EVOLUTION OF COHERENT STATES OF A CHARGED PARTICLE IN A VARIABLE MAGNETIC FIELD

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We find new invariants in the problem of a charged particle in a variable magnetic field. We introduce coherent states of a charged particle in a uniform variable magnetic field. We calculate the probabilities for transitions between Landau levels in terms of Jacobi polynomials for magnetic fields which are constant in the remote past and the remote future. We also find the evolution of the initial coherent states, and discuss the group theoretic aspect of the problem.

INTRODUCTION

RECENTLY Lewis[1] and Lewis and Riesenfeld[2] have found explicitly time-dependent integrals of motion for classical and quantal oscillators with a variable frequency. Using such an invariant they calculated the probabilities for transitions between energy states for a frequency which was varying in such a way that it is constant in the remote past (initial state) and in the remote future (final state).[2] Husimi[3] had found those transition probabilities earlier by other means and Dykhne[4] had found them in the adiabatic approximation. Crosignani et al.[7] have calculated the evolution of initial coherent states of a quantum oscillator[5,6] under the influence of a variable frequency. We[8] have introduced coherent states of a charged particle in a constant magnetic field. Lewis and Riesenfeld[2] found for the problem with a variable magnetic field an integral of motion which depended explicitly on the time and which (apart from a constant) is the same as the Hamiltonian when we go over to a constant field. The aim of the present paper is to introduce coherent states for a charged particle in a variable magnetic field, to find new invariants, and to elucidate their physical meaning.

We use the method developed in [2] to evaluate explicitly the probabilities for transitions between the Landau levels of the initial and final states that correspond to a constant magnetic field, and these probabilities are expressed in terms of Jacobi polynomials. Popov and Perelomov[9] have expressed in terms of Legendre polynomials the probabilities for transitions between excited energy levels, which had been evaluated earlier,[2-4] for an oscillator with a variable frequency. We must note that the results obtained for a charged non-relativistic particle moving in a variable magnetic field can be generalized to the case of electrical and magnetic fields ($\mathbf{E}\cdot\mathbf{H} = 0$, $E^2 - H^2 < 0$) and also to the case of a relativistic particle (Klein-Gordon and Dirac equation).

1. INVARIANTS OF THE PROBLEM

We shall consider a particle of mass M and charge e moving in a variable (classical) electromagnetic field with vector potential $\mathbf{A}(t) = \frac{1}{2}[\mathbf{H}(t)\times\mathbf{r}]$. We choose the z-axis along the magnetic field **H**. In that case $A_z = 0$ and the motion in the z-direction is free particle motion and we shall consider only the motion in the xy-plane. The Hamiltonian of such a system is

$$\mathscr{H}(t)=\frac{1}{2M}[(p_x-eA_x)^2+(p_y-eA_y)^2],\ \hbar=c=1. \tag{1}$$

We have omitted the spin part $-\mu\sigma_z H$ from (1), since what follows will be independent of it. We assume that the magnetic field H(t) in the limit as $t\to\mp\infty$ takes on the constant values H_i and H_f, respectively.

For this problem there exists the following, explicitly time-dependent, Hermitian integral of motion:[2]

$$K(t)=\frac{1}{4|e|}\left\{\left[\rho p_x-\left(M\dot\rho-\frac{ye}{x\rho}\right)x\right]^2+\left[\rho p_y-\left(M\dot\rho+\frac{xe}{y\rho}\right)y\right]^2\right\}, \tag{2}$$

where ρ is a real solution of the equation

$$(M/e)^2\ddot\rho(t)+\Omega^2(t)\rho(t)-\rho^{-3}(t)=0,\quad \Omega={}^1/_2H(t). \tag{3}$$

