ASYMPTOTIC FORM OF THE VERTEX FUNCTION IN THE $\alpha(\alpha L)^n$ APPROXIMATION

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The asymptotic form of the renornalized vertex function $\Gamma_\lambda(p, q)$ in the region $p^2 \gg q^2 \sim m^2$ is found to the accuracy of linear terms in m/p and q/p, in the so-called $\alpha(\alpha L)^n$ approximation.

1. In a previous paper [1] (hereafter cited as I) the problem is formulated of finding the asymptotic forms of the renormalized electrodynamic functions $D_{\mu\nu}(k)$, G(p), and $\Gamma_\lambda(p, q)$ in the high-momentum region in the $\alpha(\alpha L)^n$ approximation, and the corresponding asymptotic forms of the Green's functions $D_{\mu\nu}(k)$ and G(p) are found. In the present paper we shall find the $\alpha(\alpha L)^n$ asymptotic renormalized vertex function $\Gamma_\lambda(p, q)$ in the region[2)] $p^2 \gg q^2 \sim m^2$, to the accuracy of terms linear in m/p and q/p. We shall subsequently use these asymptotic forms to investigate the question of the existence and properties of solutions of the "superconducting type" in quantum electrodynamics [5] and problems of the self-consistency of the renormalized theory [6,5] when the terms of the $\alpha(\alpha L)^n$ approximation are included, but they are of course of wider significance.

2. The stated problem is here solved by the method of renormalized Dyson equations. This method was used in [2,7] to find the asymptotic forms of the electrodynamic functions in the $(\alpha L)^n$ approximation. In this approximation the only vertex diagram that contributes is the simplest (three-gamma) one, and because of this the system of Dyson equations terminates and gives a convenient basis for the asymptotic calculations. It can be shown that in the $\alpha(\alpha L)^n$ approximation there is only one diagram beyond the "three-gamma" one $\Gamma_\lambda^{(3)}$ that contributes, namely the "five-gamma" diagram $\Gamma_\lambda^{(5)}$ (Fig. 1), and therefore the system of Dyson equations again terminates. The equation for Γ_λ is then of the form

$$\Gamma_\lambda(p, q) = \gamma_\lambda + \Gamma_\lambda^{(3)}(p, q) + \Gamma_\lambda^{(5)}(p, q), \tag{1}$$

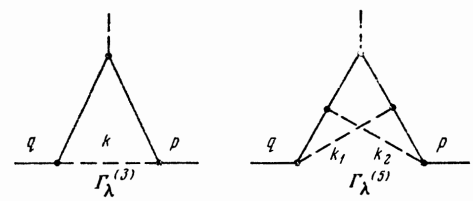
where $\Gamma_\lambda^{(3)}$ and $\Gamma_\lambda^{(5)}$ are expressed in terms of Γ_ν and the Green's functions $D_{\mu\nu}$ and G in accordance with the diagrams in the figure.

As in I, we shall represent any of the functions considered in the form

$$f = f_{(0)} + \tilde{f}, \tag{2}$$

where $f_{(0)}$ is the contribution of the $(\alpha L)^n$ approximation and $\tilde{f}$ is that of the $\alpha(\alpha L)^n$ approximation. The equation for $\tilde{\Gamma}_\lambda$ is obtained from the equation (1) for Γ_λ by separating out the $\alpha(\alpha L)^n$ parts in each term:

$$\tilde{\Gamma}_\lambda(p, q) = \tilde{\Gamma}_\lambda^{(3)}(p, q) + \tilde{\Gamma}_\lambda^{(5)}(p, q). \tag{3}$$



In accordance with the diagrams in the figure we have

$$\Gamma_\lambda^{(3)}(p, q) = \frac{\alpha}{4\pi^3} \int \Gamma_\mu(p, p-k) G(p-k) \Gamma_\lambda(p-k, q-k) G(q-k) \times \Gamma_\nu(q-k, q) D_{\mu\nu}(k)\, d^4k, \tag{4}$$

$$\Gamma_\lambda^{(5)}(p, q) = \left(\frac{\alpha}{4\pi^3}\right)^2 \int \Gamma_\mu G \Gamma_\sigma G \Gamma_\lambda G \Gamma_\nu G \Gamma_\tau D_{\mu\nu}(k_1) D_{\sigma\tau}(k_2)\, d^4k_1\, d^4k_2 \tag{5}$$

[for brevity we have omitted the arguments of the functions G and Γ in (5); they can be readily supplied from the diagrams].

The photon and electron Green's functions appearing in (4) and (5) are written in the form[3)]

$$D_{\mu\nu}(k) = \frac{d(k)}{ik^2}\left(\delta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\right) + d_l \frac{k_\mu k_\nu}{ik^4}, \tag{6}$$

$$G^{-1}(p) = -\hat{p}A(p) + imB(p). \tag{7}$$

The scalar functions d, A, and B are found in I to terms in $\alpha(\alpha L)^n$ [Eqs. (8) and (62)–(64)].

