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RADIATIVE EFFECTS AND THEIR ENHANCEMENT IN AN INTENSE ELECTROMAGNETIC FIELD

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The radiative effects—the photon mass, the change in the electron mass, and the anomalous magnetic moment of the electron—are considered in the presence of an intense electromagnetic field where they are functions of the parameter $\chi = m^{-3}\sqrt{(eF_{\mu\nu}p_\nu)}$ which is proportional to the field and to the particle momentum. The effects become optimal at $\chi \sim 1$, that is, for example, at a field strength of 4×10^8 Oe and an energy of 25 GeV. At higher energies the square of the photon mass and the change of the electron mass increase like $\chi^{2/3}$, but the anomalous magnetic moment decreases like $\chi^{-2/3}$. For $\alpha\chi^{2/3} \sim 1$ an exact theory of the interaction with the radiation field is required, which takes all radiative corrections into account. This region is more accessible than the corresponding region of logarithmically large energies for the radiative effects in vacuum. The amplitudes found for the elastic scattering of a photon and an electron have an essential singularity at $\chi = 0$.

1. INTRODUCTION

THE existence in an external electromagnetic field of the processes of pair production by a photon, $\gamma \to e^+e^-$, and the emission of a photon by an electron, $e \to e\gamma$, lead to radiative effects such as the appearance of a photon mass and a change of the electron mass in the presence of an intense field. The magnitude of these effects becomes appreciable when the intensity of the field in the proper frame of the electron is close to $B_0 = m^2c^3/e\hbar = 1.25 \times 10^{13}$ Heaviside units, which is the characteristic field in quantum electrodynamics.[1)] Since the fields existing in the laboratory are several orders of magnitude smaller than B_0, noticeable effects arise only for ultrarelativistic particles. But then any external field in the rest frame will be very close to a plane wave. If, moreover, the characteristic wavelength and period of the field are large in comparison with m/eB, which characterizes the length and time of formation of the process in a field of intensity B, then one can regard such a field as a constant crossed field ($\mathbf{E} \perp \mathbf{H}$, $E = H$). In this article the radiative effects in an intense crossed field are considered by using the method of analytic continuation, the interaction with the crossed field being taken into account exactly; however the interaction with the radiation field is considered to the lowest-order approximation of perturbation theory.

2. PHOTON MASS IN THE PRESENCE OF A FIELD

For a description of pair production in a constant field by a photon of arbitrary polarization, it is sufficient to know the probabilities for two independent polarizations, for example, parallel and perpendicular to the electric field in the system where the photon is incident in a direction opposite to the Poynting vector $\mathbf{E} \times \mathbf{H}$. These probabilities were found in[1,2]:

$$F_{\parallel,\perp}(\varkappa) = -\frac{\alpha m^2 n}{6\sqrt{\pi}k_0'}\int_1^\infty du \frac{8u+1\mp 3}{zu\sqrt{u(u-1)}}\Phi'(z), \quad z = \left(\frac{4u}{\varkappa}\right)^{2/3}. \tag{2.1}$$

The probability (2.1) is calculated per unit volume and per unit time, is an invariant quantity proportional to the ratio n/k_0' of the density of incident photons to their energy, and is in fact determined by the parameter

$$\varkappa = m^{-3}\sqrt{(eF_{\mu\nu}k_\nu')^2}, \tag{2.2}$$

in which k_ν' denotes the 4-momentum of the photon and $F_{\mu\nu}$ is the electromagnetic tensor of the external field.

The possibility of pair production by a photon in a field leads to the result that the external field plays the role of a homogeneous anisotropic medium with dispersion and absorption. One can relate the index of refraction of such a medium with the photon mass, $1 - n_{\parallel,\perp} = \mu^2_{\parallel,\perp}/2k_0'^2$, where the imaginary part of the index of refraction or the square of the mass is determined by the probability of pair production:

$$F_{\parallel,\perp}(\varkappa) = -(n/k_0')\,\mathrm{Im}\,\mu^2_{\parallel,\perp}(\varkappa). \tag{2.3}$$

The usual requirements on the index of refraction, which follow from considerations of causality, lead to the result that the square of the mass $\mu^2(\kappa)$ must be an analytic function in the upper half-plane of the complex variable κ, and it must tend to zero at least linearly upon switching-off the external field, i.e., at the point $\kappa = 0$. This means that the function $\kappa^{-2/3}\mu^2(\kappa)$ is regular in the upper half-plane and vanishes at $\kappa = 0$. But from the representation (2.1) it is seen that the analytic properties of $\kappa^{-2/3}\mu^2(\kappa)$ are determined by the derivative of the Airy function $\Phi'(z)$, $z = (4u/\kappa)^{2/3}$, which is an entire transcendental function of z. It is therefore convenient to consider the analytic properties of $\kappa^{-2/3}\mu^2$ not with respect to κ but with respect to $x = (4/\kappa)^{2/3}$, which enters into the argument of the Airy function in the representation

[1)]The units $\hbar = c = 1$, $\alpha = e^2/4\pi = 1/137$ and the metric $ab = \mathbf{a}\cdot\mathbf{b} - a_0b_0$ are used below.

