagating along the axis on the given plane t = 0 (z = 0). It is required to find w(x) and u(x) for any z.

The equations given by Eq. (1) are analogous to the hydrodynamic equations for an ideal compressible fluid with $\gamma = 2$. There is, however, the essential difference between them which is connected with the change in the sign of the "pressure". In hydrodynamics pressure always expands the flow, whereas in the case of the focusing medium considered here, nonlinear pressure tends to contract the beam. The change in the sign of the pressure makes Eq. (1) into an elliptic system in contrast to the hydrodynamic equations which are, of

1) We have confined our attention to time-independent or, more precisely, frequency-averaged perturbations of the permittivity.

course, hyperbolic. In spite of this fact, the overall similarity between the equations enables us to use hydrodynamic methods for some aspects of the solution of problems in geometric optics.

The analytic solution of Eq. (1), corresponding to a number of special conditions, was found by Talanov[7], whereas another special solution has been given by Lighthill[10] and Akhmanov, Sukhorukov, and Khokhlov.[3] In the present paper we shall investigate a broad class of analytic solutions which will enable us to investigate some general properties of nonlinear beam dynamics.

2. SELF-SIMILAR SOLUTIONS

The set of equations given by Eq. (1) can be reduced to a linear form by a suitable transformation of the variables. We shall regard w and u as the variables and x and t as the unknown functions. To transform Eq. (1) to the new variables we shall follow Landau and Lifshitz (see[8], Sec. 98) and write the partial derivatives in Eq. (1) in the Jacobian form

$$\frac{\partial(w, x)}{\partial(t, x)} + u\frac{\partial(t, w)}{\partial(t, x)} + w\frac{\partial(t, u)}{\partial(t, x)} = 0,$$
$$\frac{\partial(u, x)}{\partial(t, x)} + u\frac{\partial(t, u)}{\partial(t, x)} - \frac{\partial(t, w)}{\partial(t, x)} = 0. \qquad (3)$$

If we now multiply Eq. (3) by $\partial(t, x)/\partial(w, u)$ and evaluate the Jacobians we obtain

$$\frac{\partial x}{\partial u} - u\frac{\partial t}{\partial u} + w\frac{\partial t}{\partial w} = 0, \qquad (4)$$

$$\frac{\partial t}{\partial u} + u\frac{\partial t}{\partial w} - \frac{\partial x}{\partial w} = 0. \qquad (5)$$

These equations for the function x(u, w) and t(u, w) are now linear, and their use simplifies the derivation of the class of solutions in which we are interested.

In fact, it is readily verified that Eqs. (4) and (5) are homogeneous in w and u^2. Their special solutions can therefore be sought in the form of polynomials which are homogeneous in powers of u^2 and w. We shall assume, to begin with, that x is an even function of u:

$$x_n(u, w) = \sum_{k=0}^{n} a_{nk}u^{2k}w^{n-k}, \quad t_n(u, w) = \frac{1}{u}\sum_{k=1}^{n} b_{nk}u^{2k}w^{n-k}. \qquad (6)$$

Substituting these expressions in Eqs. (4) and (5), and equating terms with equal powers of $u^{2k}w^{n-k}$, we obtain recurrence relations between the coefficients a_{nk} and b_{nk}. Substituting $a_{n0} = 1$ and $b_{n0} = 0$, we successively find the functions x_n and t_n, which are special solutions of Eqs. (4) and (5):

$$x_1 = w + \frac{u^2}{2}, \quad t_1 = u; \qquad (7)$$

$$x_2 = w^2 - \frac{u^4}{2}, \quad t_2 = 2uw - \frac{2}{3}u^3; \qquad (8)$$

$$x_3 = w^3 - \frac{3}{2}w^2u^2 - \frac{3}{2}wu^4 + \frac{u^6}{4}, \quad t_3 = 3u\left(w^2 - wu^2 + \frac{u^4}{10}\right); \qquad (9)$$

. . . .