As $t\to-\infty$, K is the same as the Hamiltonian $\mathscr{H}_i$, if we choose the boundary conditions $\rho(-\infty)=\Omega^{-1/2}$, $\dot\rho(-\infty)=0$:

$$\mathscr{H}_i=|\omega_i|K(-\infty),\quad \omega_i=\frac{eH_i}{M}. \tag{4}$$

If Ω = constant, Eq. (3) has a first integral

$$\left(\frac{M}{e}\right)^2\dot\rho^2+\Omega^2\rho^2+\rho^{-2}=2|\Omega|\operatorname{ch}\delta \tag{5}$$

and the exact solution is

$$\rho(t)=\frac{1}{\sqrt{|\Omega|}}\left[\operatorname{ch}\delta\pm\operatorname{sh}\delta\sin\left(2\frac{e\Omega}{M}t+\varphi\right)\right]^{1/2}, \tag{5'}$$

where δ and φ are arbitrary real constants.

Although Eq. (3), used in [2] is non-linear, it is equivalent to the linear equation

$$\ddot\varepsilon(t)+\frac{\omega^2(t)}{4}\varepsilon(t)=0,\quad \omega(t)=\frac{eH(t)}{M},$$

$$\varepsilon=\rho\,\exp\left[i\frac{|e|}{M}\int^t\frac{1}{\rho^2}d\tau\right]. \tag{6}$$

In what follows we shall use the two equivalent Eqs. (3) and (6).

For the problem under consideration we can find two new, explicitly time-dependent, integrals of motion A(t) and B(t):

$$A(t)=\frac{e^{-i\gamma_a}}{2\sqrt{e}}\left[\rho p_x+i\rho p_y+\left(\frac{e}{\rho}-iM\dot{\rho}\right)(y-ix)\right],$$

$$B(t)=\frac{e^{-i\gamma_b}}{2\sqrt{e}}\left[\rho p_y+i\rho p_x+\left(\frac{e}{\rho}-iM\dot{\rho}\right)(x-iy)\right], \qquad (7)$$

where

$$\gamma_a=-\frac{|e|}{M}\int^t\left(\Omega+\frac{1}{\rho^2}\right)dt', \quad \gamma_b=\frac{|e|}{M}\int^t\left(\Omega-\frac{1}{\rho^2}\right)dt'.$$

One can easily check that the following commutator relations hold

$$[A,B]=[B,K]=0; \quad [A,A^+]=[B,B^+]=e/|e|. \qquad (8)$$

The operators A, A^+ and B, B^+ change places for oppositely charged particles. For the sake of simplicity we shall in what follows assume that $e > 0$.

The operators (7) commute with the operator $i\partial/\partial t - \mathscr{H}$. Using A and A^+ we can write the invariant K in the form $K = A^+A + \frac{1}{2}$. As we consider an axially symmetric problem (the field H(t) is along the z-axis) the z-component L_z of the angular momentum must be conserved. One can, indeed, check that one can express L_z as follows in terms of the integrals of motion A(t) and B(t):

$$B^+B-A^+A=L_z. \qquad (9)$$

To make clear the physical meaning of the invariants A(t) and B(t) we consider their limits as $t \to -\infty$:

$$A(-\infty)=A_i=\frac{e^{i\omega_i t}}{\sqrt{2eH_i}}\left[p_x+ip_y+\frac{eH_i}{2}(y-ix)\right],$$

$$B(-\infty)=B_i=\frac{1}{\sqrt{2eH_i}}\left[p_y+ip_x+\frac{eH_i}{2}(x-iy)\right]. \qquad (10)$$

The eigenvalues of the operators $A_i e^{-i\omega_i t}$ and B_i (these operators were used in [8] for a charged particle in a constant magnetic field) determine the coordinates of the relative motion $\mathbf{r}-\mathbf{r}_0$ and the coordinates $\mathbf{r}_0$ of the center of the orbit along which the charged particle moves in a constant magnetic field H_z. The eigenvalues of the operators A_i and B_i are integrals of motion; their physical meaning is that they give the initial conditions of the classical motion.

