KINETICS OF RADIATIVE POLARIZATION

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Submitted October 17, 1969

Zh. Eksp. Teor. Fiz. 58, 1695–1702 (May, 1970)

An equation describing the behavior of a spin in an external electromagnetic field is obtained in the quasiclassical approximation taking radiation effects into account. With the aid of this equation the process of radiative polarization is investigated.

1. INTRODUCTION

MAGNETIC bremsstrahlung leads to polarization of electrons and positrons moving in a field because the probability of a radiative transition with a spin flip depends on the orientation of the initial spin. The advantages of such a mechanism of polarization consist of the following:

1) This is the only method of directly obtaining polarized beams of high energy, and this avoids the very complicated problem of accelerating polarized particles (in fact it is effective beginning with an energy of several hundreds of MeV).

2) The polarization process does not introduce additional changes into the characteristics of the beam (intensity, scatter of the parameters), and this distinguishes it advantageously from, say, the method of obtaining polarized beams by means of scattering. Thus a possibility is opened up of designing experiments involving polarized electrons and positrons which enables one to expand the means for the experimental study of electromagnetic interactions at high energies.

Sokolov and Ternov[1] first indicated the existence of the mechanism of radiative polarization in a homogeneous magnetic field by calculating the probability of a radiative transition with spin-flip starting from the exact solution of the Dirac equation in a homogeneous magnetic field. The analysis of the kinetics of radiative polarization was carried out by them by means of an elementary balance equation for the number of electrons with a given component of spin along the direction of the magnetic field.

In [2,3] the probability was calculated for a radiative transition with a spin-flip in the case of an inhomogeneous magnetic field of particular interest for applications.[1)] In these papers a method was used which utilizes in an essential manner the quasiclassical nature of the motion of high energy electrons in an external field (cf., [4]). The value obtained for the probability of spin-flip accompanying radiation depends on the sign of the component of the spin vector along the $\mathbf{v}\times\dot{\mathbf{v}}$ direction, and this indicates the possibility of polarization of particles along this direction.

Of considerable interest is the detailed study of the kinetics of radiative polarization, and for this it is necessary to derive and to investigate the equation for the polarization density matrix taking into account the contribution made by the radiation. The present paper is devoted to this question. Taking into account the quasiclassical nature of the motion of high energy electrons in an external field the equation for the polarization density matrix can be conveniently represented in the form of an equation for the spin vector $\boldsymbol{\zeta}$ (the average value of the spin operator in the electron rest system). In this paper an equation of motion has been obtained for the spin vector taking into account the interaction with the radiation field which is a generalization of the well-known Bargmann–Michel–Telegdi (BMT) equation (cf., for example, [5]). Then with the aid of this equation an analysis is carried out the kinetics of radiative polarization.

2. DERIVATION OF THE EQUATION

We introduce the Heisenberg operator for the electron spin in the rest system $\boldsymbol{\sigma}(t)$ ($\boldsymbol{\sigma}^+(t) = \boldsymbol{\sigma}(t)$), the average value of which

$$\boldsymbol{\zeta}_0(t) = \langle t_0|\boldsymbol{\sigma}(t)|t_0\rangle \tag{1}$$

is the spin vector in the rest system of the particle. Without taking into account the interaction with the radiation field the variation of this vector with time for particles with a given anomalous magnetic moment is determined by the BMT equation (in the quasiclassical limit, i.e., for fields slowly varying over a length $\sim\hbar/mc$, and for narrow wave packets).

After the interaction with the radiation field has been switched on the evolution of the state vector in time is determined by the matrix $U(t, t_0)$:

$$|t\rangle = U(t, t_0)|t_0\rangle. \tag{2}$$

The variation of the average value of the spin of a Dirac electron with time taking the interaction with the radiation field into account is given by

$$\begin{aligned}\langle t|\boldsymbol{\sigma}(t)|t\rangle - \langle t_0|\boldsymbol{\sigma}(t_0)|t\rangle &= \langle t_0|U^+(t, t_0)\boldsymbol{\sigma}(t)U(t, t_0|t_0\rangle - \langle t_0|\boldsymbol{\sigma}(t_0)|t_0\rangle \\ &= \langle t_0|U^+(t, t_0)[\boldsymbol{\sigma}(t), U(t, t_0)]|t_0\rangle + \langle t_0|\boldsymbol{\sigma}(t) - \boldsymbol{\sigma}(t_0)|t_0\rangle.\end{aligned} \tag{3}$$

Here the last term determines the variation of the average spin in the absence of the radiation field. We represent the matrix $U(t, t_0)$ in the form of a perturbation theory expansion in terms of the electromagnetic coupling constant e:

$$U(t, t_0) = I + iT(t, t_0) = I + i[T_1(t, t_0) + T_2(t, t_0) + \ldots]. \tag{4}$$

[1)]The degree of inhomogeneity is limited by the following condition: the external field must undergo only a small change in a distance over which radiation is formed, and this is always satisfied in practice.