(2.1). In this connection the upper half-plane of κ corresponds to the sector $-2\pi/3 < \arg x < 0$ in the lower half-plane of x. The determination of a function $\kappa^{-2/3}\mu^2(\kappa)$, analytic in x in this sector and equal to zero at $x = \infty$, from its imaginary part on the real axis is equivalent to the construction of a function f(z), analytic in the region $-2\pi/3 < \arg z < 0$ and tending to zero as $z \to \infty$, from its imaginary part $\Phi(z)$ on the real axis. In actual fact f(z) is an analytic function over the entire lower half-plane z, falling off exponentially as $\operatorname{Im} z \to -\infty$, and therefore on the real z axis its real $\Upsilon(z)$ and imaginary $\Phi(z)$ parts are related by Hilbert transforms so that

$$\Upsilon(z) = -\frac{P}{\pi}\int_{-\infty}^{\infty}\frac{\Phi(z')\,dz'}{z'-z} = \frac{1}{\sqrt{\pi}}\int_0^{\infty} dt \sin\left(zt + \frac{t^3}{3}\right). \quad (2.4)$$

For the Airy function we have used the representation

$$\Phi(z) = \frac{1}{\sqrt{\pi}}\int_0^{\infty} dt \cos\left(zt + \frac{t^3}{3}\right). \quad (2.5)$$

The function thus determined[2)]

$$f(z) = \Upsilon(z) + i\Phi(z) = \frac{i}{\sqrt{\pi}}\int_0^{\infty} dt \exp\left\{-i\left(zt + \frac{t^3}{3}\right)\right\} \quad (2.6)$$

is an entire transcendental function, which falls off exponentially in the lower half-plane as $\operatorname{Im} z \to -\infty$. Its derivative possesses similar properties.

Thus $\mu^2(\kappa)$ possessing the proper analytic properties with respect to κ is obtained from $\operatorname{Im}\mu^2$, determined by formulas (2.3) and (2.1), by making the substitution $\Phi'(z) \to f'(z)$:

$$\mu_{\parallel,\perp}(\varkappa) = \frac{\alpha m^2}{6\sqrt{\pi}}\int_1^{\infty} du \frac{8u + 1 \mp 3}{zu\sqrt{u(u-1)}} f'(z), \quad z = \left(\frac{4u}{\varkappa}\right)^{2/3}. \quad (2.7)$$

This formula determines the function $\mu^2(\kappa)$ which is analytic in the lower half-plane of $x = (4/\kappa)^{2/3}$ (or $0 < \arg \kappa < 3\pi/2$) with a pole at the point $x = 0$ and an essential singularity at $x = \infty$.

With the aid of the representation (2.6) it is not difficult to express the function f(z) on the radial line $z = \rho e^{-2\pi i/3}$ in terms of the functions Υ and Φ, which are real for real values of ρ:

$$f(z) \equiv \Upsilon(z) + i\Phi(z) = e^{-i\pi/3}[-\Upsilon(\rho) + i\Phi(\rho)]. \quad (2.8)$$

Hence for the same radial line it follows that

$$\frac{\Upsilon'(z) + i\Phi'(z)}{z} = \frac{\Upsilon'(\rho) - i\Phi'(\rho)}{\rho}, \quad z = \rho e^{-2\pi i/3}. \quad (2.9)$$

Using this relation in the representation for $\mu^2(\kappa)$ and the connection $z(-\kappa) = z(\kappa)e^{-2\pi i/3}$, we see that the real and imaginary parts of $\mu^2(\kappa)$ are, respectively, even and odd functions of the real variable κ:

$$\operatorname{Re}\mu^2(-\varkappa) = \operatorname{Re}\mu^2(\varkappa), \quad \operatorname{Im}\mu^2(-\varkappa) = -\operatorname{Im}\mu^2(\varkappa). \quad (2.10)$$

These properties reflect the physical fact that a real electric field creates a real induction in a medium (see, for example, Sec. 62 of[4]).

The two functions $\mu^2_{\parallel,\perp}(\kappa)$ determine the poles of the photon Green's function in a constant crossed field.

[2)]One can run the contour of integration in Eq. (2.6) from zero to infinity in the sector $-\pi/3 < \arg t < 0$. See Watson's book, [3] Secs. 10.2 – 10.22, for the detailed properties of the functions Υ(z) and f(z), which were introduced by Stokes and Hardy.

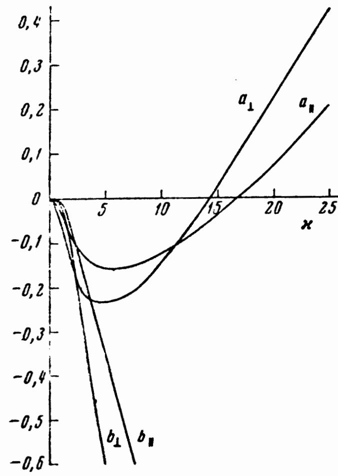

FIG. 1

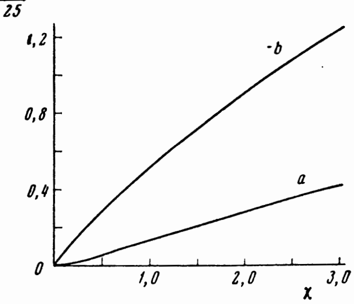

FIG. 2

FIG. 1. The dependence of the square of the photon mass, $\mu^2 = \alpha m^2(a + ib)$, on κ.

FIG. 2. The dependence of the change $\Delta m = \alpha m(a + ib)$ in the electron's mass on χ.

Such a Green's function was found by Narozhnyi,[5] and after carrying out the appropriate transformations one can show that its poles coincide with (2.7).