The boundary conditions of Eq. (2) are specified on the the t = 0 plane. The condition t = 0 determines the relation between w and u for each solution. In particular, it is clear from Eqs. (6)—(9) that the condition t = 0 is identically satisfied for any pair of x_n and t_n in the case of a plane parallel beam (u = 0). In this case, each of the solutions describes the change in the beam intensity near its boundary. In fact, consider the edge of a plane-parallel beam, i.e., assume that when t = 0

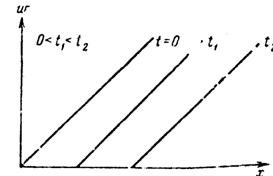

FIG. 1

$$w_0(x) = \begin{cases} x, & x > 0 \\ 0, & x < 0 \end{cases}, \quad u_0(x) = 0. \qquad (10)$$

It is readily seen that the solution x_1, t_1 satisfies the boundary conditions given by Eq. (10). Consequently, it is also valid for any t. From Eq. (7) we have

$$w = \begin{cases} x - t^2/2, & x > t^2/2 \\ 0 & x < t^2/2 \end{cases}; \quad u = \begin{cases} t, & x > t^2/2 \\ t - \sqrt{t^2 - 2x}, & x < t^2/2,\ x > 0 \\ 0, & x < 0 \end{cases} \qquad (11)$$

We have taken account here of the fact that the trivial solution w = 0, u = u_0(x − ut) will also satisfy Eq. (1), and that Eq. (1) admits of weak discontinuities. The change in the boundary of the beam is shown in Fig. 1 for different times t in accordance with Eq. (11). The solution given by Eq. (11) is self-similar and can be written in the form

$$w = t^2 f_1(\tau), \quad u = t f_2(\tau); \quad \tau = x/t^2.$$

It is readily verified that, for an arbitrary n, the solution x_n, t_n given by Eq. (6) is also self-similar and can be written in the form

$$w = t^{2/(2n-1)}f_{1n}(\tau), \ u = t^{1/(2n-1)}f_{2n}(\tau); \quad \tau = \frac{x}{t^{2n/(2n-1)}}. \qquad (12)$$

We shall consider the case where x is an even function of u. The situation where x is an odd function of w can be considered in a similar way by writing

$$x_n = u\sum_{k=0}^{n} a_{nk}u^{2k}w^{n-k}, \quad t_n = \sum_{k=0}^{n} b_{nk}u^{2k}w^{n-k} \qquad (13)$$

and substituting these expressions into Eqs. (4) and (5) to find the coefficients a_{nk} and b_{nk} for each n. The final result is

$$x_1 = uw + \frac{2}{3}u^3, \quad t_1 = -w + u^2;$$
$$x_2 = uw^2 + \frac{1}{2}u^3w - \frac{u^5}{5}, \quad t_2 = -\frac{w^2}{2} + \frac{3}{2}u^2w - \frac{u^4}{4}; \qquad (14)$$

. . . .

These solutions also represent self-similar functions w(x, t) and u(x, y):

$$w(x, t) = t^{1/n}f_1(\tau), \quad u(x, t) = t^{1/2n}f_2(\tau); \quad \tau = x/t^{(2n+1)/2n}.$$

3. APPEARANCE OF SINGULAR POINTS

We have derived a set of special self-similar solutions of Eq. (1). It is important to note that Eqs. (4) and (5) for the functions x(u, w) and t(u, w) are linear. This means that not only the functions x_n, t_n but also their arbitrary combinations

$$x = \sum_{n=1}^{\infty} C_n x_n(u, w), \quad t = \sum_{n=1}^{\infty} C_n t_n(u, w) \qquad (15)$$

will satisfy Eqs. (4) and (5). The resulting class of solutions is characterized by an infinite set of constants $C_1, C_2, \ldots, C_n$ or an arbitrary function which can be expanded into a series.