From the condition of unitarity of the scattering matrix we obtain[2)]

$$T_1 - T_1^+ = 0, \quad i(T_2^+ - T_2) = 2\operatorname{Im} T_2 = T_1^+ T_1. \tag{5}$$

Taking these relations and (1) into account we can rewrite (3) in the form

$$\zeta(t) - \zeta(t_0) = \langle t_0 | T_1^+ \sigma(t) T_1 - {}^1/_2 [\sigma(t) T_1^+ T_1 + T_1^+ T_1 \sigma(t)] + i[\sigma(t), \operatorname{Re} T_2] | t_0 \rangle + \zeta_0(t) - \zeta_0(t_0). \tag{6}$$

Here $\boldsymbol{\zeta}_0(t_0) = \boldsymbol{\zeta}(t_0)$, since the interaction with the radiation field is switched on at time t_0.

We now proceed to the evaluation of the individual terms in (6). The matrix T_1 contains the operator for the creation or the annihilation of a photon, and therefore the matrix element

$$\langle t_0 | T_1 | t_0 \rangle = 0, \tag{7}$$

since the state vector $|t_0\rangle$ describes the states of an electron in an external field in the absence of the radiation field. This circumstance has been taken into account in (6). In evaluating terms containing the combination $T_1^+ T_1$, one should take into account the fact that the only matrix elements of T_1 which are different from zero are those for the transition into the one-photon states:

$$\langle t_0 | T_1^+ T_1 | t_0 \rangle = \sum_n \langle t_0 | T_1^+ | n \rangle \langle n | T_1 | t_0 \rangle = \int d^3k \sum_{s,\lambda} \langle t_0 | T_1^+ | t_0, k \rangle \langle k, t_0 | T_1 | t_0 \rangle, \tag{8}$$

where the integration is carried out over the momenta of the photon, while the summation is carried out over the spins of the electron (s) and the polarizations of the photon (λ). In (8) $\langle k, t_0 | T_1 | t_0 \rangle$ is the matrix element for the transition into the one-photon state with the photon (k, λ). In agreement with the results of [4] it has the form

$$\langle k_1 t_0 | T_1 | t_0 \rangle = \frac{e}{(2\pi)^{3/2} \sqrt{2\hbar\omega}} \varphi_n^+ \left[\int_{t_0}^{t} Q(t_1) \exp\left\{ +i \frac{\varepsilon}{\varepsilon'} (\omega t_1 - \mathbf{k r}(t_1)) \right\} dt_1 \right] \varphi_i, \tag{9}$$

where

$$Q(t) = A(t) + i\sigma \mathbf{B}(t) \qquad A = \frac{\varepsilon + \varepsilon'}{2\varepsilon'} (\mathbf{e}^* \mathbf{v}),$$

$$\mathbf{B} = \frac{\hbar\omega}{2\varepsilon'} \left[\mathbf{e}^* \left(\mathbf{n} - \mathbf{v} + \mathbf{v} \frac{m}{\varepsilon} \right) \right]$$

$\epsilon' = \epsilon - \hbar\omega$, $\mathbf{n} = \mathbf{k}/\omega$, ω is the photon frequency, ϵ is the electron energy, $\mathbf{v}$ is the electron velocity, $\mathbf{e}$ is the polarization vector of the photon.