To obtain $\tilde{\Gamma}_\lambda^{(3)}$ from (4), after putting each of the functions in the integrand in the form (2), it is sufficient to keep: first, the product of all the factors $f_{(0)}$ (we denote this integral by I_λ), and second, the product of five factors $f_{(0)}$ times one of the factors $\tilde{f}$ (we denote these integrals by $I^{(1)}, \ldots, I^{(6)}$, where the upper index corresponds to the order number of the function taken with the tilde, counting from left to right). To get $\tilde{\Gamma}_\lambda^{(5)}$ from (5) it suffices to take all of the functions in the integral in the $(\alpha L)^n$ approximation (i.e., to take only the product of the terms $f_{(0)}$). The resulting form taken by Eq. (3) is

$$\tilde{\Gamma}_\lambda(p, q) = I_\lambda(p, q) + \sum_{i=1} I_\lambda^{(i)}(p, q) + \tilde{\Gamma}_\lambda^{(5)}(p, q). \tag{8}$$

[1)] In this approximation, in expansions with respect to the renormalized coupling constant α one includes all terms of the form $\alpha(\alpha L)^n$, where L is a general symbol for logarithms of the various large values of the momenta in question, of the type of $\ln(p^2/m^2)$, $\ln(k^2/m^2)$, and so on. It comes next after the "$(\alpha L)^n$ approximation", in which one includes all terms of the form $(\alpha L)^n$.

[2)] We confine ourselves to the consideration of $\Gamma_\lambda(p, q)$ in the region of "spacelike" vectors p and q satisfying the "triangle rule." [2] In this region there do not arise any "doubly logarithmic" terms of the form $(\alpha L^2)^n$, which are characteristic for the region of "nonspacelike" vectors, for example $pq \gg p^2, q^2$. [3,4] It can be shown [2] that the latter region "is ineffective" in the equations that determine the function $\Gamma_\lambda(p, q)$ in the region of "spacelike" vectors.

[3)] We shall conduct the calculations in an arbitrary gauge, but for simplicity take d_l independent of the momentum k. The metric and the matrices γ_μ are the same as in [8].

3. Let us consider the integral I_λ:

$$I_\lambda(p,q) = \frac{\alpha}{4\pi^3}\int \Gamma_{(0)\mu}(p, p-k) G_{(0)}(p-k)\Gamma_{(0)\lambda}(p-k, q-k) \times G_{(0)}(q-k)\Gamma_{(0)\nu}(q-k, q) D_{(0)\mu\nu}(k)\, d^4k. \quad (9)$$

This integral leads both to main $(\alpha L)^n$ terms, which arise in the "logarithmic" integration and have been found earlier (see [2,7,5]) and to the $\alpha(\alpha L)^n$ terms which we now require, and which arise in the "nonlogarithmic" integration. Confining ourselves to the calculation of the $\alpha(\alpha L)^n$ terms only, we shall not be interested in the $(\alpha L)^n$ part of the integral (9); this simplifies the further arguments.

The functions occurring in (9) are of the forms

$$D_{(0)\mu\nu}(k) = \frac{d_{(0)}(k)}{ik^2}\left(\delta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\right) + d_l\frac{k_\mu k_\nu}{ik^4}$$

$$d_{(0)}(k) = \eta^{-1}, \quad \eta = 1 - \frac{\alpha}{3\pi}\ln\frac{k^2}{m^2}, \quad k^2 \gg m^2; \quad (10)$$

$$\Gamma_{(0)\nu}(q-k, q) \approx \Gamma_{(0)\nu}(-k, q) = \gamma_\nu A_{(0)}(k) - \frac{k_\nu}{k^2}\{\hat{q}[A_{(0)}(k) - 1] - im[B_{(0)}(k) - 1]\} - \frac{\hat{q}(\hat{k}\gamma_\nu - \gamma_\nu\hat{k})}{2k^2}A_{(0)}(k)(1 - \eta^{3/4}), \quad (11)$$

$$A_{(0)}(k) = \exp\left\{-\frac{\alpha}{4\pi}d_l\ln\frac{k^2}{m^2}\right\}, \quad B_{(0)}(k) = A_{(0)}(k)\eta^{3/4};$$

$$G_{(0)}(q-k) = [(\hat{k}-\hat{q})A_{(0)}(k) + imB_{(0)}(k)]^{-1} \approx \frac{(\hat{k}-\hat{q})A_{(0)}(k) - imB_{(0)}(k)}{A^2_{(0)}(k)[(k-q)^2 + m^2]}. \quad (12)$$

In the last expression use has been made of the fact that the term $m^2B^2_{(0)}(k)$ in the denominator is important only in the region $k^2 \lesssim q^2 \sim m^2$, in which $B_{(0)}(k) \approx A_{(0)}(k)$.

As for the functions $\Gamma_{(0)\lambda}$, $G_{(0)}$, and $\Gamma_{(0)\mu}$, which contain the momentum p, they can be effectively simplified in view of the following remarks. In the integration of the logarithmic terms (containing $\ln(k^2/m^2)$ the $\alpha(\alpha L)^n$ contribution we want arises in the regions $k^2 \gg p^2$ and $p^2 \gg k^2 \gg q^2 \sim m^2$, in which the integration effectively reduces [see the corresponding analysis of the integral (20) in I] to making the replacement $\ln k^2 \to \ln p^2$ in the integrand. The form of the function $\Gamma_{(0)\lambda}(p-k, q-k)$ in these regions is[4)]

$$\Gamma_{(0)\lambda}(p-k, q-k) \approx \Gamma_{(0)\lambda}(-k, -k) = \gamma_\lambda A_{(0)}(k), \quad k^2 \gg p^2, \quad (13)$$

$$\Gamma_{(0)\lambda}(p-k, q-k) \approx \Gamma_{(0)\lambda}(p, -k) = \gamma_\lambda A_{(0)}(p) - \frac{p_\lambda}{p^2}\{\hat{k}[A_{(0)}(p) - A_{(0)}(k)] + im[B_{(0)}(p) - B_{(0)}(k)]\} - \frac{\hat{k}(\hat{p}\gamma_\lambda - \gamma_\lambda\hat{p})}{2p^2}A_{(0)}(p)\left(1 - \frac{\xi^{3/4}}{\eta^{3/4}}\right) + O\left(\frac{k^2}{p^2}\alpha\ln\frac{p^2}{k^2}\right),$$