The invariants A(t) and B(t) have the same meaning of the initial conditions of the classical motion; the eigenvalues of the operator $B(t)e^{i\gamma_b}$ determine the "moving" coordinates of the center of the orbit and the eigenvalues of the operator $A(t)e^{i\gamma_a}$ the coordinates of the relative motion which is no longer a motion along a circle.

We must note that for a quantum oscillator with a variable frequency we can also find a time-dependent invariant:

$$A'(t)=\frac{1}{\sqrt{2M}}\left[\frac{M}{\rho}q+i(\rho p-M\dot{\rho}q)\right]\exp\left[i\int^t\rho^{-2}dt'\right],$$

where $\rho(t)$ also satisfies Eq. (3), if we assume that $\Omega(t)$ is the frequency of the oscillator and put $e = M = 1$. The eigenvalues of the operator $A'(t)$ have the meaning of the initial conditions in the qp-phase plane. We must note that it is of interest to establish the connection between the exact invariants obtained above for an oscillator with a variable frequency and for the motion of a particle in a variable magnetic field and the well-known[10] adiabatic invariants for these problems.

2. TRANSITION PROBABILITIES

A method to construct solutions of the Schrödinger equation and to find the probabilities for transitions between initial and final stationary states of a quantum system was developed in [2] by using exact Hermitian, explicitly time-dependent, invariants such as the operator K(t) for a particle in a variable magnetic field. It was shown how one can construct the eigenstates of the invariant K(t) and the solution of the Schrödinger equation, but for the sake of convenience we use the A(t) and B(t) operators.

It follows from Eqs. (8) and (9) that one can diagonalize the operators K(t) and L_z simultaneously:

$$K|n_1,n_2;t\rangle=(n_1+1/2)|n_1,n_2;t\rangle,$$
$$L_z|n_1,n_2;t\rangle=(n_2-n_1)|n_1,n_2;t\rangle. \qquad (11)$$

The states $|n_1, n_2; t\rangle$ can be constructed as follows:

$$\Phi_{n_1,n_2}\equiv|n_1,n_2;t\rangle=\frac{(A^+)^{n_1}(B^+)^{n_2}}{\sqrt{n_1!\,n_2!}}|0,0;t\rangle, \qquad (12)$$

where the vacuum state is

$$|0,0;t\rangle=\frac{1}{\rho}\sqrt{\frac{e}{\pi}}\exp\left[-\frac{e-iM\dot{\rho}\rho}{2\rho^2}(x^2+y^2)-i\frac{e}{M}\int^t\frac{dt'}{\rho^2}\right]. \qquad (13)$$

The states (12) are orthonormalized:

$$\int\Phi^*_{l_1l_2}\Phi_{j_1j_2}dxdy=\delta_{l_1,j_1}\delta_{l_2,j_2},$$

and at the same time are solutions of the Schrödinger equation $i\partial\psi/\partial t = \mathscr{H}\psi$. The action of the operators A^+, B^+, A, and B (creation and annihilation of the indices n_1 and n_2) is given by the usual formulae

$$A|n_1,n_2;t\rangle=\sqrt{n_1}|n_1-1,n_2;t\rangle,$$
$$B|n_1,n_2;t\rangle=\sqrt{n_2}|n_1,n_2-1;t\rangle. \qquad (14)$$

In the far-distant past (constant field H_i) the states (12) are the same as the time-dependent eigenstates $|n_1, n_2; i\rangle$ of the Hamiltonian $\mathscr{H}_i = \omega_i(A_i^+A_i + \frac{1}{2})$ and the angular momentum component $L_z = B_i^+B_i - A_i^+A_i$. The probability for a transition from the initial state of the system $|n_1, n_2; i\rangle$ to a final, time-dependent, state $|k_1, k_2; f\rangle$ is

$$W_{n_1,n_2}^{k_1,k_2}=|A_{n_1,n_2}^{k_1,k_2}|^2=|\langle k_1,k_2;f|n_1,n_2;\infty\rangle|^2. \qquad (15)$$