The characteristic time for the variation of the matrix elements of the operator T_1 is the radiation time $\tau \sim T_0/\gamma$, while the characteristic time for the variation of $\sigma(t)$ ($\boldsymbol{\zeta}(t)$) is T_0 (T_0, for example, is the period of revolution of the particle in the field), and, therefore, with an accuracy up to terms $\sim 1/\gamma$ ($\gamma = \epsilon/m$) we can neglect the variation of $\sigma(t)$ during the time of formation of the radiation. Taking this circumstance into account we have

$$\Delta\zeta_1 = \langle t_0 | T_1^+ \sigma(t) T_1 - {}^1/_2 (\sigma(t) T_1^+ T_1 + T_1^+ T_1 \sigma(t)) | t_0 \rangle$$
$$= \frac{e^2}{4\pi\hbar} \int \frac{d^3k}{2\pi\omega} \int_{t_0}^{t} dt_1 \int_{t_0}^{t} dt_2 \sum_\lambda \mathrm{N} \exp\left\{ -\frac{i\varepsilon}{\varepsilon'} [\omega(t_2 - t_1) - \mathbf{k}(\mathbf{r}_2 - \mathbf{r}_1)] \right\}, \tag{10}$$
$$\mathrm{N} = \mathrm{Sp} \left\{ \left[Q_2^+ \sigma(t) Q_1 - \frac{\sigma(t) Q_2^+ Q_1 + Q_2^+ Q_1 \sigma(t)}{2} \right] \frac{1 + \zeta\sigma}{2} \right\}.$$

Utilizing the relation $\mathbf{Q}\boldsymbol{\sigma} = \boldsymbol{\sigma}\mathbf{Q} + 2[\mathbf{B} \times \boldsymbol{\sigma}]$, it is not difficult to evaluate the trace in (10): $\mathbf{N} = \mathbf{N}_A + \mathbf{N}_B$, where

$$\mathrm{N}_A = -[(A_2 \mathbf{B}_1 + A_1 \mathbf{B}_2) \zeta],$$
$$\mathrm{N}_B = -2i[\mathbf{B}_2 \mathbf{B}_1] + \mathbf{B}_2(\zeta \mathbf{B}_1) + \mathbf{B}_1(\zeta \mathbf{B}_2) - 2\zeta(\mathbf{B}_1 \mathbf{B}_2). \tag{11*}$$

The expression obtained for $\mathbf{N}$ contains terms of two types: quadratic in $\mathbf{B}$ ($\mathbf{N}_B \sim \hbar^2$) and linear in $\mathbf{B}$ ($\mathbf{N}_A \sim \hbar$); these terms lead to different physical consequences, and, therefore, we investigate them separately. We multiply $\mathbf{N}$ in (10) by $\boldsymbol{\zeta}$:

$$\mathrm{N}\zeta = \mathrm{N}_B \zeta = 2\{(\mathbf{B}_1 \zeta)(\mathbf{B}_2 \zeta) - \zeta^2 (\mathbf{B}_1 \mathbf{B}_2) - i\zeta [\mathbf{B}_2 \mathbf{B}_1]\}. \tag{12}$$

it can be seen (cf., [3]), that the term $\mathbf{N}_B \boldsymbol{\zeta}$ is expressed in terms of the square of the matrix element of the radiative transition with spin-flip. We note that here we are considering an ensemble of electrons (in the language of one of the representations of the density matrix) so that, generally speaking, $|\boldsymbol{\zeta}| \neq 1$. The further evaluation of the integral (10) coincides with that carried out in [3] 3) since terms involving given structures in $\boldsymbol{\zeta}$ can be picked out uniquely. Thus, the answer follows directly from the last formula in [3]:

$$\frac{\Delta\zeta_{1,B}}{\Delta t} = -\frac{1}{T} \left\{ \zeta - \frac{2}{9} \mathbf{v}(\zeta \mathbf{v}) + \frac{8}{5\sqrt{3}} \frac{[\mathbf{v}\dot{\mathbf{v}}]}{|\dot{\mathbf{v}}|} \right\}, \tag{13}$$

where

$$\frac{1}{T} = \frac{5\sqrt{3}}{8} \alpha \frac{\hbar^2}{m^2} \gamma^5 |\dot{\mathbf{v}}|^3, \quad \alpha = \frac{e^2}{4\pi\hbar} = \frac{1}{137}.$$

A direct evaluation of $\Delta\boldsymbol{\zeta}_{1,A}/\Delta t$ presents no difficulties, but one should have in mind the following circumstance: a) the structure of this term is $[\mathbf{F} \times \boldsymbol{\zeta}]$, where $\mathbf{F}$ is an axial vector constructed from the vectors of the problem. Terms of this type lead to a rotation of the vector $\boldsymbol{\zeta}$ (about $\mathbf{F}$), but not to a change in its modulus; b) the term $\Delta\boldsymbol{\zeta}_{1,A}/\Delta t$ is proportional to Planck's constant $\hbar$, since it is linear in $\mathbf{B}$, and, at the same time, as will be seen below, there are rotational terms which do not contain $\hbar$. In this sense the term $\Delta\boldsymbol{\zeta}_{1,A}/\Delta t$ is a correction term to the description of rotation.

