GEOMETRICAL OPTICS AND CONSERVATION OF THE ADIABATIC INVARIANT

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The conditions for the conservation of the adiabatic invariant $d\mathscr{E}/\omega$ are determined for a wave packet moving in a smoothly inhomogeneous and slowly nonstationary dispersive medium. Corrections to the "quasistationary" value of the dielectric tensor of a nonabsorbing medium are derived on the basis of the requirement of conservation of the adiabatic invariant.

1. In this paper, making use of the geometrical-optics approximation, we determine the conditions under which the adiabatic invariant is conserved for a wave packet propagating in an inhomogeneous and nonstationary dispersive medium (at the same time, we refine the concept of adiabatic invariant for wave packets). In addition, we obtain the correction to the "quasistationary" value of the dielectric tensor in a non-absorbing medium under more general conditions than obtained by Pitaevskiĭ[1].

2. The geometrical-optics asymptotic form of the solutions of Maxwell's equations was constructed in the general case in[2] under the assumption that the kernel $\epsilon_{\alpha\beta}(t - t', t; \mathbf{r} - \mathbf{r}', \mathbf{r})$ of the material equation

$$D_\alpha(\mathbf{r}, t) = \int_{-\infty}^{t} dt' \int d^3r' \varepsilon_{\alpha\beta}(t-t', t; \mathbf{r}-\mathbf{r}', \mathbf{r}) E_\beta(\mathbf{r}', t') \tag{1}$$

is specified. This is equivalent to specifying the dielectric tensor

$$\varepsilon_{\alpha\beta}(\omega, t; \mathbf{k}, \mathbf{r}) = \int_{-\infty}^{t} dt' \int d^3r' \varepsilon_{\alpha\beta}(t-t', t; \mathbf{r}-\mathbf{r}', \mathbf{r}) \exp\{i\omega(t-t') - i\mathbf{k}(\mathbf{r}-\mathbf{r}')\}. \tag{2}$$

In[2] it was shown that the density of the electromagnetic energy W in an inhomogeneous and nonstationary dispersive medium

$$W = \frac{1}{16\pi} \frac{\partial(\omega^2 \varepsilon_{\alpha\beta}^{h})}{\omega \partial\omega} E_\alpha E_\beta^*, \tag{3}$$

satisfies the following transport equation

$$\frac{\partial W}{\partial t} + \operatorname{div} \mathbf{S} = \varkappa W \left[2i\omega \varepsilon_{\alpha\beta}^{a} - \frac{\partial \varepsilon_{\alpha\beta}^{h}}{\partial t} + \omega \left(\frac{\partial^2 \varepsilon_{\alpha\beta}^{h}}{\partial\omega \partial t} - \frac{\partial^2 \varepsilon_{\alpha\beta}^{h}}{\partial k_j \partial x_j} \right) \right] f_\alpha f_\beta^*, \tag{4}$$

where $\mathbf{S} = \mathbf{u}W$ is the Poynting vector, $\mathbf{u}$ is the group velocity, $\epsilon_{\alpha\beta}^{h}$ and $\epsilon_{\alpha\beta}^{a}$ are the Hermitian and anti-Hermitian parts of the tensor (2), and

$$\varkappa = \left[\frac{\partial(\omega^2 \varepsilon_{\alpha\beta}^{h})}{\omega \partial\omega} f_\alpha f_\beta^* \right]^{-1}.$$

The polarization vector $\mathbf{f}$ normalized to unity is determined from the system of homogeneous equations

$$g_{\alpha\beta} f_\beta = 0, \quad g_{\alpha\beta} \equiv k^2 \delta_{\alpha\beta} - k_\alpha k_\beta - \frac{\omega^2}{c^2} \varepsilon_{\alpha\beta}^{\mathrm{э}}, \tag{5}$$

where ω and $\mathbf{k}$ stand for the corresponding partial derivatives of the phase φ: $k_j \equiv \partial\varphi/\partial x_j$ and $\omega \equiv -\partial\varphi/\partial t$.