$$\xi = 1 - \frac{\alpha}{3\pi}\ln\frac{p^2}{m^2}. \quad (13')$$

where

$$p^2 \gg k^2 \gg m^2,$$

When we also use the effective replacement $\ln k^2 \to \ln p^2$ in the expression (13′), we can neglect all the terms except the first, to the accuracy required. In view of the fact that we also have $A_{(0)}(k) \to A_{(0)}(p)$ in (13), we conclude that in the entire region of integration in (9) we can effectively set

$$\Gamma_{(0)\lambda}(p-k, q-k) \to \gamma_\lambda A_{(0)}(p). \quad (14)$$

Similarly,

$$\Gamma_{(0)\mu}(p, p-k) \to \gamma_\mu A_{(0)}(p), \quad (15)$$

$$G_{(0)}(p-k) \to [(\hat{k}-\hat{p})A_{(0)}(p) + imB_{(0)}(p)]^{-1} \approx \frac{(\hat{k}-\hat{p})A_{(0)}(p) - imB_{(0)}(p)}{A_{(0)}^2(p)(k-p)^2}. \quad (16)$$

We note that since we are calculating $\tilde{\Gamma}_\lambda(p,q)$ to the accuracy of terms linear in q/p and m/p all the functions in (9) can be taken only to this same accuracy.

As the result of all these simplifications the integral (9) takes the form

$$I_\lambda(p,q) = \frac{\alpha}{4\pi^3}A^2_{(0)}(p)\int \gamma_\mu G_{(0)}(p-k)\gamma_\lambda G_{(0)}(q-k)\Gamma_{(0)\nu}(-k,q)D_{(0)\mu\nu}(k)\,d^4k. \quad (17)$$

Substituting the expressions (10)–(12) and (16) in (17) and performing the integration by means of the formulas in Appendix I, we get

$$I_\lambda(p,q) = \frac{\alpha}{4\pi}\frac{A_{(0)}(p)}{p^2}\left\{\left(\frac{1}{2}\gamma_\lambda p^2 + \hat{p}\gamma_\lambda\hat{p}\right)\frac{1}{\xi} - \left(1 - \frac{m^2}{q^2} + \frac{d_l}{2}\right)\hat{q}\gamma_\lambda\hat{p} - \frac{d_l}{2A_{(0)}(p)}\hat{p}\gamma_\lambda\hat{q} + \left[\xi^{-1/4} + \frac{d_l}{A_{(0)}(p)} + (d_l - 1)\left(1 + 2\frac{m^2}{q^2}\right)\right]\frac{\hat{q}p_\lambda}{2} + \xi^{-1/4}p_\lambda\frac{\hat{p}\hat{q}\hat{p}}{p^2} + \frac{im}{2}\left[(d_l-1)\left(1 + 2\frac{m^2}{q^2}\right)\frac{\hat{q}\hat{p}\gamma_\lambda\hat{q}}{q^2} + \left(\xi^{5/4} - 2\xi^{3/4}d_l + \frac{d_l}{A_{(0)}(p)}\right)\hat{p}\gamma_\lambda\right] + \frac{imp_\lambda}{2}\left[7(1-\xi^{5/4}) + d_l(1 - A^{-1}_{(0)}(p)) + 2(1-d_l)\frac{m^2}{q^2}\right]\right\}. \quad (18)$$

4. Before going on to the consideration of the integrals $I^{(i)}$ and $\tilde{\Gamma}^{(5)}_\lambda$, let us separate out in $\tilde{\Gamma}_\lambda(p,q)$ the terms of zeroth order in the parameters m/p and q/p:

$$\tilde{\Gamma}_\lambda(p,q) = \gamma_\lambda A_1(p) + \frac{\hat{p}\gamma_\lambda\hat{p}}{p^2}A_2(p) + \left(\text{terms} \sim \frac{m}{p}, \frac{q}{p}\right). \quad (19)$$

It is not hard to show that $I^{(i)}_\lambda$ and $\tilde{\Gamma}^{(5)}_\lambda$ do not contribute to $A_2(p)$ (this is because the integration in them is logarithmic). Therefore from (18) we have

$$A_2(p) = \frac{\alpha}{4\pi}A_{(0)}(p)\xi^{-1}. \quad (20)$$

We now use the generalized Ward identity [9] for the renormalized functions, and separate out the $\alpha(\alpha L)^n$ terms in them:

$$(p-q)_\lambda\tilde{\Gamma}_\lambda(p,q) = \tilde{G}^{-1}(q) - \tilde{G}^{-1}(p) = \hat{p}\tilde{A}(p) - \hat{q}\tilde{A}(q) - im(\tilde{B}(p) - \tilde{B}(q)). \quad (21)$$

Substituting in (21) the expression (19) for $\tilde{\Gamma}_\lambda$ and using (20) and the result (62) of I for A(p), we find the explicit form of $A_1(p)$ (we note that in this way we at once get the renormalized function):

$$A_1(p) = \tilde{A}(p) - A_2(p) = \frac{\alpha}{4\pi}A_{(0)}(p)\left[\sigma - \frac{21}{8} + \frac{13}{8}\xi^{-1}\right],$$

$$\sigma = (d_l - 3)\ln\frac{\lambda^2}{m^2} - 3. \quad (22)$$

Accordingly, we have no further interest in the terms in $I^{(i)}_\lambda$ and $\Gamma^{(5)}_\lambda$ that contribute to $A_1(p)$, and shall calculate only the terms ~m/p, q/p.