We now proceed to the term in (6) with $\operatorname{Re} T_2$. In order to evaluate it, it is necessary to know the Green's function for the electron in an external electromagnetic field. We utilize the Green's function evaluated by Schwinger.[6] In his paper he has given a formal expression for the Green's function which is valid in all orders in $\hbar$, and the Green's function is calculated explicitly in the approximation linear in the field (in fact the terms $\sim\hbar^0$). As is shown in the Appendix, starting with this Green's function it is possible to obtain

$$\langle t_0 | \operatorname{Re} T_2 | t_0 \rangle = \int_{t_0}^{t} \left\langle t_0 \left| a - \frac{\mu e}{2\varepsilon} (\mathbf{H}_R \sigma) \right| t_0 \right\rangle dt. \tag{14}$$

2)We note that the state vector introduced above is a two-component spinor, whilte $U(t, t_0)$ is a 2 × 2 matrix operating in the space of these spinors. $\operatorname{Re}T_2$ and $\operatorname{Im}T_2$ denote respectively the hermitean part and the antihermitean part divided by i of the operator T_2.

*$[\mathbf{B}_2 \mathbf{B}_1] \equiv \mathbf{B}_2 \times \mathbf{B}_1$.

3)It is necessary that the difference of times should satisfy $t - t_0 \gg \tau$.

The quantity a contains a divergent integral related to mass renormalization, in future we shall not need it; $\mu = \alpha/2\pi$;

$$\mathbf{H}_R = \gamma\left\{\mathbf{H} - \frac{\mathbf{v}(\mathbf{vH})}{1+1/\gamma} - [\mathbf{vE}]\right\}$$

is the magnetic field in the rest system of the electron, if **H** and **E** are the fields in the laboratory system. Carrying out simple calculations and utilizing considerations concerning the dependence of σ on the time analogous to those presented above we obtain

$$\langle t_0|i[\sigma, \mathrm{Re}T_2]|t_0\rangle = \int_{t_0}^{t} \frac{\mu e}{\varepsilon}[\zeta\mathbf{H}_R]\,dt. \tag{15}$$

Thus, we have obtained that

$$\frac{\Delta\zeta_2}{\Delta t} = \frac{\mu e}{\varepsilon}[\zeta\mathbf{H}_R], \tag{16}$$

i.e., we have obtained a rotational term proportional to the anomalous magnetic moment of the electron.

Finally, the difference appearing in (6)

$$\zeta_0(t) - \zeta_0(t_0) = \frac{\Delta\zeta_0}{\Delta t}\Delta t$$

describes the variation of the spin vector of the electron in an external field in the absence of an interaction with the radiation field. In the quasiclassical limit one can obtain directly from the equation for the spin operator of the Dirac equation (cf., for example, [5])

$$\frac{\Delta\zeta_0}{\Delta t} = \frac{e}{\varepsilon}[\zeta\mathbf{H}_E], \quad \mathbf{H}_E = \mathbf{H} + \frac{[\mathbf{Ev}]}{1+1/\gamma}. \tag{17}$$

Thus, the picture of the phenomenon under consideration is the following. In the absence of an interaction with the radiation field the spin precesses in accordance with Eq. (17). The switching on of the interaction with the radiation field leads to two types of effects. First, new forms of rotational terms appear which are related to the acquisition by the electrons as a result of the interaction with the radiation field of an anomalous magnetic moment (16). The sum of (16) and (17) gives an equation for the variation of the vector $\boldsymbol{\zeta}$ for an electron with an anomalous magnetic moment in an external field (the BMT equation)[4] Moreover terms of a new type appear—those that change $|\boldsymbol{\zeta}|$. As a result we obtain the following equation of motion for the spin of an ensemble of electrons in an external field taking radiation effects (13), (16), and (17) into account:

$$\frac{d\zeta}{dt} = \frac{e}{\varepsilon}[\zeta(\mu\mathbf{H}_R + \mathbf{H}_E)] - \frac{1}{T}\left(\zeta - \frac{2}{9}\mathbf{v}(\zeta\mathbf{v}) + \frac{8}{5\sqrt{3}}\frac{[\mathbf{v}\dot{\mathbf{v}}]}{|\dot{\mathbf{v}}|}\right), \tag{18}$$

where $\mathbf{H}_R$ is defined by (14), $\mathbf{H}_E$ is defined by (17), 1/T is defined by (13). One should keep in mind that the rotational terms in (18) are of order $\hbar^0$ (we do not take into account corrections of higher order in $\hbar$ to the rotational terms since they are negligibly small, and do not lead to new qualitative effects). The terms altering $|\boldsymbol{\zeta}|$ are of order $\hbar^2$, yet they should be retained, since they lead to new qualitative effects—the variation of $|\boldsymbol{\zeta}|$. Nevertheless, such a distinction in the orders of magnitudes simplifies the solution of the kinetic equation (18) and enables one in many cases to consider separately the effects of rotation and of the variation of $|\boldsymbol{\zeta}|$.

3. THE ANALYSIS OF THE KINETIC EQUATION

For the sake of simplicity we consider the case of a purely magnetic field, i.e., we set E = 0 in (18). Equation (18) can be rewritten in the form of a system of equations for the components of the vector $\boldsymbol{\zeta}$ along the $\mathbf{e}_1 = \mathbf{v}/|\mathbf{v}|$, $\mathbf{e}_2 = \dot{\mathbf{v}}/|\dot{\mathbf{v}}|$ and $\mathbf{e}_3 = [\mathbf{e}_1\times\mathbf{e}_2]$ axes. We now have

$$\dot{\zeta}_1 = -\frac{7}{9}\frac{\zeta_1}{T} - \Omega\zeta_2, \qquad \dot{\zeta}_2 = -\frac{\zeta_2}{T} + \Omega\zeta_1 + \omega\zeta_3, \qquad \dot{\zeta}_3 = -\frac{1}{T}\left(\zeta_3 + \frac{8}{5\sqrt{3}}\right) - \omega\zeta_2, \tag{19}$$

where

$$\Omega = \gamma\mu|\dot{\mathbf{v}}|, \qquad \omega = \frac{\mu e}{\varepsilon}H_{||} + \frac{e}{\varepsilon}\frac{\dot{\mathbf{v}}\dot{\mathbf{H}}}{|\dot{\mathbf{v}}|^2}$$

$$H_{||} = \mathbf{vH}, \qquad \dot{\mathbf{H}} = (\mathbf{v}\nabla)\mathbf{H}$$

(It is assumed that **H** does not explicitly depend upon the time). If the particles move in a homogeneous field along a circular trajectory, then $\omega = 0$: the coefficients Ω and 1/T do not depend upon the time. We write out the resulting solution (cf., also [7]):

$$\begin{aligned}\zeta_1(t) &= \zeta_\perp(0)\cos(\Omega t + \alpha)\exp(-{}^8/_9 t/T),\\ \zeta_2(t) &= \zeta_\perp(0)\sin(\Omega t + \alpha)\exp(-{}^8/_9 t/T),\\ \zeta_3(t) &= -8/5\sqrt{3} + (\zeta_3(0) + 8/5\sqrt{3})e^{-t/T};\end{aligned} \tag{20}$$

here $\zeta_\perp(t) = \sqrt{\zeta_1^2(t) + \zeta_2^2(t)}$.

It can be seen that the spin rotates about the $\mathbf{e}_3$ axis, while the ζ_1 and ζ_2 components decay during a time ~ T, while the nondecaying term $-8/5\sqrt{3}$ in ζ_3 gives a finite polarization which does not depend on the initial value of the vector $\boldsymbol{\zeta}$. In this manner asymptotically (i.e., after a time $t \gg T$) one obtains a spin vector of length $8/5\sqrt{3}$, directed along $-[\mathbf{v}\times\dot{\mathbf{v}}]$, i.e., against the field for electrons and along the field for positrons.