One can easily express the operators A(t) and B(t) in terms of the operators A_f, B_f, A_f^+, and B_f^+ corresponding to a constant field H_f:

$$A(t)=A_f\xi(t)\exp\{-i(\omega_f t+\gamma_a-\gamma_b)\}+\eta(t)B_f^+,$$
$$B(t)=A_f^+\eta(t)\exp\{-i(\gamma_b-\gamma_a-\omega_f t)\}+\xi(t)B_f, \qquad (16)$$

where

$$\xi(t)=\frac{1}{2\sqrt{\Omega_f}}\left[\Omega_f\rho+\frac{1}{\rho}-i\frac{M}{e}\dot{\rho}\right]e^{-i\gamma_b},$$

$$\eta(t)=\frac{1}{2\sqrt{\Omega_f}}\left[i\left(\Omega_f\rho-\frac{1}{\rho}\right)-\frac{M}{e}\dot{\rho}\right]e^{-i\gamma_a}. \qquad (17)$$

The commutation relations (8) impose the condition

$$|\eta|^2-|\xi|^2=-1, \qquad (18)$$

which is identically satisfied.

As $t \to \infty$ the invariant K(t) remains time-dependent and $[K(\infty), \mathscr{H}_f] \neq 0$. The states $|n_1, n_2; \infty\rangle$ are therefore superpositions of the states $|k_1, k_2; f\rangle$:

$$|n_1, n_2; \infty\rangle = \sum_{k_1, k_2=0}^{\infty} A_{n_1, n_2}^{k_1, k_2} |k_1, k_2; f\rangle. \quad (19)$$

It is more convenient to find first the amplitude $A_{00}^{k_1 k_2}$. Putting $n_1 = n_2 = 0$ in (19) and operating from the left with the annihilation operators $A(\infty)$ and $B(\infty)$ we easily find for the probability for the transition $|0, 0: i\rangle \rightarrow |k_1, k_2; f\rangle$

$$W_{0,0}^{k_1, k_2} = \frac{2(\text{ch}\,\delta - 1)^{k_1}}{(\text{ch}\,\delta + 1)^{k_1+1}} \delta_{k_1, k_2}, \quad (20)$$

where we have used the formulae $|\xi|^2 = \frac{1}{2}(\cosh\delta + 1)$ and $|\eta|^2 = \frac{1}{2}(\cosh\delta - 1)$ which follow easily from (17), if we use (5). The final expression for the state $|0, 0; \infty\rangle$ can be written in the form

$$|0, 0; \infty\rangle = \frac{1}{|\xi|} \sum_{k_1=0}^{\infty} \left(-\frac{\eta}{\xi}\right)^{k_1} |k_1, k_2; f\rangle\, \delta_{k_1, k_2}. \quad (21)$$

Here and henceforth we shall understand by ξ, η the constants $\xi(\infty)$, $\eta(\infty)$.

Using (16), letting the operators $[A^+(\infty)]^{n_1}[B^+(\infty)]^{n_2}$ act upon (21), and summing the appropriate series we can now obtain the amplitudes $A_{n_1,n_2}^{k_1,k_2}$:

a) $n_1 > n_2, \quad k_2 \geqslant n_2,$

$$A_{n_1,n_2}^{k_1,k_2} = (-1)^{k_1-n_1} \frac{\eta^{k_1-n_1}\xi^{-k_2-n_1}}{|\xi|\sqrt{n_1!k_2!/n_2!k_1!}} P_{n_2}^{k_1-n_1,\,-k_2-n_1-1}(\text{ch}\,\delta); \quad (22a)$$