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4)The expression (13′) for $\Gamma_{(0)\lambda}(p, -k)$ in the region $p^2 \gg k^2 \gg m^2$ can easily be obtained by a method analogous to that with which Eq. (A.19) of [5] was derived. For this it is sufficient to make obvious changes, associated with the condition $k^2 \gg m^2$, in Eqs. (28)–(32) and (A.14) of [5] [here the lower limit zero in the integral of Eq. (A.14) is replaced by η].

5. Let us consider $\widetilde{\Gamma}_\lambda^{(5)}$. The corresponding integral is given by the expression (5), in which all functions are to be taken in the $(\alpha L)^n$ approximation, and the calculation is to be made to accuracy $\alpha(\alpha L)^n$. Here the integration over one of the variables in (5) is "logarithmic" and the second is "nonlogarithmic." An analysis analogous to that carried out for the integral (9) enables us to bring the part of the integral (5) that gives an $\alpha(\alpha L)^n$ contribution of order m/p and q/p into the following form:

$$\widetilde{\Gamma}_\lambda^{(5)}(p,q)=\left(\frac{\alpha}{4\pi^3}\right)^2 A_{(0)}(p)\int\gamma_\mu S(p-k_1)\gamma_\sigma S(p-k_1)\gamma_\lambda S(-k_1)$$
$$\times\gamma_\nu D_{(0)\mu\nu}(k_1)\,d^4k_1\cdot J_\sigma(p,q)=\left\{\frac{\partial}{\partial p_\sigma}[\Gamma_{(0)\lambda}(p,0)+I_\lambda(p,0)]\right\}J_\sigma(p,q), \quad (23)$$

where

$$S(p)=(im-\hat{p})^{-1},$$
$$J_\sigma(p,q)=\frac{\alpha}{4\pi^3}\int_q^p G_{(0)}(q-k_2)\Gamma_{(0)\tau}(q-k_2,q)D_{(0)\sigma\tau}(k_2)\,d^4k_2$$
$$=\frac{im}{4}\gamma_\sigma\left(\xi^{3/4}-\frac{1}{A_{(0)}(p)}\right)+\frac{1}{2}\left(q_\sigma+\frac{1}{2}\hat{q}\gamma_\sigma\right)(\xi^{3/4}-1)$$
$$+\frac{1}{4}\gamma_\sigma\hat{q}\left(\frac{1}{A_{(0)}(p)}-1\right) \quad (24)$$

(in the last integral we have kept only the $(\alpha L)^n$ terms); here $I_\lambda(p,0)$ is given by the first two terms of the expression (18), and $\Gamma_{(0)\lambda}(p,0) = \gamma_\lambda A_{(0)}(p)$. The final expression for $\widetilde{\Gamma}_\lambda^{(5)}$ (with only the terms ~m/p, q/p) is

$$\widetilde{\Gamma}_\lambda^{(5)}(p,q)=\frac{\alpha}{4\pi}\frac{A_{(0)}(p)}{p^2}\Big\{\hat{q}p_\lambda[\gamma(\xi)d_+ +(A_{(0)}^{-1}(p)-1)d_-]$$
$$+\frac{\hat{p}\gamma_\lambda\hat{q}}{2}[\gamma(\xi)d_- +(A_{(0)}^{-1}(p)-1)d_+]-\left[\hat{q}\gamma_\lambda\hat{p}d_l+2\hat{p}q_\lambda d_- -2p_\lambda\frac{\hat{p}\hat{q}\hat{p}}{p^2}\xi^{-1}\right]\gamma(\xi)$$
$$+im\left[\frac{\hat{p}\gamma_\gamma}{2}d_+ +p_\lambda d_-\right](\xi^{3/4}-A_{(0)}^{-1}(p))\Big\},$$
$$\gamma(\xi)\equiv 1-\xi^{3/4},\quad d_\pm\equiv\xi^{-1}\pm d_l. \quad (25)$$

6. We now go on to $I_\lambda^{(i)}(p,q)$. We write out explicitly only the expression for $I_\lambda^{(1)}$:

$$I_\lambda^{(1)}(p,q)=\frac{\alpha}{4\pi^3}\int\widetilde{\Gamma}_\mu(p,p-k)G_{(0)}(p-k)\Gamma_{(0)\lambda}(p-k,q-k)$$
$$\times G_{(0)}(q-k)\Gamma_{(0)\nu}(q-k,q)D_{(0)\mu\nu}(k)\,d^4k.$$

The other $I_\lambda^{(i)}$ differ from this only in having the tilde on the function whose position corresponds to the value of the index (1), instead of on the Γ_μ (i = 1).

The $I_\lambda^{(i)}$ contain one more factor α than the I_λ, and therefore the $\alpha(\alpha L)^n$ contribution to them is given by a "logarithmic" integration which arises in the regions $k^2 \gg p^2$ and $p^2 \gg k^2 \gg q^2 \sim m^2$. The contribution of the region $k^2 \gg p^2$ is $\sim\gamma_\mu$, and consequently can be neglected. When in the region $p^2 \gg k^2 \gg q^2 \sim m^2$ we expand the integrand in terms of the parameters m/p, q/p, and k/p, we can easily estimate that in the functions $\Gamma_\mu(p, p-k)$, $G(p-k)$ and $\Gamma_\lambda(p-k, q-k)$ all of the terms of the expansion except the first give no contribution which we need. This allows us to write