Using the properties of the functions $\Upsilon(z)$ and $\Phi(z)$ it is not difficult to find asymptotic expressions for the square of the mass for small and large values of κ:

$$\mu_{\parallel,\perp}(\varkappa) = \alpha m^2 \begin{cases} -\dfrac{11 \mp 3}{90\pi}\varkappa^2 - i\sqrt{\dfrac{3}{2}}\dfrac{3 \mp 1}{16}\varkappa e^{-8/3\varkappa}, & \varkappa \ll 1 \\ \dfrac{5 \mp 1}{28\pi^2}\sqrt{3}\,\Gamma^4\left(\dfrac{2}{3}\right)(1 - i\sqrt{3})(3\varkappa)^{2/3}, & \varkappa \gg 1 \end{cases} \quad (2.11)$$

The dependences of the real and imaginary parts of the square of the photon mass on κ, determined numerically according to formula (2.7) and by the tabulated functions $\Upsilon'(z)$ and $\Phi'(z)$, are presented in Fig. 1.

Let us turn our attention to whether $\operatorname{Re}\mu^2(\kappa)$ possesses a negative minimum and changes sign with an increase of κ. For large values of κ the real and imaginary parts of μ^2 are of the same order of magnitude and increase with the photon energy and with the field like $\kappa^{2/3}$, i.e., more strongly than the radiative corrections in vacuum, where they increase with energy to a maximum, like the square of the logarithm. For $\alpha\kappa^{2/3} \sim 1$ the photon mass becomes the same order of magnitude as the electron mass, and expression (2.7), obtained to lowest order in the interaction with the radiation field, becomes inapplicable: it is necessary to take all orders of perturbation theory into account. Thus, the obtained results are valid as long as $\alpha\kappa^{2/3} \ll 1$.

3. CHANGE OF THE ELECTRON MASS IN AN INTENSE FIELD

The total probability for the emission of a photon by an unpolarized electron in a constant crossed field is found in[1,2] and has the form

$$F(\chi) = -\frac{\alpha m^2 c}{3\sqrt{\pi}}\int_0^{\infty} du \frac{5 + 7u + 5u^2}{z(1+u)^3}\Phi'(z), \quad z = \left(\frac{u}{\chi}\right)^{2/3} \quad (3.1)$$

This probability is calculated per unit volume and per unit time, is an invariant quantity, and is determined by the parameter

$$\chi = m^{-3}\sqrt{(eF_{\mu\nu}p_\nu)^2}, \tag{3.2}$$

in which the constant 4-vector "momentum" p_μ, $p^2 = -m^2$, characterizes the state of the electron in the field.[3)] The invariant factor c is the ratio of the number density of incident electrons to their kinetic energy or the ratio of the number density of electrons in the rest system to the electron's mass. Therefore $F(\chi)/cm$ is the inverse proper lifetime of an electron with respect to photon emission.

The existence of emission leads to elastic scattering of an electron in the field, the amplitude for which is related to the total probability of emission by the optical theorem, $2\,\mathrm{Im}\,M_{pp} = F(\chi)VT$ (the amplitude is determined from the S-matrix according to the relation $S = 1 + iM$). On the other hand, this amplitude is proportional to the change of the electron's mass, $M_{pp} = -cm\Delta mVT$, and therefore one can regard $-F/2cm$ as an imaginary correction to the electron's mass

$$\mathrm{Im}\,\Delta m = -F(\chi)\,/\,2cm. \tag{3.3}$$

One can regenerate the real correction to the electron's mass from the imaginary part by proceeding from considerations of the analyticity of the amplitude M_{pp} or of the correction Δm in the upper half-plane of the complex variable χ, and the fact that it vanishes at least linearly for $\chi = 0$. In such a case the function $\chi^{-2/3}\Delta m(\chi)$ will be analytic with respect to $x = \chi^{-2/3}$ in the sector $-2\pi/3 < \arg x < 0$ and tends to zero as $x \to \infty$. But then the construction of Δm from its imaginary part, just like in Sec. 2, reduces to the substitution $\Phi'(z) \to f'(z)$, that is,

$$\Delta m = \frac{\alpha m}{6\sqrt{\pi}}\int_0^\infty du\,\frac{5+7u+5u^2}{z(1+u)^3}f'(z), \quad z = \left(\frac{u}{\chi}\right)^{2/3}. \tag{3.4}$$

By virtue of (2.9) the real and imaginary parts of Δm are even and odd functions of the real variable χ.

Using the properties of the functions $\Psi(z)$ and $\Phi(z)$, it is not difficult to find asymptotic expressions for Δm for small and large values of χ:

$$\mathrm{Re}\,\Delta m = \frac{4\alpha m}{3\pi}\chi^2\left(\ln\frac{1}{\chi} + C + \frac{1}{2}\ln 3 - \frac{33}{16}\right) + \ldots$$
$$\mathrm{Im}\,\Delta m = -\frac{5\alpha m}{4\sqrt{3}}\chi\left(1 - \frac{8\sqrt{3}}{15}\chi + \frac{7}{2}\chi^2 - \ldots\right), \quad \chi \ll 1; \tag{3.5}$$

$$\Delta m = \frac{7\Gamma(2/3)(1 - i\sqrt{3})\alpha m}{27\sqrt{3}}(3\chi)^{2/3}, \quad \chi \gg 1. \tag{3.6}$$

Here $C = 0.577\ldots$ is Euler's constant.

Let us turn our attention to the fact that $\mathrm{Re}\,\Delta m > 0$ and increases monotonically with χ (see Fig. 2). Again, just as in the case of the photon mass, for large values of χ the real and imaginary parts of Δm differ only by the factor $-\sqrt{3}$, and they increase with energy or field like $\chi^{2/3}$. The obtained results are valid as long as $\alpha\chi^{2/3} \ll 1$. For $\alpha\chi^{2/3} \sim 1$ the change in the electron mass becomes comparable with the electron mass in vacuum, and it is necessary to take all radiative corrections into consideration.