3. We show first that under the condition

$$2i\varepsilon_{\alpha\beta}^{a} + \left(\frac{\partial^2 \varepsilon_{\alpha\beta}^{\mathrm{э}}}{\partial\omega \partial t} - \frac{\partial^2 \varepsilon_{\alpha\beta}^{\mathrm{э}}}{\partial k_j \partial x_j} \right) = 0 \tag{6}$$

it follows from the transport equation (4) that the adiabatic invariant is conserved. To this end, we take into consideration the fact that the three-parameter family of ray trajectories $\mathbf{r} = \mathbf{r}(t, \sigma_1, \sigma_2, \sigma_3)$, to which the vector $\mathbf{u} = d\mathbf{r}/dt$ is tangent, satisfies the following relation[3]:

$$\operatorname{div} \mathbf{u} = \frac{d}{dt} \ln j(t), \quad j(t) \equiv \frac{\partial(x_1, x_2, x_3)}{\partial(\sigma_1, \sigma_2, \sigma_3)}. \tag{7}$$

We further take into account the fact that, in accordance with Hamilton's canonical equations corresponding to the eikonal equation $g \equiv \det \| g_{\alpha\beta} \| = 0$, the derivative $d\omega/dt$ on the trajectory $\mathbf{r} = \mathbf{r}(t)$ is equal to

$$\frac{d\omega}{dt} = -\frac{\partial g}{\partial t} \Big/ \frac{\partial g}{\partial \omega}. \tag{8}$$

Using (5) and (8), we can verify that

$$\frac{d\omega}{dt} = -\omega\varkappa \frac{\partial \varepsilon_{\alpha\beta}^{\mathrm{э}}}{\partial t} f_\alpha f_\beta^*.$$

As the result, the transport equation (4) takes the form

$$\frac{dW}{dt} + W \frac{d}{dt} \ln j(t) = W \frac{d}{dt} \ln \omega. \tag{9}$$

Its integration along the ray $\mathbf{r} = \mathbf{r}(t)$ gives the relation $Wj\omega^{-1} = \text{const}$, which when multiplied by $d\sigma_1 d\sigma_2 d\sigma_3$ yields

$$\frac{d\mathscr{E}}{\omega} = \frac{WdV}{\omega} = \text{const} = \frac{W^0 dV^0}{\omega^0}, \tag{10}$$

where $dV = j(t) d\sigma_1 d\sigma_2 d\sigma_3 = dx_1 dx_2 dx_3$ is a wave-packet volume element moving together with the wave (the time variation of dV is in accord with the behavior of the ray trajectories $\mathbf{r}(t)$), and dV^0 is the initial element of the volume at the initial instant of time t^0.

Thus, when condition (6) is satisfied (and the condition for the applicability of the geometrical approximation holds, i.e., neglecting diffraction effects and the momentum spread due to dispersion) the ratio $d\mathscr{E}/\omega$ (of the energy $d\mathscr{E} = WdV$ concentrated in the wave-packet volume element dV to the frequency ω) is a conserved quantity, which can naturally be called the adiabatic invariant (it is usually assumed that $\omega = \text{const}$ within the limits of the packet, and in analogy with the harmonic oscillator, the adiabatic invariant is defined as $\mathscr{E}/\omega$, where $\mathscr{E} = \int WdV$ is the total energy of the packet[4,5]; in the case of wave packets with variable frequency, however, the quantity $\mathscr{E}/\omega$ is not conserved whereas $d\mathscr{E}/\omega = \text{const}$).

4. It is obvious that the entire procedure can be carried out in reverse order, and then the condition that $d\mathscr{E}/\omega$ be conserved leads to (6).

For nonabsorbing media (zero Joule losses) the requirement of conservation of the adiabatic invariant $d\mathcal{E}/\omega$ denotes in essence the condition for conservation of the number of quanta dN in the wave packet:

$$\frac{d\mathcal{E}}{\omega}=\frac{\hbar\omega dN}{\omega}=\hbar dN=\text{const.} \quad (11)$$

in the case of slow variation of the properties of the medium, this requirement is perfectly natural[1], since the quantum-mechanical state of the system remains practically unchanged in this case, and consequently the probability of the transition to other levels is exponentially small. In order of magnitude, this probability is proportional to $e^{-\omega T}$, where T is the characteristic time of the nonstationary processes, and is large compared with the period of the oscillation $\tau \sim 1/\omega$. In the geometrical-optics approximation, exponentially small terms of order $e^{-\omega T}$ are not taken into account (the field amplitudes are expanded only in powers of the small parameter $\mu \sim 1/\omega T$, whereas at $\mu \to 0$ the exponential $e^{-\omega T} \sim e^{-1/\mu}$ tends to zero more rapidly than any power of μ), so that the quantum-mechanical condition (11) agrees well with the geometrical approximation of the solutions of Maxwell's equations.