In fact, consider a plane-parallel beam, i.e., assume that when t = 0

$$u(x, 0) = 0, \qquad w(x, 0) = w_0(x). \tag{16}$$

According to Eqs. (6)—(9), we have $x^n = w^n$ for u = 0. Therefore, if we expand the function $x_0(w)$, which is the inverse of $w_0(x)$, into the Taylor series

$$x_0(w) = x_0 + \sum_{n=1}^{\infty} C_n w^n, \quad C_n = \frac{1}{n!}\left(\frac{\partial^n x_0}{\partial w^n}\right)_{w=0}, \tag{17}$$

we can determine the constants C_n. In other words, the formulas of Eq. (15) with the coefficients C_n determined in accordance with Eq. (17), give the solution of Eq. (1) in an implicit form which satisfies the boundary conditions of Eq. (16). The above class of solutions of the equations of geometric optics is equivalent to simple waves in the hydrodynamics of an ideal compressible fluid.2) A combination of self-similar solutions, Eq. (15), is no longer a self-similar function. In this sense, we again have an analogy with simple waves in hydrodynamics. We shall show that, as in the case of a simple wave, the development of the solution given by Eq. (15) leads to a gradual sharpening of the front and the appearance of singular points.

As an example, consider the simplest case when

$$w(x, 0) = \begin{cases} \sqrt{\alpha^2/4 + x} - \alpha/2 & \text{for } x > 0 \\ 0 & \text{for } x < 0 \end{cases}, \quad u(x, 0) = 0. \tag{18}$$

In these expressions, α is an arbitrary constant ($\alpha > 0$). The function w(x; 0) is shown in Fig. 2. The inverse function $x_0(w) = \alpha w + w^2$ and, consequently, the solution of Eqs. (4) and (5) satisfying the boundary conditions of Eq. (18), is in this case a combination of only two self-similar solutions x_1, t_1 and x_2, t_2:

$$x = \alpha\left(w + \frac{u^2}{2}\right) + w^2 - \frac{u^4}{2}, \quad t = u\left[\alpha + 2\left(w - \frac{u^2}{3}\right)\right], \quad w \geqslant 0. \tag{19}$$

The dependence of w on x for different t, which is defined by Eq. (19), is shown in Fig. 2. It is clear that the beam boundary shifts in the direction of increasing intensity and w(x) becomes increasingly steeper. At some value $t = t_c$ the slope of the gradient $\partial w/\partial x$ becomes infinite and we have a singularity. The condition for the appearance of the singularity on the beam boundary is

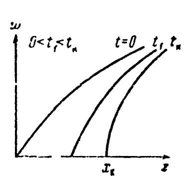

FIG. 2

$$\left(\frac{\partial x}{\partial w}\right)_t = 0, \quad w = 0. \tag{20}$$

Differentiating Eq. (19) with respect to w, we obtain

$$\left(\frac{\partial x}{\partial w}\right)_t = \alpha + \alpha u\left(\frac{\partial u}{\partial w}\right)_t + 2w - 2u^3\left(\frac{\partial u}{\partial w}\right)_t,$$

$$\alpha\left(\frac{\partial u}{\partial w}\right)_t + 2u + 2w\left(\frac{\partial u}{\partial w}\right)_t - 2u^2\left(\frac{\partial u}{\partial w}\right)_t = 0.$$

Using Eq. (20), we find that at the singular point w = 0, $u^2 = \alpha/2$. Consequently, the boundary singularity appears if $\alpha > 0$. From Eq. (19) we obtain the explicit coordinates of the singular points: $x_c = \alpha^2/8$, $t_c = \sqrt{2}\,\alpha^{3/2}/3$. It is readily verified that the derivative $(\partial u/\partial x)_t$ becomes infinite at the same point.

The above singularity is analogous to that found in the hydrodynamics of an ideal compressible fluid on the boundary of a simple wave and a stationary gas (see [8], p. 454). We note also that the above analysis of the boundary singularity ("quadratic" singularity, $w \sim x^{1/2}$, $u \sim x^{1/2}$) is quite general, since near the beam boundary, i.e., for $w \to 0$, we can confine our attention to the first terms in the expansion of $x_0(w)$ into a series of powers of w: $x_0(w) = aw + bw^2$. Consequently, $\alpha = a/b^{1/2}$, $x_c = a^2/8b$, $t_c = 2^{1/2}a^{3/2}/3b^{3/4}$.