b) $n_1 > n_2, \quad k_2 < n_2,$

$$A_{n_1, n_2}^{k_1, k_2} = \frac{(\eta^*)^{n_1-k_1}\xi^{-k_2-n_1}}{|\xi|\sqrt{n_2!\,k_1!/n_1!\,k_2!}} P_{k_2}^{n_1-k_1,\,-k_2-n_1-1}(\text{ch}\,\delta), \quad (22b)$$

where the $P_n^{\alpha,\beta}(x)$ are Jacobi polynomials and where we have used the relation $n_1 - n_2 = k_1 - k_2 = -L_z$. The amplitudes (22a) and (22b) are, of course, time-independent. There remain two cases: c) $n_1 \leq n_2$, $k_2 \geq n_2$, and $n_1 \leq n_2$, $k_2 < n_2$; they can be described by the same formulae by swapping the indices $1 \rightleftarrows 2$

The transition probability (15) depends solely on $\cosh\delta$:

$$W_{n_1, n_2}^{k_1, k_2} = 2^{2n_1+k_2-k_1+1} \frac{k_1!\,n_2!\,(\text{ch}\,\delta - 1)^{k_1-n_1}}{k_2!\,n_1!\,(\text{ch}\,\delta + 1)^{k_2+n_1+1}} |P_{n_2}^{k_1-n_1,\,-k_2-n_1-1}(\text{ch}\,\delta)|^2. \quad (23)$$

The case b) is obtained from a) by the switch $n_i \rightleftarrows k_i$. The symmetry of Eq. (23), $W_{n_1,n_2}^{k_1,k_2} = W_{n_2,n_1}^{k_2,k_1}$ and $W_{n_1,n_2}^{k_1,k_2} = W_{k_2,k_1}^{n_2,n_1}$ corresponds to symmetry under time reversal. This also occurs, even if $H(-t) \neq H(t)$ as $\cosh\delta$ does not change, as we shall show below, when the time is reversed. Equation (23) gives us an expression for the probabilities for a transition from an arbitrary initial to an arbitrary final state in terms of a single parameter—the probability to remain in the ground state, as $\cosh\delta$ can be expressed in terms of $W_{0,0}^{0,0}$, as is clear from Eq. (20). Using the analogy of Eq. (6) with the Schrödinger equation for one-dimensional motion (if we replace t by x[1]) we can connect $\cosh\delta$ with coefficient R for reflection from an appropriate one-dimensional potential barrier. The solution of Eq. (6) when $t \rightarrow \infty$ is

$$\varepsilon(t) = \frac{\xi}{\sqrt{\Omega_f}} \exp\left(i\frac{\omega_f}{2}t\right) - \frac{i\eta}{\sqrt{\Omega_f}} \exp\left(-i\frac{\omega_f}{2}t\right), \quad (24)$$

where ξ and η are arbitrary constants, $|\xi|^2 - |\eta|^2 = 1$. We easily see that $R = |\eta/\xi|^2$ after which we find from (24) and (17)

$$\text{ch}\,\delta = (1 + R)\,/\,(1 - R). \quad (25)$$

Time reversal ($\omega(t) \rightarrow \omega(-t)$) corresponds to changing to a wave incident from the right, but in that case the reflection coefficient R is unchanged[11] which explains the additional symmetry $W_{n_1,n_2}^{k_1,k_2} = W_{k_2,k_1}^{n_2,n_1}$.

It follows from the equivalence of Eqs. (3) and (6) that the solution of the problem of the motion in a variable magnetic field is completely determined by the constants ξ and η and the transition probabilities by the reflection coefficient R. We give without derivation the values for some particular cases for the time-dependence $\omega(t)$ of the frequency. In the adiabatic limit ($\dot{\omega}/\omega^2 \ll 1$) $R = 0$. There is also no reflection in the case when $\omega^2(t) = g^{-4}(t) - \ddot{g}(t)/g(t)$, where the arbitrary function g(t) satisfies the conditions $g(-\infty) = (\frac{1}{2}\omega_i)^{-1/2}$, $g(\infty) = (\frac{1}{2}\omega_f)^{-1/2}$. In that case Eq. (6) has the exact solution

$$\varepsilon(t) = \text{const}\cdot g(t) \exp\left[i\int^t g^{-2}\,dt'\right].$$