Although the expression obtained for the mass of an electron in the presence of a field is caused by the emission of an unpolarized electron, it contains spin effects: for a spinless particle Δm would differ from (3.4) by the replacement in the numerator of $5u^2$ by $2u^2$. In order to obtain the mass shift associated with a specific orientation of the spin with respect to the field, let us consider the emission of a polarized electron.

4. EMISSION OF A POLARIZED ELECTRON

The amplitude for the emission of a photon by an electron in a crossed field is obtained in[6]:

$$M = \frac{ie}{\sqrt{2k_0'}}\int_{-\infty}^{\infty} ds\,\bar{u}_{p'r'}\left\{\hat{e}' A + ie\left(\frac{\hat{a}\hat{k}\hat{e}'}{2kp'} + \frac{\hat{e}'\hat{k}\hat{a}}{2kp}\right)A' + \frac{e^2a^2(ke')\hat{k}}{2kp\cdot kp'}A''\right\}u_{pr}$$
$$\times(2\pi)^4\delta(sk + p - p' - k'). \tag{4.1}$$

See[6] for an explanation of all notation. The probability for the emission of a polarized electron is determined by the following trace:

$$\frac{\mathrm{Sp}(M\rho M^+)}{VT} = \frac{(2\pi)^5 e^2}{L_\varphi 2k_0'}\int_{-\infty}^{\infty} ds\,\delta(sk + p - p' - k') \tag{4.2}$$
$$\times\mathrm{Sp}\left[O\frac{m - i\hat{p}}{4p_0}(1 - i\hat{s}\gamma_5)\beta O^+\beta\frac{m - i\hat{p}'}{2p_0'}\right],$$

where ρ is the density matrix of rank two for the incident electron, which is polarized in the direction $\boldsymbol{\zeta}$ in the coordinate system where the vector "momentum" $\mathbf{p} = 0$; s_μ is the 4-vector polarization of the electron, which is related to $\boldsymbol{\zeta}$ by the relation

$$s_\mu = \left(\boldsymbol{\zeta} + \frac{\mathbf{p}(\boldsymbol{\zeta}\mathbf{p})}{m(p_0 + m)},\quad i\frac{\mathbf{p}\boldsymbol{\zeta}}{m}\right) \tag{4.3}$$

and which satisfies the conditions $sp = 0$ and $s^2 = 1$; O denotes the matrix of rank four appearing inside the curly brackets in expression (4.1); finally L_φ denotes a large interval of the phase $\varphi = kx$, corresponding to the space-time volume VT in which the process is being considered.

The trace on the right-hand side of expression (4.2) is given by

$$\mathrm{Sp}[\;] = \frac{1}{p_0p_0'}\Big\{|pe''A - ie(ae'')A'|^2 - 2\beta(kk')[|A'|^2 + \mathrm{Re}\,AA''^*]$$
$$+ em\,\mathrm{Re}\,AA'^*\Big[i\frac{ae''}{kp}\varepsilon_{\mu\nu\lambda\sigma}k_\mu k_\nu' e_\lambda'' s_\sigma \tag{4.4}$$
$$-\left(\frac{1}{kp} - \frac{1}{kp'}\right)\left(e''p'F_{\lambda\sigma}^* e_\lambda'' s_\sigma + \frac{1}{2}F_{\lambda\sigma}^* k_\lambda' s_\sigma\right)\Big]\Big\}.$$

Here

$$F_{\lambda\sigma}^* = i\varepsilon_{\mu\nu\lambda\sigma}k_\mu a_\nu = \frac{i}{2}\varepsilon_{\mu\nu\lambda\sigma}F_{\mu\nu}, \qquad e'' = e' - k'\frac{ke'}{kk'};$$

for the remaining notation, see[6]. If expression (4.4) is summed over the polarizations e′ of the emitted photon, then the following result is obtained

$$\sum_{e'}\mathrm{Sp}[\;] = \frac{m^2}{\pi p_0p_0'x^2}\left(\frac{2\chi\chi'}{\varkappa}\right)^{2/3}\Big\{-\Phi^2(y)$$

[3)]The kinetic momentum is not conserved in a crossed field, but the motion of an electron is characterized by a constant 4-vector p_μ with three independent components ($p^2 = -m^2$) having the following physical meaning: in a system where E and H are located in the (1, 2) plane, and the Poynting vector is directed along the 3-axis, the components $p_{1,2}$ are the generalized momenta along the 1, 2-axes, and $p_0 - p_3$ is the difference between the total energy and the generalized momentum along the 3-axis; see [1].

$$+\left(\frac{2\chi\chi'}{\varkappa}\right)^{1/3}\left(1+\frac{\varkappa^2}{2\chi\chi'}\right)(y\Phi^2+\Phi'^2)-\left(\frac{2\chi\chi'}{\varkappa}\right)^{1/3}\frac{\varkappa\Phi\Phi'}{\chi^2}$$
$$\times\left[\frac{eF_{\mu\nu}{}^{*}p_\mu s_\nu}{m^3}+\left(\frac{1}{\chi}+\frac{1}{\chi'}\right)\frac{e^2F_{\mu\nu}{}^{*}F_{\mu\lambda}{}^{*}p_\lambda s_\nu}{m^5}\tau\right]\Big\}, \quad (4.5)$$

where $x = ea/m$, the invariant variables χ, χ', and κ are related to the momenta p, p′, and k′ by relations (3.2) and (2.2), and the function A is expressed in terms of the Airy function $\Phi(y)$ with argument

$$y=\left(\frac{\varkappa}{2\chi\chi'}\right)^{2/3}(1+\tau^2),\quad \tau=-\frac{eF_{\mu\nu}{}^{*}p_\mu'p_\nu}{m^4\varkappa}. \quad (4.6)$$