5. Pitaevskiĭ[1] obtained from the condition of the conservation of the adiabatic invariant[1)] a correction to the "quasistationary" value of the dielectric constant of a non-absorbing nonstationary medium with frequency dispersion. Using (6), we can generalize Pitaevskiĭ's results to the case of inhomogeneous and nonstationary media with both frequency and spatial dispersion.

We note beforehand that the calculation of the tensor $\epsilon_{\alpha\beta}(\omega, t, \mathbf{k}, \mathbf{r})$ requires, generally speaking, an analysis of the microscopic processes occurring in the medium (this can be done within the framework of the geometrical-optics method by considering self-consistent solutions of the field equations and the charge motion equations). However, in the absence of absorption, it is possible to determine the tensor $\epsilon_{\alpha\beta}$ by a purely phenomenological method.

Obviously, in the zeroth approximation of the geometrical-optics method

$$\varepsilon_{\alpha\beta}(\omega,\ t;\ \mathbf{k},\ \mathbf{r})=\varepsilon_{\alpha\beta}{}^{0}(\omega,\ \mathbf{k},\ \gamma_j(\mathbf{r},\ t))+O(\mu), \quad (12)$$

where $\mu \sim 1/\omega T \ll 1$ is the small parameter of the method, and $\epsilon^{0}_{\alpha\beta}$ is the "quasistationary" value of the dielectric tensor, calculated for a homogeneous and stationary medium having the same values of the macroscopic parameter γ_j (pressure, temperature[2)], etc.) as in the considered inhomogeneous and nonstationary medium at the given instant of time t and at the given point of space $\mathbf{r}$.

For non-absorbing media, $\epsilon^{0}_{\alpha\beta}$ is Hermitian, so that

$$\varepsilon_{\alpha\beta}{}^{\mathrm{э}}=\varepsilon_{\alpha\beta}{}^{0}+O(\mu). \quad (13)$$

Substituting (13) in (6), we obtain for the anti-Hermitian part of the tensor $\epsilon_{\alpha\beta}$, which is of the order of $O(\mu)$ in accordance with the initial premises of the geometrical approximation[2],

$$\varepsilon^{\mathrm{a}}_{\alpha\beta}=\frac{i}{2}\Big(\frac{\partial^2\varepsilon_{\alpha\beta}{}^{0}}{\partial\omega\,\partial t}-\frac{\partial^2\varepsilon_{\alpha\beta}{}^{0}}{\partial k_j\,\partial x_j}\Big)+O(\mu^2). \quad (14)$$

Not being interested in first-order corrections to $\epsilon^{0}_{\alpha\beta}$, which are significant perhaps only in the determination of the total phase shift, we obtain from (12)–(14)

$$\varepsilon_{\alpha\beta}=\varepsilon^{0}_{\alpha\beta}+\frac{i}{2}\Big(\frac{\partial^2\varepsilon_{\alpha\beta}}{\partial\omega\partial t}-\frac{\partial^2\varepsilon^{0}_{\alpha\beta}}{\partial x_j\,\partial k_j}\Big)+O(\mu^2). \quad (15)$$

Although the analysis has been carried out for electromagnetic waves in an anisotropic medium, the results can be readily transferred to the case of longitudinal and transverse waves in an isotropic medium (to this end it is necessary to replace $\epsilon_{\alpha\beta}$ by $\epsilon_{\perp}\delta_{\alpha\beta}$ or $\epsilon_{\parallel}\delta_{\alpha\beta}$ throughout). In particular, in the absence of spatial dispersion in a homogeneous isotropic medium we get from (15) Pitaevskii's result[1]:

$$\varepsilon=\varepsilon^0(\omega,t)+\frac{i}{2}\Big(\frac{\partial^2\varepsilon^0(\omega,t)}{\partial\omega\,\partial t}\Big)+O(\mu^2).$$