The reason for the appearance of the singularity can be readily understood. Thus, in the case we are considering, $\alpha > 0$, and the slope $\partial w/\partial x$ is a maximum on the beam boundary, $w \to 0$ (see Fig. 2). At this point, the "pressure" of the nonlinear medium is therefore also a maximum. It shifts the rays and gradually "presses" the beam against its boundary.

There is, of course, another possible case when the initial slope $\partial w/\partial x$ is a maximum not on the boundary but at some internal point of the beam. Here, the "pressure" ensures that the slopes $\partial w/\partial x$ and $\partial u/\partial x$ will also increase and will become infinite at some time. Here the singularity is "cubic" ($w \sim x^{1/3}$, $u \sim x^{1/3}$) and appears inside the beam and not on its boundary. This is analogous to the usual spillover of a simple wave in hydrodynamics (see [8], Sec. 94). Let us illustrate this by an example.

Suppose that the intensity distribution in a plane-parallel beam on the initial plane t = 0 is defined by the cubic curve (for $w > 0$):

$$x_0(w) = \alpha w - \beta w^2 + w^3. \tag{21}$$

The condition for the absence of spillover on the initial curve ($\partial x_0/\partial w > 0$) is satisfied if $\alpha > \beta^2/3$. The inflection point is $w_p = \beta/3$ and at this point the slope $\partial w_0/\partial x$ of the initial function is a maximum.

The solution of Eqs. (4) and (5) which satisfies the boundary conditions given by Eq. (21) is given by Eqs. (7)—(9) and (15)

$$x = \alpha\left(w + \frac{u^2}{2}\right) - \beta\left(w^2 - \frac{u^4}{2}\right) + \left(w^3 - \frac{3}{2}w^2u^2 - \frac{3}{2}wu^4 + \frac{u^6}{4}\right),$$

$$t = \alpha u - 2\beta u\left(w - \frac{u^2}{3}\right) + 3u\left(w^2 - wu^2 + \frac{u^4}{10}\right). \tag{22}$$

The variation of the intensity w(x) defined by these equations is shown in Fig. 3. The conditions for the appearance of the singularity in the function w(x, t) are

$$\left(\frac{\partial x}{\partial w}\right)_t = 0, \quad \left(\frac{\partial^2 x}{\partial w^2}\right)_t = 0.$$

If we now use Eq. (22) to find $(\partial x/\partial w)$ and $(\partial^2 x/\partial w^2)_t$ we

2) The class of simple waves is, strictly speaking, broader because it includes functions which cannot be expanded into series. However, it will be shown in Sec. 4 that the family of self-similar solutions discussed here can be extended so that it will be characterized not by a discrete series but by a continuously varying parameter λ. Accordingly, a linear combination of these solutions will be characterized by an arbitrary function $C(\lambda)$.

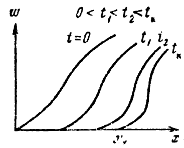

FIG. 3

obtain the following algebraic equations which determine the values of u and w at the singular point

$$(\alpha - 2\beta w + 3w^2 - 3wu^2 - {}^3/_2 u^4)^2 + u^2(3u^2 - 6w + 2\beta)[2\alpha + 2\beta u^2 - 2\beta w - 9wu^2] = 0,$$

$$-A^2(2\beta - 6w + 3u^2) - A^2u^2[12(w + u^2)(2\beta - 6w + 3u^2) + 6(\alpha + 2\beta u^2 - 3w^2 - 6wu^2 + 3u^4/2)] + Au^2(2\beta - 6w + 3u^2) \cdot [(2\beta - 6w + 3u^2)(\alpha + 6\beta u^2 - 3w^2 - 18wu^2 + {}^{15}/_2 u^4) + 2(2\beta - 6w + 9u^2)(\alpha + 2\beta u^2 - 3w^2 - 6wu^2 + 3u^4/2)] - 2u^4(2\beta - 6w + 3u^2)^2(2\beta - 9w + 3u^2)(\alpha + 2\beta u^2 - 3w^2 - 6wu^2 + 3u^4/2) = 0,$$