In Eq. (4.5) we utilized the conservation law $sk + p = p' + k'$, which reduces, for example, to the relation $\chi = \chi' + \kappa$, and a decomposition of the vector k' into the vectors p_μ, ea_μ, k_μ and $ie\epsilon_{\mu\nu\lambda\sigma}p_\nu k_\lambda a_\sigma \equiv eF^*_{\mu\nu}p_\nu$:

$$k_\mu'=\left[p_\mu-e\left(a_\mu-k_\mu\frac{ap}{kp}\right)\rho-k_\mu\frac{e^2a^2}{2kp}\left(\rho^2+\frac{\tau^2-1}{x^2}\right)+\frac{eF_{\mu\nu}{}^{*}p_\nu}{m^2\chi}\tau\right]\frac{\varkappa}{\chi}. \quad (4.7)$$

It is convenient to carry out the integration (4.2) over the final states $d^3p'd^3k'(2\pi)^{-6}$ of the electron and photon with the aid of the formula

$$\int_{-\infty}^{\infty}ds\,\delta(sk+p-p'-k')\frac{d^3p'\,d^3k'}{p_0'k_0'}\ldots$$
$$=-\frac{d^3k'}{(kp')k_0'}\ldots=\frac{x^2u}{\chi(1+u)^2}d\rho\,d\tau\,du\ldots, \quad (4.8)$$

in which

$$u=\frac{\varkappa}{\chi'},\quad \rho=\frac{eF_{\mu\nu}p_\mu'p_\nu}{m^4x\varkappa}. \quad (4.9)$$

Since the differential probability does not depend on k_1' or ρ, then the integration over ρ turns out to be a "dummy" integration. However, as shown in[6,7] the variable ρ characterizes the space-time region from which emission with a given k' occurs: the differential probability is formed in a region which has phase and extent given by

$$\varphi=\rho,\quad \Delta\varphi\sim\frac{1}{x}\left(\frac{2\chi\chi'}{\varkappa}\right)^{1/3}\max(1,\sqrt{y}). \quad (4.10)$$

Therefore the integration over ρ is equivalent to a summation of the contributions from all possible regions, so that the integral over ρ is equal to L_φ. The large phase interval L_φ which arises is cancelled by the same L_φ appearing in the denominator of (4.2), and consequently we obtain the following result for the differential distribution with respect to the variables τ and u:

$$dF=\frac{e^2m^2c}{2\pi^2(1+u)^2}\left(\frac{u}{2\chi}\right)^{1/3}\Big\{-\Phi^2(y)+\left(\frac{2\chi}{u}\right)^{2/3}\left(1+\frac{u^2}{2(1+u)}\right)$$
$$\times(y\Phi^2+\Phi'^2)-\left(\frac{u}{2\chi}\right)^{2/3}\frac{2\Phi\Phi'}{1+u}\Big[\frac{eF_{\mu\nu}{}^{*}p_\mu s_\nu}{m^3}$$
$$+\frac{e^2F_{\mu\nu}{}^{*}F_{\mu\lambda}{}^{*}p_\lambda s_\nu}{m^5\chi}(2+u)\tau\Big]\Big\}d\tau\,du. \quad (4.11)$$

Upon integration with respect to τ (in the limits from $-\infty$ to ∞) the last term, which is odd with respect to τ, vanishes. In the remaining terms one can reduce the bilinear combinations of Airy functions to linear combinations if one uses the change of variable $t = (u/2\chi)^{2/3}\tau^2$ and the formula

$$\int_0^\infty\frac{dt}{\sqrt{t}}\Phi^2(a+t)=\frac{\sqrt{\pi}}{2}\int_{2^{2/3}a}^{\infty}dx\,\Phi(x).$$

(For more details about this procedure, see[2].) As a result we obtain the following expression for the distribution with respect to u:

$$\frac{dF}{du}=-\frac{e^2m^2c}{4\pi\sqrt{\pi}(1+u)^2}\Big\{\int_z^\infty dx\,\Phi(x)+\frac{2}{z}\left(1+\frac{u^2}{2(1+u)}\right)\Phi'(z)$$
$$-\gamma\frac{2z}{1+u}\Phi(z)\Big\},\quad z=\left(\frac{u}{\chi}\right)^{2/3}, \quad (4.12)$$

in which the last term, which is proportional to the invariant

$$\gamma=eF_{\mu\nu}{}^{*}p_\mu s_\nu/2m^3, \quad (4.13)$$

describes the effect of the electron's polarization (compare with formula (46) in[2]). The invariant $-\gamma m$ is the interaction energy of the spin magnetic moment with the field in the rest system (in this system it is equal to $-\mu_0 H$, where $\mu_0 = e\zeta/2m$).

The integration of the distribution (4.12) over u (within the limits from 0 to ∞) gives the total probability for the emission of a polarized electron, $F(\chi) = F_0(\chi) + F_s(\chi)$, where F_0 now denotes the probability for the emission of an unpolarized electron, which is determined by formula (3.1), and F_s is the contribution to the total probability due to the electron's polarization:

$$F_s(\chi)=\frac{2\alpha m^2c\gamma}{\sqrt{\pi}}\int_0^\infty du\frac{z\Phi(z)}{(1+u)^3},\quad z=\left(\frac{u}{\chi}\right)^{2/3} \quad (4.14)$$