6. Physically, the appearance of anti-Hermitian corrections to the tensor $\epsilon_{\alpha\beta}$ is equivalent to a certain additional phase shift between the components of the vectors $\mathbf{D}(\mathbf{r}, t)$ and $\mathbf{E}(\mathbf{r}, t)$, and corresponds to the fact that no "stationary" phase relation, determined by the equality $D_\alpha = \epsilon^{0}_{\alpha\beta}E_\beta$, can be established in a medium with varying parameters. For example, for high-frequency waves in an isotropic plasma, in the absence of collisions between the electrons and the ions we have

$$\varepsilon^0=1-\omega_L{}^2/\omega^2, \quad (16)$$

where $\omega_L^2 = 4\pi e^2 n/m$, e and m are the charge and mass of the electrons, and n is their concentration. In the stationary case (n = const), the oscillations of $\mathbf{D}(\mathbf{r}, t)$ and $\mathbf{E}(\mathbf{r}, t)$ are in phase. However, when the concentration changes with time as the result of the finite time of establishment of the polarization of the medium, the oscillations of D either lag or lead (depending on the sign of $\partial n/\partial t$) the oscillations of the field intensity E at the same point. In a medium with spatial dispersion, owing to the processes of matter transport, such a phase disparity can be due also to the inhomogeneity of the medium.

7. The foregoing analysis agrees with the results of a number of previously published papers. In particular, for electromagnetic waves in a magnetoactive plasma drifting with velocity $\mathbf{U}$ in the direction of the external magnetic field $\mathbf{H}_0$, as is well known, the following relations hold (when $\mathbf{k} \parallel \mathbf{H}_0$)[6]

$$\varepsilon^{0}_{\alpha\beta}=\begin{pmatrix}\varepsilon_1 & ig_1 & 0\\ -ig_1 & \varepsilon_1 & 0\\ 0 & 0 & \varepsilon_2\end{pmatrix}, \quad (17)$$

where

$$\varepsilon_1=1-\frac{\omega_L{}^2(\omega-\mathbf{kU})^2}{\omega^2[(\omega-\mathbf{kU})^2-\omega_H{}^2]},\quad \omega_H=\frac{eH_0}{mc},$$

$$\varepsilon_2=1-\frac{\omega_L{}^2}{(\omega-\mathbf{kU})^2},\quad g_1=\frac{\omega_H(\omega-\mathbf{kU})\,\omega_L{}^2}{\omega^2[(\omega-\mathbf{kU})^2-\omega_H{}^2]}.$$

Substituting (17) in (15), we can easily find the anti-Hermitian part of the tensor $\epsilon_{\alpha\beta}$, which depends here on the change of the parameters $n(\mathbf{r}, t)$, $\mathbf{U}(\mathbf{r}, t)$, and $\mathbf{H}_0(\mathbf{r}, t)$. At the same time, it is easy to prove that a

1) The fields considered in [1] were monochromatic.

2) Generally speaking, the set of parameters γ_j can be continual, for example the set of values of the unperturbed distribution function $f_0(p)$ in the case of plasmalike media.

calculation of the amplitudes and of the wave energy in accordance with formulas (4) and (15)—(17) gives results that coincide with those obtained in[4,5] by directly using the equations of motion of the electrons instead of the phenomenological relation (1).

It seems that the corrections to the tensor $\epsilon^0_{\alpha\beta}$. Which are determined from (15), are valid for a rather wide class of media.

8. In conclusion we note that the transport equation (4) in the absence of absorption can be rewritten, as follows from (6), (8), and (9) in the form

$$\frac{d}{dt}\left(\frac{W}{\omega}\right)+\frac{W}{\omega}\operatorname{div}\mathbf{u}=0. \quad (18)$$

For hydrodynamics, this form of the transport equation was proposed by Bretherton and Garnet[7]. To be sure, in[7], unlike in our case, they considered waves in a medium moving with velocity $\mathbf{U}$, so that ω was replaced in[7] by the frequency $\omega' = \omega - \mathbf{k}\cdot\mathbf{U}$, which differs from ω by the Doppler shift $\mathbf{k}\cdot\mathbf{U}$.

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