$$A = \alpha - 2\beta w + 2\beta u^2 + 3w^2 - 9wu^2 + {}^3/_2 u^4.$$

The analytic solution of this equation is readily found for $\Delta = \alpha - \beta^2/3 \ll \alpha$ when the slope of the initial front is large. In this case, $w_c = \beta/3 + \Delta/2\beta$, $u_c^2 = \Delta/\beta$ and the coordinates of the singular point are $x_c = \beta^3/27$, $t_c = 2\Delta^{3/2}/3\beta^{1/2}$. The functions w(x, t) and u(x, t) are single-valued up to the singular point $t = t_c$. Therefore, they are triple-valued in the region $x \sim x_c$.

The third essential type of singularity involves singularities appearing at maximum beam intensity. In contrast to the other two discussed earlier, these singularities have no analogs in the hydrodynamics of simple waves in an ideal fluid. They are described by self-similar solutions of the second class given by Eq. (13). Figure 4 shows the functions w(x, t) and u(x, t) near the beam maximum described by the solution $x = x_2 - \alpha x_1$, $t = t_2 - \alpha t_1$ [see Eq. (14)]. When $t < \alpha^2/2$ the intensity w is a smooth function of x:

$$w = w_0 - \frac{(3w_0 - 2\alpha)}{2w_0^2(\alpha - w_0)^3}x^2; \quad w_0(t) = \alpha - \sqrt{\alpha^2 - 2t}.$$

When $t = \alpha^2/2$ we have a singularity on the axis (x = 0): near this singularity $w_{t=\alpha^2/2} = \alpha - \alpha^{-1/4}|x|^{1/2}$. The function u(x) has a singularity at the same point: when $t < \alpha^2/2$, $u \approx x/w_0(w_0 - \alpha)$; when $t = \alpha^2/2$, $u(x) = -\alpha^{-3/4}|x|^{1/2}\operatorname{sign} x$. The physical significance of this singularity is that the rays begin to cut the beam axis at the point $t = \alpha^2/2$. The conditions for the appearance of the singularity on the beam axis are

$$x = 0, \quad \left(\frac{\partial x}{\partial u}\right)_t = 0, \quad \left(\frac{\partial x}{\partial w}\right)_t = 0. \tag{23}$$

Thus, as the smooth initial distribution develops, we find singularities connected with the possible intersection of rays propagating through the nonlinear medium. The main types of singularity are: quadratic ($w \sim x^{1/2}$, $u \sim x^{1/2}$) on the boundary of the beam, cubic ($w \sim x^{1/3}$, $u \sim x^{1/3}$) inside the beam, and axial singularity at maximum intensity on the beam axis ($w \sim |x|^{1/2}$, $u \sim |x|^{1/2}\operatorname{sign} x$). The multiray regions and strong discontinuities in the intensity and angular distributions of rays in the beam appear after the singular points. Equation (1) is, of course, invalid in these regions.

4. ARBITRARY PLANE BEAMS

Let us now consider the general problem of propagation of an arbitrary plane beam of electromagnetic waves through a slightly nonlinear medium. Equations (4) and (5) which describe the propagation of the beam, can be reduced to a single linear second-order equation as in the case of hydrodynamics.[10] Consider the function (cf.[8], Sec. 98)

$$\psi = S - ux + t(u^2/2 - w), \quad S = \int u\,dx,$$

where S is the eikonal. We shall look upon ψ as a function of the two variables u and w. It follows from the definition of ψ that

$$d\psi = -x\,du + t\,d(u^2/2 - w) = -t\,dw + (ut - x)\,du.$$

If we compare this with

$$d\psi = \frac{\partial\psi}{\partial w}dw + \frac{\partial\psi}{\partial u}du,$$

we find that

$$t = -\frac{\partial\psi}{\partial w}, \quad x = -u\frac{\partial\psi}{\partial w} - \frac{\partial\psi}{\partial u}. \tag{24}$$