We now assume that the particle in a homogeneous field moves along a helical line, then the second term in ω which depends on the field gradients naturally vanishes and we have for ω: $\omega = \mu e H_{||}/\epsilon$. In the expression for $\boldsymbol{\zeta}(t)$ the essentially new feature compared to (20) consists of the fact that a nondecaying part appears also in the component ζ_1, so that asymptotically a vector is obtained situated in the (1, 3) plane and rotated from the $\mathbf{e}_3$ axis by an angle $\sim \omega/\Omega = \gamma^{-1}H_{||}/H_\perp$. The motion along a helical line can be obtained from a circular motion by a Lorentz transformation parallel to the magnetic field. Since $-\boldsymbol{\zeta}^2 = s^2$—the square of the spin 4-vector, then the degree of polarization of the particles (measured by $|\boldsymbol{\zeta}|$) must be the same in both these cases. The same result also follows naturally from the solution of the system (19) if one takes into account that in deriving Eq. (18) in that part of it, in front of which the factor 1/T appears, terms of order $1/\gamma^2$ were omitted compared to those that were retained; the role played by these terms leads only to a change in ζ_3^{as} and, consequently, in the degree of polarization by a quantity of the order of $1/\gamma^2$.

In an inhomogeneous field it is usually possible to neglect the first term in comparison to the second.

[4] In this sense the calculation carried out above represents a direct derivation of the BMT equation. One could also argue in the opposite direction: starting with the general representation for $\mathrm{Re}T_2$ (14), independently of the coefficients, it can be easily seen that this term is of a rotational type, but then it can be equal only to the term with the anomalous magnetic moment in the BMT equation.

Generally speaking, the degree of polarization changes (compared to the case of the homogeneous field), since a specific physical mechanism—the presence of real field gradients—comes into play. This change can be found, if $\omega/\Omega \gg 1/\gamma$; in the opposite case the coefficients in (18) in front of 1/T are insufficiently accurate (however, then the corrections to the degree of polarization are negligibly small—of the order of $1/\gamma^2$).

Further, we consider the practically interesting case of an inhomogeneous field when the orbit of the particle differs insignificantly from a plane circular one. Then the ratio of the frequencies ω/Ω turns out to be small—of the order of the ratio of the dimensions of the deviation from a plane circular orbit to its mean radius. The system (19) was solved by approximations in terms of this small parameter. In the linear approximation the following expressions are obtained for the spin components

$$\zeta_+(t) = e^{g(t)}\left[\zeta_+(0) + i\int_0^t \omega(\tau)\zeta_3^{(0)}(\tau)e^{-g(\tau)}d\tau\right],$$

$$\zeta_3(t) = \zeta_3^{(0)}(t) - \exp\left(-\int_0^t \frac{d\tau}{T}\right)\operatorname{Im}\zeta_+(0)\int_0^t \omega(\tau)e^{f(\tau)}d\tau; \qquad (21)$$

here

$$\zeta_+(t) = \zeta_1(t) + i\zeta_2(t) \quad \zeta_3^{(0)}(t) = -\frac{8}{5\sqrt{3}} + \left(\zeta_3(0) + \frac{8}{5\sqrt{3}}\right)\exp\left(-\int_0^t \frac{d\tau}{T}\right)$$

$$g(t) = \int_0^t \left(-\frac{8}{9}\frac{1}{T} + i\Omega\right)d\tau, \quad f(t) = g(t) + \int_0^t \frac{d\tau}{T}.$$

The difference from the case of the homogeneous field consists of the fact that in ζ_1 and ζ_2 nondecaying terms of small ($\sim\omega/\Omega$) amplitude appear; in ζ_3 there is a decaying term linear in ω/Ω. The nondecaying correction to ζ_3 appears only in the next approximation, and has the form:

$$\Delta\zeta_3 = \frac{8}{5\sqrt{3}}\exp\left(-\int_0^t \frac{d\tau}{T}\right)\operatorname{Re}\int_0^t d\tau\,\omega(\tau)e^{f(\tau)}\int_0^\tau \omega(\tau_1)e^{-g(\tau_1)}d\tau_1. \qquad (22)$$

In a specific case the configuration of the field determines the law of motion for the particle which determines the dependence on the time of the functions $\Omega(t)$, $\omega(t)$, and $T(t)$. After evaluating the integrals appearing in (21), (22), we obtain the desired behavior of the spin.