When

$$R = 0 \qquad W_{n_1, n_2}^{k_1, k_2} = \delta_{n_1, k_1}\delta_{n_2, k_2},$$

i.e., the system remains in the initial quantum state. For an Eckart potential [12]

$$R = \left[\text{ch}\left(\frac{2MlA}{\omega_i + \omega_f}\right) + \cos\sqrt{\pi^2 - 2Ml^2B}\right] \cdot \left[\text{ch}\left(\frac{2MlA}{\omega_i - \omega_f}\right) + \cos\sqrt{\pi^2 - 2Ml^2B}\right]^{-1},$$

where A, B, and l are parameters of the potential. When $l \ll \omega_i, \omega_f$ we can go over from the Eckart potential to a steep discontinuity in $\omega(t)$. In that case $R = (\omega_f - \omega_i)^2/(\omega_f + \omega_i)^2$. For an oscillator with a variable frequency this case was given in [9] (the solutions of the problem of an oscillator are also determined by Eq. (3) of [2] or Eq. (6) of [4]).

3. COHERENT STATES AND THEIR EVOLUTION

Normalized coherent states of a charged particle moving in a uniform constant magnetic field were introduced in [8]. They can be written in the form

$$|\alpha_0, \beta_0\rangle = \exp\left[-\frac{|\alpha_0|^2}{2} - \frac{|\beta_0|^2}{2}\right] \sum_{n_1, n_2=0}^{\infty} \frac{\alpha_0^{n_1}\beta_0^{n_2}}{\sqrt{n_1!\,n_2!}} |n_1, n_2\rangle, \quad (26)$$

where the $|n_1, n_2\rangle$ are the time-dependent eigenstates of the Hamiltonian and the z-component of the angular momentum, and α_0 and β_0 are arbitrary complex numbers with the following physical meaning: $|\alpha_0|$ determines the radius of the classical orbit, and β_0 the coordinates of the center of the orbit.

Using the operators (7) and acting upon $|0, 0; t\rangle$ with the Weyl operators $D(\alpha_0)D(\beta_0)$ we can construct normalized coherent states for the case with a variable

[1] This is an idea of L. P. Pitaevskii's (see reference to this in [4]).

magnetic field

$$|\alpha_0, \beta_0; t\rangle = \exp\left[-\frac{|\alpha_0|^2}{2} - \frac{|\beta_0|^2}{2}\right] \sum_{n_1, n_2=0} \frac{\alpha_0^{n_1}\beta_0^{n_2}}{\sqrt{n_1!\,n_2!}} |n_1, n_2; t\rangle, \quad (27)$$

which in Cartesian coordinates have the form

$$|\alpha_0, \beta_0; t\rangle = |0, 0; t\rangle \exp\left[-\frac{|\alpha|^2}{2} - \frac{|\beta|^2}{2} + \frac{\sqrt{e}}{\rho}\beta(x+iy) \quad (27') + i\frac{\sqrt{e}}{\rho}\alpha(x-iy) - i\alpha\beta\right],$$