If $\psi(u, w)$ is known, then Eq. (24) defines in an implicit form the dependence of u and w on x and t. The equation for ψ is obtained by direct substitution of Eq. (24) into the continuity equation given by Eq. (4) [Eq. (5) then becomes an identity]

$$\frac{\partial^2\psi}{\partial v^2} + \frac{1}{p}\frac{\partial}{\partial p}\left(p\frac{\partial\psi}{\partial p}\right) = 0, \quad p = \sqrt{w}, \quad v = \frac{u}{2}. \tag{25}$$

The boundary conditions of Eq. (2) are specified on the t = 0 plane in the form of the initial intensity distribution $w_0(x)$ and the angular distribution of rays in the beam, $u_0(x)$. In our case, the conditions

$$p^2 = w_0(x), \quad v = u_0(x)/2 \tag{26}$$

define in an implicit form an axially symmetric surface S in the p, v space. On this surface

$$\left(\frac{\partial\psi}{\partial p}\right)_S = 0, \quad \left(\frac{\partial\psi}{\partial v}\right)_S = -2x_0(p^2), \tag{27}$$

where $x_0(w)$ is the inverse of $w_0(x)$.

The function ψ is described by the Laplace equation (25). There is, therefore, a direct analogy between problems on the propagation of arbitrary plane beams in a nonlinear medium, on the one hand, and the boundary-value problems of electrostatics, on the other. The conditions given by Eq. (27) signify that the radial component of the "field" $E_p = -\partial\psi/\partial p$ is zero on the axially symmetric surface S, whereas the component along the v axis, namely, $E_v = -\partial\psi/\partial v$ is a given function of p and v.

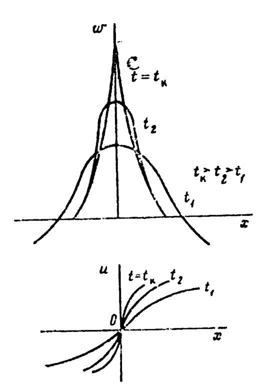

FIG. 4

Let us take a plane-parallel beam $u_0(x) \equiv 0$ as an example. Here, the surface S is a disk of unit radius lying on the v = 0 plane. To find ψ from the Laplace equation it is convenient to transform to oblate spheroidal coordinates ϵ and η defined by

$$v = \varepsilon\eta, \quad p^2 = (\varepsilon^2 + 1)(1 - \eta^2). \tag{28}$$

The Laplace equation assumes the form

$$\frac{\partial}{\partial\varepsilon}\left[(\varepsilon^2+1)\frac{\partial\psi}{\partial\varepsilon}\right] + \frac{\partial}{\partial\eta}\left[(1-\eta^2)\frac{\partial\psi}{\partial\eta}\right] = 0. \tag{29}$$

The coordinate η ranges from 0 to 1 and ϵ from 0 to ∞. The $\epsilon = 0$ surface is a disk of unit radius in ϵ, η space on which the boundary conditions of Eq. (27) are satisfied. In terms of ϵ, η they take the form

$$\left(\frac{\partial\psi}{\partial\eta}\right)_{\varepsilon=0} = 0, \quad \left(\frac{\partial\psi}{\partial\varepsilon}\right)_{\varepsilon=0} = -2x_0(1-\eta^2). \tag{30}$$

In these expressions $x_0(w)$ is the inverse of $w_0(x)$.

The solution of Eq. (29) is naturally sought in the form of a series in terms of the oblate spheroidal harmonics of the first and second kind, P_n and Q_n:

$$\psi = \sum_n \{Q_n(\eta)[A_nQ_n(i\varepsilon) + B_nP_n(i\varepsilon)] + P_n(\eta)[C_nQ_n(i\varepsilon) + D_nP_n(i\varepsilon)]\}. \tag{31}$$

The coefficients A_n, B_n, C_n, and D_n are determined from the boundary conditions, Eq. (30).