$$I_\lambda^{(1)}(p,q)+I_\lambda^{(2)}(p,q)+I_\lambda^{(3)}(p,q)$$
$$=\{\widetilde{\Gamma}_\mu(p,p)S(p)\gamma_\lambda+A_{(0)}^2(p)\gamma_\mu\widetilde{G}(p)\gamma_\lambda+\gamma_\mu S(p)\widetilde{\Gamma}_\lambda(p,0)\}\,J_\mu(p,q),$$

$$I_\lambda^{(4)}(p,q)=\frac{\alpha}{4\pi^3}A_{(0)}(p)\gamma_\mu S(p)\gamma_\lambda\int_q \widetilde{G}(q-k)\Gamma_{(0)\nu}(-k,q)D_{(0)\mu\nu}(k)\,d^4k,$$

$$I_\lambda^{(5)}(p,q)=\frac{\alpha}{4\pi^3}A_{(0)}(p)\gamma_\mu S(p)\gamma_\lambda\int_q^p G_{(0)}(q-k)\widetilde{\Gamma}_\nu(q-k,q)D_{(0)\mu\nu}(k)\,d^4k,$$

$$I_\lambda^{(6)}(p,q)=\frac{\alpha}{4\pi^3}A_{(0)}(p)\gamma_\mu S(p)\gamma_\lambda\int_q^p G_{(0)}(q-k)\Gamma_{(0)\nu}(q-k,q)\widetilde{D}_{\mu\nu}(k)\,d^4k. \quad (26)$$

We shall write out the functions with a tilde which appear in (26) to the required accuracy. $\Gamma_\mu(p, p)$ is determined from G(p) by means of the Ward identity:

$$\widetilde{\Gamma}_\mu(p,p)=-\frac{\partial}{\partial p_\mu}G^{-1}(p)-\gamma_\mu A_{(0)}(p)$$
$$=\frac{\partial}{\partial p_\mu}\{\hat{p}[A_{(0)}(p)+\widetilde{A}(p)]-imB(p)\}-\gamma_\mu A_{(0)}(p)$$
$$=\gamma_\mu\widetilde{A}(p)-\frac{\alpha}{2\pi}d_l A_{(0)}(p)\frac{\hat{p}p_\mu}{p^2}-im\frac{\partial}{\partial p}B(p), \quad (27)$$

and in (26) we need to substitute only the first two terms in (27), dropping the term ~m/p. The function $\widetilde{\Gamma}_\lambda(p, 0)$ is determined by the first two terms of the expression (19). The functions $\widetilde{G}$ and $\widetilde{D}_{\mu\nu}$ must be taken to the following accuracy:

$$\widetilde{G}(k-q)\equiv[(\hat{k}-\hat{q})(A_{(0)}+\widetilde{A})+im(B_{(0)}+\widetilde{B})]^{-1}$$
$$-[(\hat{k}-\hat{q})A_{(0)}+imB_{(0)}]^{-1}\approx(k-q)^{-2}A_{(0)}^{-2}(k)\{(\hat{q}-\hat{k})\widetilde{A}(k)$$
$$-im[\widetilde{B}(k)-2\widetilde{A}(k)B_{(0)}(k)A_{(0)}^{-1}(k)]\},$$

$$\widetilde{G}(p)\approx\frac{\hat{p}}{p^2}\frac{\widetilde{A}(p)}{A_{(0)}^2(p)},\qquad \widetilde{D}_{\mu\nu}(k)=\frac{\widetilde{d}(k)}{ik^2}\left(\delta_{\mu\nu}-\frac{k_\mu k_\nu}{k^2}\right) \quad (28)$$

where $\widetilde{A}$, $\widetilde{B}$, and $\widetilde{d}$ are determined by Eqs. (62), (63), and (8) of I. When we use (8) and the matrix structure of the results (18), (19), and (25) and the integrals (26), we can write the general matrix structure for $\widetilde{\Gamma}_\lambda(p, q)$:

$$\widetilde{\Gamma}_\lambda(p,q)=\gamma_\lambda A_1+\frac{1}{p^2}\Big(\hat{p}\gamma_\lambda\hat{p}A_2+\hat{q}\gamma_\lambda\hat{p}A_3+\hat{q}p_\lambda A_4+\hat{p}\gamma_\lambda\hat{q}A_5$$
$$+\hat{p}q_\lambda A_6+p_\lambda\frac{\hat{p}\hat{q}\hat{p}}{p^2}A_7\Big)+\frac{im}{p^2}\left(\hat{p}\gamma_\lambda B_1+p_\lambda B_2+\frac{\hat{q}\hat{p}\gamma_\lambda\hat{q}}{q^2}B_3\right), \quad (29)$$

where A_i and B_i are scalar functions which, in virtue of the identity (21), satisfy the relations

$$A_1+A_2=\widetilde{A}(p),\quad A_1-A_3-A_4-A_5-{}^1\!/_2A_6=\widetilde{A}(q),$$
$$A_2-{}^1\!/_2A_6-A_7=0,\quad B_1+B_2+B_3=\widetilde{B}(q)-\widetilde{B}(p), \quad (30)$$

where $\widetilde{A}(q)$ and $\widetilde{B}(q)$ are given by Eq. (A.8) of Appendix II, and $\widetilde{A}(p)$ and $\widetilde{B}(p)$ are given by Eqs. (62) and (63) of I.