In order to investigate this function for small and large values of χ it is convenient to transform it, with the aid of the formula

$$\frac{1}{(1+u)^3}=\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}dk\frac{\Gamma(-k)\Gamma(3+k)}{\Gamma(3)}u^k$$

and subsequent integration over u, to the form

$$F_s(\chi)=\frac{9\alpha m^2c\gamma}{4\pi\sqrt{3}}\chi\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}dk\,\Gamma(-k)\Gamma(3+k)$$
$$\times\Gamma\left(\frac{k}{2}+\frac{5}{6}\right)\Gamma\left(\frac{k}{2}+\frac{7}{6}\right)(3\chi)^k. \quad (4.15)$$

For small values of χ one can close the path of integration to the right, reducing the integral to a sum of the residues at the poles $k = 0, 1, 2, \ldots$ of the integrand which lie in the right half-plane. Then we obtain the asymptotic series

$$F_s(\chi)=\frac{9\alpha m^2c\gamma}{4\pi\sqrt{3}}\chi\sum_{k=0}(-1)^k(k+1)(k+2)$$
$$\times\Gamma\left(\frac{k}{2}+\frac{5}{6}\right)\Gamma\left(\frac{k}{2}+\frac{7}{6}\right)(3\chi)^k. \quad (4.16)$$

For $\chi \gg 1$ the path of integration can be closed to the left, reducing the integral to a sum of the residues at the poles of the integrand which lie in the left half-plane: $k = -n - 3, -2n - (5/3), -2n - (7/3)$; $n = 0, 1, 2, \ldots$ Then we obtain the series

$$F_s(\chi)=\sqrt{3}\pi\alpha m^2c\gamma\sum_{n=0}^{\infty}\Big\{-\frac{4(6n-1)(3n+1)(3\chi)^{-2n-2/3}}{27n!\,\Gamma(n+2/3)}$$
$$-\frac{4(6n+1)(3n+2)(3\chi)^{-2n-4/3}}{27n!\,\Gamma(n+4/3)}$$
$$+\frac{(n+1)(n+2)(3\chi)^{-n-2}}{((-1)^n+2)\Gamma(n/2+4/3)\Gamma(n/2+5/3)}\Big\}, \quad (4.17)$$

which converges for all values of χ and demonstrates that $F_S(\chi)$ is analytic everywhere, except for an essential singularity and branching at the point $\chi = 0$.

In concluding this section we note that the obtained formulas also determine other polarization effects. Thus, the differential probability of emission, associated with a polarization s'_μ of the final electron, is obtained from (4.5) by making the following substitutions in the last term, $p \rightleftarrows -p'$, $s \to s'$, and by reversing its sign. The differential probability for pair formation by a photon, associated with polarization s_μ of the positron, is obtained from (4.5) by the substitution $p \to -p$, $k' \to -k'$, and by reversal of its overall sign.

5. ANOMALOUS MAGNETIC MOMENT OF AN ELECTRON IN AN INTENSE FIELD

The contribution Δm_S to the electron's mass, due to a specific orientation of the spin with respect to the field, may be found from the function $F_S(\chi)$ in the same way as in Secs. 2 and 3, that is, by the substitution $\Phi(z) \to f(z)$. Here we shall use the physical property that the function $\Delta m_S/\gamma$, which is closely related to the electron's anomalous magnetic moment, is finite for $\chi = 0$, and consequently the function $\chi^{2/3}\Delta m_S/\gamma$ vanishes at $\chi = 0$, and one can write a dispersion relation for it with respect to $x = \chi^{-2/3}$. Then we obtain

$$\Delta m_s = -\frac{\alpha m\gamma}{\sqrt{\pi}}\int_0^\infty du \frac{zf(z)}{(1+u)^3}, \quad z = \left(\frac{u}{\chi}\right)^{2/3}. \tag{5.1}$$

The real and imaginary parts of the integral to the right are positive for all physical values of χ; therefore the signs of $\mathrm{Re}\,\Delta m_S$ and $\mathrm{Im}\,\Delta m_S$ agree with the sign of the energy $-\gamma m$ of the spin magnetic moment in a field.

For $\chi \ll 1$ it is not difficult to obtain the expansion

$$\mathrm{Re}\,\Delta m_s = -\frac{\alpha m\gamma}{2\pi}\left[1 - 12\chi^2\left(\ln\chi^{-1} + C + \frac{1}{2}\ln 3 - \frac{37}{12}\right) + \ldots\right],$$

$$\mathrm{Im}\,\Delta m_s = -\frac{\sqrt{3}\,\alpha m\gamma}{4}\chi\left(1 - 4\sqrt{3}\chi + \frac{105}{2}\chi^2 - \ldots\right). \tag{5.2}$$

For $\chi \gg 1$ we have

$$\mathrm{Re}\,\Delta m_s = -\frac{\Gamma(1/3)\,\alpha m\gamma}{9\sqrt{3}(3\chi)^{2/3}}\left(1 + \frac{6\Gamma(2/3)}{\Gamma(1/3)}(3\chi)^{-2/3} + \ldots\right),$$

$$\mathrm{Im}\,\Delta m_s = -\frac{\Gamma(1/3)\,\alpha m\gamma}{9(3\chi)^{2/3}}\left(1 - \frac{6\Gamma(2/3)}{\Gamma(1/3)}(3\chi)^{-2/3} + \ldots\right). \tag{5.3}$$

In contrast to Δm which increases with energy and field like $\alpha m\chi^{2/3}$, the mass correction associated with the direction of the spin increases more weakly: $\Delta m_S \propto \alpha m\chi^{1/3}$ (in order of magnitude $\gamma \sim \chi$).