$$\alpha = \alpha_0 e^{i\gamma_a}, \quad \beta = \beta_0 e^{i\gamma_b}.$$

We construct in this way also the Green function for our problem*

$$G(\mathbf{r}_2 z_2 t_2;\, \mathbf{r}_1 z_1 t_1) = (2\pi i(t_2 - t_1)/M)^{-1/2}(2\pi i \rho_2 \rho_1 \sin\gamma)^{-1}$$
$$\times \exp\frac{i}{2}\{[\rho_2\dot{\rho}_2 R_2^2 - \rho_1\dot{\rho}_1 R_1^2]M/e + 2[\mathbf{R}_1\mathbf{R}_2]_z\}$$
$$\times \exp\frac{i}{2}\{M(z_2 - z_1)^2/(t_2 - t_1) + (\mathbf{R}_2 - \mathbf{R}_1)^2 \operatorname{ctg}\gamma\};$$
$$\gamma = \int_{t_1}^{t_2} d\tau/\rho^2; \quad R_k = \sqrt{e}\, r_k/\rho_k; \quad \varphi_{R_k} = \varphi_{r_k} + \gamma_b(t_k), \quad k = 1, 2.$$

The operators A(t) and B(t) act upon the states (27) according to the formulae

$$A|\alpha_0, \beta_0; t\rangle = \alpha_0|\alpha_0, \beta_0; t\rangle, \quad B|\alpha_0, \beta_0; t\rangle = \beta_0|\alpha_0, \beta_0; t\rangle. \quad (28)$$

These coherent states satisfy the Schrödinger equation in time and describe the most "classical" state in the same sense as the coherent states of a charged particle in a constant magnetic field. The center of the packet (27) moves along the classical orbit.

For a constant field H_i ($t \to -\infty$) the states (27) are the same as the initial coherent states $|\alpha_0, \beta_0; i\rangle$. We get at once from Eq. (21) the final expression for the coherent states (27) in the form

$$|\alpha_0, \beta_0; \infty\rangle = \frac{1}{|\xi|}\exp\left[-\frac{|\alpha_0|^2}{2} - \frac{|\beta_0|^2}{2} + \alpha_0 A^+(\infty) + \beta_0 B^+(\infty)\right]$$
$$\times \sum_{m=0}^{\infty}\left(-\frac{\eta}{\xi}\right)^m \frac{1}{m!}(A_f^+)^m (B_f^+)^m |0, 0; f\rangle. \quad (29)$$

One verifies easily that the following formula holds:

$$[A^+{}_f B^+{}_f]^m \exp[{}^1\!/_2(|\alpha_0|^2 + |\beta_0|^2)]|\alpha_0, \beta_0; f\rangle$$
$$= \left\{\left(\frac{d^2}{dz_a\,dz_b}\right)^m \exp[{}^1\!/_2(|z_a|^2 + |z_b|^2)]\,|z_a, z_b; f\rangle\right\}_{z_a=\alpha_0, z_b=\beta_0} \quad (30)$$

Using (16), the definition (26), and Eq. (30) we get after a few transformations

$$|\alpha_0, \beta_0; \infty\rangle = \frac{1}{|\xi|}\exp\left[-\frac{|\alpha_0|^2}{2} - \frac{|\beta_0|^2}{2} - \alpha_0\beta_0\xi^*\eta^*\right]\cdot\left\{\exp\left(-\frac{\eta}{\xi}\frac{d^2}{dz_a dz_b}\right)\right.$$
$$\left.\times \exp[\alpha_0\eta^* z_b + \beta_0\eta^* z_a + {}^1\!/_2(|z_a|^2 + |z_b|^2)]\,|(z_a, z_b; f\rangle\right\}_{z_a=\alpha_0\xi^* e^{i(\gamma_a - \gamma_b)},\ z_b=\beta_0\xi^*}. \quad (31)$$

The final expression for the state $|\alpha_0, \beta_0; i\rangle$ was obtained as a series in the parameter η/ξ. Equation (31) also gives an expansion of $|\alpha_0, \beta_0; \infty\rangle$ in terms of the final coherent states $|\alpha_0', \beta_0'; f\rangle$. In the adiabatic approximation we must put[2] $\rho(t) = \Omega^{-1/2}(t)$ and Eq. (31) then gives

$$|\alpha_0, \beta_0; \infty\rangle = \left|\alpha_0 \exp\left[-\int^t \omega(t')\,dt'\right], \ \beta_0; f\right\rangle, \quad (32)$$

whence it follows that in the adiabatic limit the initial coherent states change to coherent final states, where the center of the orbit does not change, while the radius of the orbit changes in correspondence with the change in frequency, while α_0 is invariant.