Let us take the initial profile $w_0(x) = \cosh^{-2}(x)$ as an example. In this case,

$$A_n = D_n = 0, \quad B_1 = C_0 = 2i, \quad B_k = 0 \quad (k \neq 1), \quad C_k = 0 \quad (k \neq 0).$$

The function ψ is of the form

$$\psi = -\varepsilon\eta\ln\frac{1+\eta}{1-\eta} + 2\varepsilon + 2\,\mathrm{arctg}\frac{1}{\varepsilon}.$$

From Eq. (24) we now find the explicit formulas

$$x = -\ln\frac{1+\eta}{1-\eta} + \frac{2\eta\varepsilon^2}{(1+\varepsilon^2)(1-\eta^2)}, \quad t = \frac{\varepsilon}{(1+\varepsilon^2)(1-\eta^2)}; \tag{32}$$

where ϵ and η are related to w and u by Eqs. (28) and (25). It is readily verified that this solution is the same as that given in[3]. The smooth initial beam profile is rapidly deformed and an axial singularity appears for t = 1/2. In fact, the appearance of the axial singularity is determined by Eq. (23). From Eq. (32) we find that these conditions are satisfied for $\epsilon = 1$, $\eta = 0$ (i.e., for x = 0, t = 1/2); w = 2, u = 0.

Consider now the parabolic beam profile

$$w_0(x) = \begin{cases} 1 - x^2 & \text{for } |x| < 1 \\ 0 & \text{for } |x| > 1 \end{cases}. \tag{33}$$

Here, the inverse function is $x_0(w) = \sqrt{1-w} = \eta$, and the coefficients are $C_0 = 2i/3$, $D_2 = \pi/3$, $C_2 = -2i/3$; $A_n = B_n = C_i = D_k = 0$; $i \neq 0.2$; $k \neq 2$. Consequently,

$$\psi = \frac{2}{3}\left\{\mathrm{arctg}\frac{1}{\varepsilon} - \frac{3\eta^2-1}{2}\left[\frac{\pi(3\varepsilon^2+1)}{4} - \frac{3\varepsilon^2+1}{2}\mathrm{arctg}\frac{1}{\varepsilon} + \frac{3\varepsilon}{2}\right]\right\}.$$

If we evaluate the derivatives in Eq. (24) we find that

$$x = \frac{\eta}{1+\varepsilon^2}, \quad t = \frac{1}{2}\left[\frac{\pi}{2} - \mathrm{arctg}\frac{1}{\varepsilon} + \frac{\varepsilon}{1+\varepsilon^2}\right]. \tag{34}$$

It is important to note here that ϵ is a function of t only. If we use the last equation to find $\eta = x(1 + \epsilon^2)$ and substitute in Eq. (28), we find that

$$w = \frac{1}{f(t)}\left(1 - \frac{x^2}{f^2}\right), \quad u = -x\varepsilon(1+\varepsilon^2) = -x\frac{d\ln f}{dt}, \quad f(t) = \frac{1}{1+\varepsilon^2(t)}. \tag{35}$$

It is clear that the initial parabolic beam persists for all t. Bearing this in mind, we can readily find the solution given by Eq. (35) directly from Eq. (1), which was, in fact, done by Talanov.[7] An important property of the parabolic beam is that it is focused down to a point. It is clear from Eq. (34) that as $t \to \pi/4$ we have $\epsilon \to \infty$ and $f(t) \to 0$. When $t \to \pi/4$ the intensity on the beam axis increases without limit.

The behavior of beams with near-parabolic initial profiles is therefore of considerable interest:

$$w_0(x) = 1 - x^2 + \alpha x^4, \quad |\alpha| \ll 1. \tag{36}$$

For the inverse function $x_0(w)$ we obtain, retaining the only linear term in α,

$$x_0(w) = \sqrt{1-w} + \alpha(1-w)^{3/2}/2.$$

From the boundary conditions of Eq. (30) we find the expansion coefficients in Eq. (31):

$$C_0 = \frac{2i}{3}, \; C_2 = -\frac{2i}{3}, \; D_2 = \frac{\pi}{3}, \; C_4 = \frac{3i\alpha}{35}, \; D_4 = -\frac{3\pi}{70}\alpha. \tag{37}$$