In the rest system one can regard $\mathrm{Re}\,\Delta m_S$ as the interaction energy $-\Delta\mu\boldsymbol{\zeta}\cdot\mathbf{H}$ of the anomalous magnetic moment with the magnetic field, which in an arbitrary system can be written in the form $-\Delta\mu m\gamma/\mu_0$. In such a case it follows from Eq. (5.1) that the anomalous magnetic moment $\Delta\mu$, due to radiative effects of second order and the interaction with an intense external field, is given by

$$\frac{\Delta\mu}{\mu_0} = \frac{\alpha}{\sqrt{\pi}}\int_0^\infty du \frac{z\Upsilon(z)}{(1+u)^3} =$$

$$= \begin{cases} \dfrac{\alpha}{2\pi}\left[1 - 12\chi^2\left(\ln\dfrac{1}{\chi} + C + \dfrac{1}{2}\ln 3 - \dfrac{37}{12}\right) + \ldots\right], & \chi \ll 1 \\ \dfrac{\Gamma(1/3)\,\alpha}{9\sqrt{3}(3\chi)^{2/3}}\left[1 + \dfrac{6\Gamma(2/3)}{\Gamma(1/3)}(3\chi)^{-2/3} + \ldots\right], & \chi \gg 1 \end{cases} \tag{5.4}$$

The ratio $\Delta\mu/\mu_0$ is positive and as χ increases it decreases monotonically from Schwinger's value $\alpha/2\pi$[8] to zero (see Fig. 3).

6. CONCLUSION

In the radiative effects which have been considered, we have essentially used the fact that upon switching-off the field μ^2 and Δm tend to zero at least linearly, but the anomalous magnetic moment tends to a finite value. This reduced the problem to the construction of a function $f(z)$ (see Eq. (2.6)) from its imaginary part $\Phi(z)$ on the real axis. The Airy function describes the motion of a charged particle in a constant field and obeys the homogeneous equation $\Phi'' - z\Phi = 0$. The epsilon function $\Upsilon(z)$ associated with it obeys the inhomogeneous equation $\Upsilon'' - z\Upsilon = -\pi^{-1/2}$. Both axes make up the function $f(z)$ in the same way that the well-known singular homogeneous and inhomogeneous functions Δ_1 and Δ_P form the causal Green's function $\Delta_C = \Delta_P + (i/2)\Delta_1$.[9] Let us show that there is a basis for this analogy.

It is well-known that in a plane-wave field with potential $A_\mu = A_\mu(kx)$, $k^2 = kA = 0$, the Fourier transform of the causal Green's function with respect to the difference of coordinates is given by[4)]

$$\Delta_C(x,p) = i\int_0^\infty d\tau \exp\left\{-i\int_0^\tau d\tau' [(p - eA(\varphi - 2kp\tau'))^2 + m^2 - i\varepsilon]\right\},$$

$$\varphi = kx. \tag{6.1}$$

For a constant crossed field $A_\mu = a_\mu(kx)$, one can write the function (6.1) in the form

$$\Delta_C(x,p) = \frac{i}{m^2(2\chi)^{2/3}}\exp\left\{i\left(zu + \frac{u^3}{3}\right)\right\}\int_u^\infty dt \exp\left\{-i\left(zt + \frac{t^3}{3}\right)\right\},$$

where

$$z = \frac{p^2 + m^2 - (ap)^2 a^{-2}}{m^2(2\chi)^{2/3}}, \quad u = -\frac{(a, p - eA)}{am(2\chi)^{1/3}}, \tag{6.3}$$

and χ is related to p_μ by formula (3.2). The dependence on the coordinates enters into u only in terms of the phase $\varphi = kx$. Upon integration over the coordinates, an interval of phases lying near $\varphi = (ap)/ea^2$ and having an extent $\Delta\varphi \sim (m/ea)(2\kappa)^{1/3}\max(1, \sqrt{z})$ gives the basic contribution to the matrix elements (see[6,7]). This interval corresponds to $u = 0$ and $\Delta u \sim \max(1, \sqrt{z})$. Thus, in the center of the effective interval the Green's function (6.2) reduces to $f(z)$:

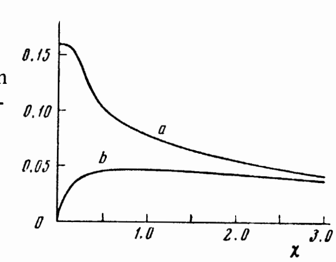

FIG. 3. The radiative correction $\Delta m_S = -\alpha m\gamma(a + ib)$ to the interaction energy of the spin with the field and the anomalous magnetic moment $\Delta\mu = \alpha\mu_0 a$ as functions of χ.

4)Upon switching-off the field, $\Delta_C(x, p) \to (p^2 + m^2 - i\epsilon)^{-1}$.

$$\Delta_c(x,p)\,|_{u=0} = \frac{\sqrt{\pi}}{m^2(2\chi)^{2/3}} f(z). \qquad (6.4)$$

The obtained formulas will be valid if the dimensions of the field appreciably exceed the length m/eB of the forming region; for $B \sim 10^8$ Heaviside units this length amounts to 4×10^{-6} cm. The effects attain their optimal values for $\kappa, \chi \sim 1$ which, for a field of 10^8 Heaviside units corresponds to an energy of photons and electrons of 25 GeV.

The connection of Re μ^2, Re Δm, and $\Delta\mu$ with observed effects requires individual consideration. Here we mention only the Cerenkov radiation of electrons in a field, the angle and intensity of which are related to Re μ^2 by the relations

$$\theta_{\parallel,\perp} = \left[-\frac{\mathrm{Re}\,\mu^2_{\parallel,\perp}(\varkappa)}{\omega'^2} - \frac{m^2}{p_0^2}\right]^{1/2},$$

$$dI_{\parallel,\perp} = \alpha\left(-\mathrm{Re}\,\mu^2_{\parallel,\perp} - \frac{m^2\omega'^2}{p_0^2}\right)\frac{d\omega'}{\omega'}$$

where $k' = (\mathbf{k}, i\omega')$ denotes the momentum and frequency of the radiation, and p_0 denotes the energy of the electrons.