As an illustration we write out the asymptotic form of the solution for the spin vector in the case when the particle performs small oscillations in an axially symmetric magnetic field with an inhomogeneity index n:

$$\zeta_1^{as} = -\frac{8}{5\sqrt{3}}\frac{z_0}{R}\gamma\mu\frac{n^{3/2}}{n-(\gamma\mu)^2}\sin\sqrt{n}\,\omega_0 t,$$

$$\zeta_2^{as} = \frac{8}{5\sqrt{3}}\frac{z_0}{R}\frac{n^2}{n-(\gamma\mu)^2}\cos\sqrt{n}\,\omega_0 t \qquad (23)$$

$$\zeta_3^{as} = -\frac{8}{5\sqrt{3}} + \frac{2}{5\sqrt{3}}\left(\frac{z_0}{R}\right)^2\frac{n^3}{n-(\gamma\mu)^2}\cos 2\sqrt{n}\,\omega_0 t.$$

Here $\omega_0 = 1/R$; R is the equilibrium radius; z_0 is the amplitude of the vertical oscillations.

APPENDIX

EVALUATION OF $\langle t_0|\operatorname{Re}T_2|t_0\rangle$

In the interaction representation in the Furry picture the scattering matrix formally has the same form as in quantum electrodynamics of free particles:

$$R(t,t_0) = T\exp\left\{-ie\int_{t_0}^t dt\int d^3x\,\bar\psi_F\gamma^\mu\psi_F A_\mu\right\}, \qquad (A.1)$$

where $\psi_F(x)$ is the operator for the electron-positron field in the given external field. As before, Wick's theorem holds for the expansion of the T-products into a sum of normal products, but the contraction of the fermion operators will now be the Green's function in the given field. Since we are evaluating the average value of Re T_2 between one-electron states, it is clear that the photon operators must be contracted, i.e., we are dealing with a self-energy diagram. We write out the average value of $R_2(t, t_0)$ over the one-electron states. From what has been said above it follows that

$$\langle p|R_2(t,t_0)|p\rangle = ie^2\int d^4x\int d^4x'\Phi_p^+(x)\bar u_s(p)\gamma^\mu G(x,x')$$
$$\times\gamma^\nu D_{\mu\nu}(x-x')u_s(p)\Phi_p(x). \qquad (A.2)$$

Here G(x, x′) is the Green's function for the electron in an electromagnetic field, $D_{\mu\nu}(x-x')$ is the photon propagator. In formula (A.2) we have also utilized the quasiclassic representation for $|p\rangle$ (cf., [4]).

The explicit form for the Green's function G(x, x′) we find in [6] in an approximation linear in the field. Substituting this Green's function into (A.2) we have

$$\langle p|R_2(t,t_0)|p\rangle = \frac{e^2}{(4\pi)^2}\int_{t_0}^t dt\int d^3x\Phi_p^+(x)\bar u_s(p)\left[\int_0^\infty\frac{ds}{s^2}\int_0^s e^{-im^2(s-w)}dw\right.$$
$$\left.\times\left[2m\left(2-\frac{w}{s}\right) - 2mw\left(1-\frac{w}{s}\right)\frac{i}{2}e\sigma^{\mu\nu}F_{\mu\nu}\right]u_s(p)\Phi_p(x). \qquad (A.3)$$

Utilizing the explicit form for u(p) from [4] one can rewrite this expression in the two-component form. Then taking into account the fact that

$$\int d^3x(\Phi^+\ldots\Phi) = \langle i|\ldots|i\rangle, \quad |t_0\rangle = \varphi|i\rangle \qquad (A.4)$$

(φ is a two-component spinor), we have

$$\langle p|R_2(t,t_0)|p\rangle = \langle t_0|T_2|t_0\rangle = \int_{t_0}^t dt\left\langle t_0\left|a - \frac{\mu e}{2\varepsilon}(\mathbf{H}_R\sigma)\right|t_0\right\rangle, \qquad (A.5)$$

where[6]

$$a = \frac{\alpha}{2\pi}\frac{m^2}{\varepsilon}\int_0^\infty\frac{ds}{s}\int_0^1 du(1+u)\exp[-m^2us], \qquad (A.6)$$

$$\mu = im^2\frac{\alpha}{\pi}\int_0^\infty ds\int_0^s\frac{dw}{s}\frac{w}{s}\left(1-\frac{w}{s}\right)\exp[-im^2(s-w)] = \frac{\alpha}{2\pi} \qquad (A.7)$$

H_R was defined in (14). Thus, in the approximation linear in the external field the matrix T_2 is hermitian.

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Translated by G. Volkoff
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