The remaining coefficients are all zero. If we now evaluate the derivatives of ψ, we find the required solution in an implicit form. We shall reproduce it only for the region of high intensities ($\epsilon \gg 1$)

$$\begin{aligned} x &= \eta\left[\frac{1}{1+\varepsilon^2} - \frac{9\pi\alpha\varepsilon^3}{32}\left(1 + \frac{5}{3}\eta^2\right)\right], \\ t &= \frac{1}{2}\left[\frac{\pi}{2} - \mathrm{arctg}\frac{1}{\varepsilon} + \frac{\varepsilon}{1+\varepsilon^2} - \frac{9\pi\alpha\varepsilon^2}{32}(1 - 5\eta^2)\right]. \end{aligned} \tag{38}$$

The departures of the beam profile from the parabolic form for small t are unimportant. On the contrary, for $t \to \pi/4$, i.e., near the focal point, they play the dominant role. As a result, we have singularities on the axis ($\alpha > 0$) or on the beam boundary (for $\alpha < 0$). In particular, when $\alpha > 0$ we find from Eqs. (23) and (38) that the maximum beam intensity at the axial singularity is

$$w_m = (32/9\pi\alpha)^{2/5}.$$

The distribution of w and u with x near this point is

$$w = w_m - w_m^{-1/4}\frac{|x|^{1/2}}{A}, \quad u = -\frac{w_m^{-3/4}}{A}|x|^{1/2}\,\mathrm{sign}\,x, \quad A = \frac{5^{1/2}3^{7/5}\pi^{7/10}}{2^4}\alpha^{7/10}. \tag{39}$$

The shift of the ray intersection point is $t_c - \pi/4 = \frac{5}{3}w_m^{-3/2}$. The behavior of w and u near the ray intersection point constitutes an axial singularity ($w \sim |x|^{1/2}$, $u \sim |x|^{1/2}\,\mathrm{sign}\,x$), in accordance with the classification given in Sec. 3. If we expand the solution given by Eq. (38) near the singularity into a series in powers of $w_m - w$ and u, we can express them in terms of the self-similar solutions of Eq. (14): $x = A^2(x_2 - w_m x_1)$, $t = 4^{-1}\pi + A^2(t_2 - w_m t_1)$. In precisely the same way, by expanding the solution given by Eq. (32) around the singularity we obtain $x = \frac{1}{4}(x_2 - 2x_1)$, $t = \frac{1}{4}(t_2 - 2t_1)$. Therefore, the behavior of the solution given by Eqs. (14), (32), and (38) near the axial singularity is described (to within a factor) by the same functions.

Equation (25) will, in general, admit of the self-similar solutions

$$\psi = p^\lambda F_\lambda(\tau), \quad \tau = v/p, \tag{40}$$

where F_λ is described by

$$(1+\tau^2)\frac{d^2F_\lambda}{d\tau^2}-(2\lambda-1)\tau\frac{dF_\lambda}{d\tau}+\lambda^2F_\lambda=0. \qquad (41)$$

By substituting $F_\lambda = (1+\tau^2)^{\lambda/2}f$ and changing the variable so that $z = \frac{1}{2}(1-\tau/\sqrt{1+\tau^2})$, we can transform Eq. (41) to the hypergeometric form:

$$\frac{d^2f}{dz^2}+\frac{2z-1}{z(z-1)}\frac{df}{dz}-\frac{\lambda(\lambda+1)}{z(z-1)}f=0. \qquad (42)$$

For integral values $\lambda = n$ the solutions of the last equation are Legendre polynomials, so that

$$F_n(\tau) = (1+\tau^2)^{n/2}P_n(\tau/\sqrt{1+\tau^2}).$$

These solutions were discussed above in Sec. 2. When $\lambda = n - \frac{1}{2}$ the function f can be expressed in terms of the complete elliptic integrals of the first and second kinds. For arbitrary λ the solution of Eq. (42) is given by the hypergeometric series $f = w(\alpha, \beta, \gamma; z)$, where $\alpha = (1+\sqrt{1+4\lambda(1+\lambda)})/2$, $\beta = (1-\sqrt{1+4\lambda(1+\lambda)})/2$, $\gamma = 1$. The series converges for all τ since $|z| = \frac{1}{2}|1-\tau/\sqrt{\tau^2+1}| < 1$.

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