When we substitute all of the necessary functions in (26), integrate over the angles, and use (30), we get

$$I_\lambda^{(1)}(p,q)+I_\lambda^{(2)}(p,q)+I_\lambda^{(3)}(p,q)=\frac{A_{(0)}(p)}{p^2}\Big\{(\hat{q}\gamma_\lambda\hat{p}-\hat{q}p_\lambda)$$
$$\times\left(\frac{A_1(p)-A_2(p)}{A_{(0)}(p)}-\frac{\alpha}{4\pi}d_l\right)(\xi^{3/4}-1)+\hat{q}p_\lambda\left(\frac{\widetilde{A}(p)}{A_{(0)}(p)}-\frac{\alpha}{4\pi}d_l\right)$$
$$\times(1-A_{(0)}^{-1}(p))+\hat{p}\gamma_\lambda\hat{q}\frac{\alpha}{8\pi}d_l(2-A_{(0)}^{-1}(p)-\xi^{3/4})+\hat{p}q_\lambda\frac{\alpha}{2\pi}d_l(\xi^{3/4}-1)$$
$$-im\left[\hat{p}\gamma_\lambda\frac{\alpha}{8\pi}d_l+p_\lambda\left(\frac{\widetilde{A}(p)}{A_{(0)}(p)}-\frac{\alpha}{4\pi}d_l\right)\right](\xi^{3/4}-A_{(0)}^{-1}(p))\Big\}, \quad (31)$$

$$I_\lambda^{(4)}(p,q)=\frac{3}{4}\frac{A_{(0)}(p)}{p^2}\int_1^\xi\frac{d\eta}{A_{(0)}}\Big\{\Big[(\hat{q}p_\lambda-\hat{q}\gamma_\lambda\hat{p})\eta^{-1/4}-\hat{q}p_\lambda\frac{d_l}{A_{(0)}}\Big]\widetilde{A}$$
$$-imp_\lambda\Big[3\widetilde{B}\eta^{-1}-6\widetilde{A}\eta^{3/4}+\Big(\widetilde{B}-\widetilde{A}\eta^{3/4}-\frac{\widetilde{A}}{A_{(0)}}\Big)d_l\Big]\Big\}, \quad (32)$$

$$I_\lambda\ (p,q)=\frac{3}{4}\frac{A_{(0)}(p)}{p^2}\int_1^\xi\frac{d\eta}{\eta A_{(0)}}\Big\{\hat{q}\gamma_\lambda\hat{p}\left(A_1+\frac{1}{2}A_2+A_3+\frac{3}{4}A_6\right)$$

$$-\hat{q}p_\lambda\left[A_1+2A_2+A_3+3A_5+\frac{3}{2}A_6-\tilde{A}(q)\,d_l\eta\right]+im\frac{\hat{q}\hat{p}\gamma_\lambda\hat{q}}{q^2}B_3$$

$$-imp_\lambda[3\eta^{3/4}(A_1-A_2)+3B_1+B_3+(\tilde{A}\eta^{3/4}+\tilde{B}(q)-\tilde{B})\,d_l\eta]\Big\}, \quad (33)$$

$$J_\lambda^{(6)}(p,q)=\frac{3}{4}\frac{A_{(0)}(p)}{p^2}\int_1 d\eta\,[(\hat{q}\gamma_\lambda\hat{p}-\hat{q}p_\lambda)\eta^{3/4}-3imp_\lambda\eta^{3/4}]\,\tilde{d}. \quad (34)$$

where

$$\xi=1-\frac{\alpha}{3\pi}\ln\frac{p^2}{m^2},\quad \eta=1-\frac{\alpha}{3\pi}\ln\frac{k^2}{m^2}.$$

In Eqs. (32)–(34) we have for brevity omitted the arguments of the functions in the integrands. All of them [except the explicitly written $\tilde{A}(q)$ and $\tilde{B}(q)$] have the argument η. In general the functions A_i and B_i also depend on the parameter q^2. These statements also apply to the integrals (35) and (36).

7. On substituting these results in (8) and equating the linearly independent combinations, we get a series of relations which determine the functions A_i and B_i. For A_3 and B_3 these are integral equations, but the other functions are determined explicitly [see below, Eqs. (38) and (39)]. The equations for B_3 and A_3, when A_3 is expressed by means of the combination

$$A_+=[A_3+(A_1-A_2)(1-\xi^{3/4})]A_{(0)}^{-1},$$

are as follows:

$$B_3(\xi)=\frac{\alpha}{8\pi}A_{(0)}(p)\left(1+2\frac{m^2}{q^2}\right)(d_l-1)+\frac{3}{4}A_{(0)}(p)\int_1^\xi\frac{B_3}{A_{(0)}}\frac{d\eta}{\eta}, \quad (35)$$

$$A_+(\xi)=\frac{\alpha}{4\pi}\left(\frac{m^2}{q^2}-1-\frac{d_l}{2}\right)+\frac{3}{4}\int_1^\xi\frac{d\eta}{\eta}$$

$$\times\left[A_++\left(\frac{3}{2}-2\eta^{3/4}\right)\frac{A_2}{A_{(0)}}+\frac{3}{4}\frac{A_6}{A_{(0)}}+\eta^{3/4}\tilde{d}\right]. \quad (36)$$

The search for their solutions does not present any particular difficulties.