Using (21) and (27) we can easily evaluate the following amplitudes:

$$A^{0,0}_{\alpha_0, \beta_0} = \langle 0, 0; f|\alpha_0, \beta_0; t\rangle = \frac{1}{|\xi|}\exp\left[-\frac{|\alpha_0|^2}{2} - \frac{|\beta_0|^2}{2} + \alpha_0\beta_0\frac{\eta^*}{\xi}\right], \quad (33)$$

and also

$$A^{\mu_0, \nu_0}_{\alpha_0, \beta_0} = \langle \mu_0, \nu_0; f|\alpha_0, \beta_0; t\rangle = A^{0,0}_{\alpha', \beta'}\exp[i\,\mathrm{Im}(\alpha_0\mu_0^*\xi^* + \alpha_0\nu_0\eta^* + \beta_0\mu_0\eta^* + \beta_0\nu_0^*\xi^*)], \quad (34)$$

where $\alpha' = \alpha_0 - \mu_0\xi - \nu_0^*\eta$, $\beta' = \beta_0 - \mu_0^*\eta - \nu_0\xi$.

Using the explicit form of the coherent states (27′) and comparing it with the coherent states for a constant field[8] we find easily the explicit form of the functions (compare [2])

$$|n_1, n_2; t\rangle = |0, 0; t\rangle(-1)^p i^{n_1}\sqrt{\frac{p!}{(p + |n_1 - n_2|)!}}\left(\frac{e}{\rho^2}r^2\right)^{|n_1 - n_2|/2}$$
$$\times L_p^{|n_1 - n_2|}\left(\frac{e}{\rho^2}r^2\right)\exp\{i[(n_2 - n_1)\varphi + n_1\gamma_a + n_2\gamma_b]\}, \quad (35)$$

where $p = \frac{1}{2}(n_1 - |n_2 - n_1| + n_2)$; φ, r are polar coordinates, and $L_n^s(x)$ Laguerre polynomials. We can also calculate the transition amplitude for the from a coherent state to a Landau level:

a) $n_1 \geq n_2$,

$$A^{n_1, n_2}_{\alpha_0, \beta_0} = A^{0,0}_{\alpha_0, \beta_0}\sqrt{\frac{n_2!}{n_1!}}\left(-\frac{\eta}{\xi}\right)^{n_2}\left(\frac{\alpha_0}{\xi}\right)^{n_1 - n_2} L_{n_2}^{n_1 - n_2}\left(\frac{\alpha_0\beta_0}{\xi\eta}\right). \quad (36)$$

b) When $n_1 < n_2$ Eq. (36) holds with the substitution $\alpha_0 \rightleftarrows \beta_0$, $n_1 \rightleftarrows n_2$. In the coherent state (27) the distributions with respect to the quantum numbers n_1 and n_2 which do not change with time are Poisson distributions (compare [8]).

In conclusion we note that in the problem discussed here there is a magneto-translational symmetry group, the operators of which are the Weyl operators $D(\hat{A})$ and $D(\hat{B})$ and, moreover, the whole spectrum of the problem is combined in one irreducible representation of the dynamic U(2, 1) group. These statements are a generalization of the case of a constant magnetic field.[13] On the basis of what has been said we can give Eq. (23) for the probability a group theoretic meaning and also construct the Bloch and Wannier functions of the given problem. Details of these results will be published in another paper.

By diagonalizing the operators $\frac{1}{2}(B + B^+)$ and K one can easily construct a generalization of the Landau solutions for a constant field.[11]

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*$[\mathbf{R}_1\mathbf{R}_2] \equiv \mathbf{R}_1 \times \mathbf{R}_2$.

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