The most interesting property is the enhancement of the radiative corrections in a constant field at large energies and fields, which for $\alpha\kappa^{2/3} \sim 1$ leads to the necessity for an exact amount of the interaction with the radiation field. For fields of 10^8 Heaviside units, this corresponds to an energy of 25×10^3 GeV. In the classical theory one can treat the radiation as a perturbation as long as the radiative damping force is small in comparison with the Lorentz force in the particle's rest system, i.e., for $\alpha\chi \ll 1$. We see that the quantum theory changes this condition, and apparently $\alpha\chi^{2/3}$ is the universal parameter characterizing the applicability of perturbation theory to radiation in the presence of an external field for large energies and fields.

The enhancement of the radiative corrections is closely related to the behavior of the probabilities (2.1) and (3.1) for large values of κ and χ. In this connection we note that in a variable field $A_\mu = a_\mu \cos(kx)$, when the parameter $x \equiv ea/m \lesssim 1$ but $\kappa \gg 1$, the probability of pair production depends on κ logarithmically, i.e., in exactly the same way as in perturbation theory but with the replacement of the electron's mass by an effective mass, $m \to m_*$ (see[7]). Therefore, one can assume that in a variable field the radiative corrections will increase with energy logarithmically, but a power law dependence is characteristic for their growth in a constant field, where the interaction with quanta of a field of arbitrarily small frequency turns out to be essential. Although the probabilities require radiative corrections in the region κ, $\chi \sim \alpha^{-3/2}$, these formulas by themselves are internally consistent since over the coherence interval the probabilities, as usual, are small, $\sim\alpha$. For example, the probability for pair production is given by

$$\frac{F(\varkappa)}{n}\Delta t \sim \frac{\alpha m^2 \varkappa^{2/3}}{k_0'}\frac{m}{eB}\varkappa^{1/3} \sim \alpha;$$

an estimate of the coherence time Δt is written down here on the basis of Eq. (4.10), where it has been taken into consideration that values $\kappa^2/\chi\chi' \sim 1$ and $y \lesssim 1$ are effective for large values of κ.

In an arbitrary constant field the effects are determined, in addition to χ, by two more purely field invariants, for example, the quantities ϵ, η of the electric and magnetic fields in a coordinate system where they are parallel, relative to B_0.[5)] For actually existing fields ϵ and η are very small in comparison with unity, but for ultrarelativistic electrons they are also small in comparison with χ. Their neglect is equivalent to a transformation to a crossed field.

The change of the electron mass in the presence of a magnetic field was considered by Newton[10] in the approximation $\chi^{2/3} \lesssim \eta \ll 1$; as a consequence of this approximation it is difficult to compare his results with ours. Nevertheless, the coefficient of $\chi^2 \ln \chi^{-1}$ in the expansion (3.5) for Re Δm agrees with Newton's coefficient for $\chi^2 \ln \eta^{-1}$. The anomalous magnetic moment of an electron in a magnetic field was considered by Gupta,[11] Demeur,[12] Newton,[10] and Ternov and collaborators.[13] Comparison is possible only with the results of the last article, which should agree with our results in the limit $\eta = 0$. However, such agreement does not exist although the expressions for $\Delta\mu$ for $\chi \gg 1$ differ only by a coefficient.

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Note (September 5, 1969). Another way to obtain the mass shift of an electron and its anomalous magnetic moment is a direct evaluation of the amplitude for the elastic scattering of an electron in an external field according to the formula

$$M_{pp} = -ie^2 \int \overline{\psi}_p(x')\gamma_\mu S^c(x',x)\gamma_\mu\psi_p(x) D^c(x'-x)\,d^4x\,d^4x',$$

where ψ_p and S^c are the wave function and Green's function of the electron in the field, and D^c is the photon Green's function. The amplitude determined by this formula diverges and requires renormalization. For this reason, one should subtract from it a similar amplitude corresponding to a field B_1 which tends to zero. The renormalized amplitude which is thus obtained does not contain any divergences and vanishes when the field is turned off. One can represent it in the form (compare with Eq. (4.12))

$$M_{pp} = -VT\frac{\alpha m^2}{2\sqrt{\pi}p_0}\int_0^\infty \frac{du}{(1+u)^2}\left\{\int_z^\infty dt\left[f(t) - \frac{1}{\sqrt{\pi}t}\right] + \frac{2+2u+u^2}{z(1+u)}f'(z) - 2\gamma\frac{zf(z)}{1+u}\right\}, \quad z = \left(\frac{u}{\chi}\right)^{2/3},$$

where the first two terms determine the part of the amplitude which does not depend on the spin direction, leading to the change in the electron mass given by Eq. (3.4), and the third term – the part of the amplitude which depends on the spin direction – gives formulas (5.1) and (5.4) for the mass shift Δm_S and for the anomalous magnetic moment $\Delta\mu$.

An important consequence of the direct calculation of the amplitude is a proof of its analytic properties which, in the method of analytic continuation, were postulated on the basis of causality arguments.

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[5)] These quantities are given by $\epsilon, \eta = (\sqrt{f^2+g^2} \mp f)^{1/2}$, where $f = F_{\mu\nu}^2/4B_0^2$ and $g = F_{\mu\nu}F^*_{\mu\nu}/4B_0^2$.

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