8. The final result for the asymptotic form of the vertex function $\Gamma_\lambda(p,q)$ in the region $p^2\gg q^2\sim m^2$, to the accuracy of terms linear in m/p and q/p, with both the $(\alpha L)^n$ and the $\alpha(\alpha L)^n$ approximations taken into account, is of the form

$$\Gamma_\lambda(p,q)=\gamma_\lambda(A_{(0)}+A_1)+\frac{1}{p^2}\Big\{\hat{p}\gamma_\lambda\hat{p}A_2+\hat{q}\gamma_\lambda\hat{p}\,[(\xi^{3/4}-1)A_{(0)}+A_3]$$

$$+\hat{q}p_\lambda[(2-\xi^{3/4})A_{(0)}-1+A_4]+\hat{p}\gamma_\lambda\hat{q}A_5+\hat{p}q_\lambda A_6+p_\lambda\frac{\hat{p}\hat{q}\hat{p}}{p^2}A_7\Big\}$$

$$+\frac{im}{p^2}\left[\hat{p}\gamma_\lambda B_1+p_\lambda(1-B_{(0)}+B_2)+\frac{\hat{q}\hat{p}\gamma_\lambda\hat{q}}{q^2}B_3\right], \quad (37)$$

where

$$A_{(0)}=\exp\left\{-\frac{\alpha}{4\pi}d_l\ln\frac{p^2}{m^2}\right\}\quad (d_l - \text{is a number!}),$$

$$B_{(0)}=A_{(0)}\xi^{3/4},\quad \xi=1-\frac{\alpha}{3\pi}\ln\frac{p^2}{m^2};$$

$$A_1=\frac{\alpha}{4\pi}A_{(0)}\left(\sigma-\frac{21}{8}+\frac{13}{8}\xi^{-1}\right),\quad \sigma=(d_l-3)\ln\frac{\lambda^2}{m^2}-3,$$

$$A_2=\frac{\alpha}{4\pi}A_{(0)}\xi^{-1},\quad A_1+A_2=\tilde{A}(p).$$

$$A_3=\frac{\alpha}{8\pi}A_{(0)}\left[\frac{21}{4}-2\sigma-\frac{5}{4}\xi^{-1}+\frac{59}{6}\xi^{-1/4}+\frac{3}{2}\xi^{-1/4}\ln\xi\right.$$

$$\left.+\left(2\sigma+2\frac{m^2}{q^2}-\frac{95}{6}-d_l\right)\xi^{3/4}\right], \quad (38)$$

$$A_4=A_1-A_3-A_5-\frac{1}{2}A_6-\tilde{A}(q),\quad \tilde{A}(q)=\frac{\alpha}{4\pi}\left(\sigma-\frac{m^2}{q^2}d_l\right),$$

$$A_5=\frac{\alpha}{8\pi}(\xi^{-1}-A_{(0)}\xi^{-1/4}-d_l),$$

$$A_6=\frac{\alpha}{2\pi}A_{(0)}(\xi^{-1/4}-\xi^{-1}),\quad A_7=A_2-\frac{1}{2}A_6;$$

$$B_1=\frac{\alpha}{4\pi}\left(A_{(0)}\xi^{3/4}-\frac{1}{2}\right)(\xi^{-1}-d_l),$$

$$B_2=\tilde{B}(q)-\tilde{B}(p)-B_1-B_3,\quad \tilde{B}(q)=\frac{\alpha}{4\pi}(\sigma+d_l),$$

$$\tilde{B}(p)=\frac{\alpha}{4\pi}B_{(0)}\left[\sigma+d_l+\frac{59}{4}(\xi^{-1}-1)+\frac{27}{4}\xi^{-1}\ln\xi\right],$$

$$B_3=\frac{\alpha}{4\pi}A_{(0)}\left(\frac{1}{2}+\frac{m^2}{q^2}\right)(d_l-1)\xi^{3/4}. \quad (39)$$

APPENDIX I

Two types of integrals arise in the calculation of $I_\lambda(p,q)$ [see the expression (17)]. First, there are integrals of the type

$$\int\frac{k_\mu\,d^4k}{k^2(k-p)^2[(q-k)^2+m^2]}$$

and so on, which are calculated in the standard way by means of the Feynman parametrization.[8] Second, there are integrals of analogous form, but containing in addition functions of the following type:

$$f(k)\equiv f\left(\alpha\ln\frac{k^2}{m^2}\right)=\begin{cases}\sum_{n=1}^{\infty}f_n\left(\alpha\ln\frac{k^2}{m^2}\right)^n, & k^2\gg m^2\\ O(\alpha), & k^2\lesssim m^2\end{cases}$$

In this case the contribution in which we are interested arises in the region $k^2\gg q^2\sim m^2$, in which we can make an expansion in terms of the parameter q/k and keep only the linear terms. It is convenient to do the integration of the resulting expressions in spherical coordinates φ, ϑ, θ, r ($d^4k=ir^3dr\sin^2\theta d\theta\sin\vartheta\,d\vartheta\,d\varphi$) in the four-dimensional Euclidean (after rotation of the contour) momentum space. Here it is helpful to use the following formulas for the integration over the angles:

$$\overline{k_\mu}^{\vartheta\varphi}=\frac{(kp)}{p^2}p_\mu,$$

$$\overline{k_\mu k_\nu}^{\vartheta\varphi}=\frac{\delta_{\mu\nu}}{3}\left[k^2-\frac{(kp)^2}{p^2}\right]+\frac{1}{3}\frac{p_\mu p_\nu}{p^2}\left[4\frac{(kp)^2}{p^2}-k^2\right],$$

$$\overline{k_\mu k_\nu k_\lambda}^{\vartheta\varphi}=\frac{(kp)}{p^2}\left[2\frac{(kp)^2}{p^4}-\frac{k^2}{p^2}\right]p_\mu p_\nu p_\lambda$$

$$+\frac{(kp)}{3}\left[\frac{k^2}{p^2}-\frac{(kp)^2}{p^4}\right](p_\mu\delta_{\nu\lambda}+p_\nu\delta_{\mu\lambda}+p_\lambda\delta_{\mu\nu}). \quad (A.1)$$

where

$$\overline{k_\mu\ldots k_\lambda}^{\vartheta\varphi}\equiv\frac{1}{4\pi}\int_0^{2\pi}d\varphi\int_0^\pi\sin\vartheta\,d\vartheta\,(k_\mu\ldots k_\lambda)$$

and the vector p_μ defines the direction from which the angle θ is measured;

$$I_0=\frac{\pi}{2}\left[\frac{\theta_{pr}}{p^2}+\frac{\theta_{rp}}{r^2}\right],\quad I_1=\frac{\pi}{4r\cdot p}\left[\frac{r^2}{p^2}\theta_{pr}+\frac{p^2}{r^2}\theta_{rp}\right],$$

$$I_2=\frac{\pi}{8}(r^2+p^2)\left[\frac{\theta_{pr}}{p^4}+\frac{\theta_{rp}}{r^4}\right],$$

$$I_3=\frac{\pi}{16rp}\left[\left(\frac{r^2}{p^2}+1\right)^2\theta_{pr}+\left(\frac{p^2}{r^2}+1\right)^2\theta_{rp}-1\right]. \quad (A.2)$$

where

$$I_n = \int_0^\pi \frac{\cos^n\theta \sin^2\theta\, d\theta}{r^2+p^2-2rp\cos\theta}, \quad \theta_{pr} = \begin{cases} 1, & p^2 > r^2 \\ 0, & p^2 < r^2 \end{cases}$$

The remaining integration over r can be done without difficulty. We write the final results in the following form:

$$\int \frac{d^4k}{F(k)}\{1; f(k)\} = \frac{\pi^2 i}{p^2}\{2; f(p)\} + \ldots, \quad (A.3)$$

$$\int \frac{k_\mu d^4k}{F(k)}\{1; f(k)\} = \frac{\pi^2 i}{p^2}\left\{\left[p_\mu\left(1+\frac{pq}{p^2}\right)+\frac{q_\mu}{4}\right]\{1; f(p)\} + \frac{q_\mu}{4}\left(1+2\frac{m^2}{q^2}\right)\{1; 0\}\right\} + \ldots, \quad (A.4)$$

$$\int \frac{k_\mu k_\nu d^4k}{F(k)}\{1; f(k)\} = \frac{\pi^2 i}{2}\left[\frac{\delta_{\mu\nu}}{2}\left(\frac{1}{2}+\frac{pq}{p^2}\right)+\frac{p_\mu p_\nu}{p^2}\left(1+\frac{pq}{p^2}\right) + \frac{p_\mu q_\nu + p_\nu q_\mu}{2p^2}\right]\{1; f(p)\} + \ldots, \quad (A.5)$$

$$\int \frac{k_\mu k_\nu d^4k}{k^2F(k)}\{1; f(k)\} = \frac{\pi^2 i}{2p^2}\left\{\left(\frac{\delta_{\mu\nu}}{4}+\frac{p_\mu p_\nu}{p^2}\right)\{1; f(p)\} + \left(\frac{\delta_{\mu\nu}}{2}\left(\frac{1}{2}-\frac{m^2}{q^2}\right)+\frac{q_\mu q_\nu}{q^2}\left(1+2\frac{m^2}{q^2}\right)\right]\{1; 0\}\right\} + \ldots, \quad (A.6)$$

$$\int \frac{k_\mu k_\nu k_\lambda d^4k}{k^2F(k)}\{f(k)\} = \frac{\pi^2 i}{8p^2}\left[2\frac{p_\mu p_\nu p_\lambda}{p^2} + p_\mu\delta_{\nu\lambda} + p_\nu\delta_{\mu\lambda} + p_\lambda\delta_{\mu\nu}\right]\{f(p)\} + \ldots,$$

$$F(k) = k^2(k-p)^2[(q-k^2)+m^2], \quad p^2 \gg q^2 \sim m^2. \quad (A.7)$$

We note that, corresponding to their role in the integral (17), the integrals (A.4) and (A.5) have been calculated to the accuracy of terms linear in q/p, and in (A.3), (A.6) and (A.7) only the terms of zeroth order in q/p have been kept. Moreover, in all of the integrals considered logarithmic terms of the form $\int dk^2 k^{-2}\{1; f(k)\}$ have been dropped; they contribute only in the $(\alpha L)^n$ approximation and are of no interest to us here.

APPENDIX II

Here we shall find $\tilde{A}(q)$ and $\tilde{B}(q)$ for $q^2 \sim m^2$. By definition

$$G(q)^{-1} = -\hat{q}[A_{(0)}(q)+\tilde{A}(q)] + im[B_{(0)}(q)+\tilde{B}(q)] = im - \hat{q} - \Sigma_R(q).$$

For $q^2 \sim m^2$ we have $A_{(0)}(q) = B_{(0)}(q) = 1$, and $\tilde{A}(q)$ and $\tilde{B}(q)$ contain only terms $\sim\alpha$. To find them we need only calculate $\Sigma_R(q)$ in first order in α. The calculations required can be done in the standard way.[8] The result is

$$\Sigma_R^{(2)}(q) = \frac{\alpha}{4\pi}\left[\hat{q}\left(\sigma - \frac{m^2}{q^2}d_l\right) - im(\sigma+d_l)\right],$$

from which we have

$$\tilde{A}(q) = \frac{\alpha}{4\pi}\left(\sigma - \frac{m^2}{q^2}d_l\right), \quad \tilde{B}(q) = \frac{\alpha}{4\pi}(\sigma + d_l), \quad (A.8)$$

where [see Eq. (64) of I]

$$\sigma = (d_l - 3)\ln\frac{\lambda^2}{m^2} - 3,$$

λ being the "mass" of the photon.

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