PARAMETRIC EXCITATION OF UPPER AND LOWER HYBRID RESONANCES

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The stability with respect to excitation of potential oscillations of a plasma located in a stationary magnetic and high-frequency electric fields is considered. It is shown that the plasma is unstable if the overtones of the HF field $p\omega_0$ approach the sum of the upper and lower hybrid frequencies ($p\omega_0 \approx \omega_+ + \omega_-$). Dissipative effects determining the threshold of instability are taken into account.

A plasma interacting with a high-frequency field $\mathbf{E} = \mathbf{E}_0 \sin \omega_0 t$ turns out to be, under certain conditions, unstable against the buildup of potential oscillations. The additional presence of a constant magnetic field $\mathbf{B}_0$ greatly extends the possibility of appearance of such an instability. References 1 and 2 are devoted to an investigation of the behavior of a plasma under precisely such conditions. In this paper we point out a new type of instability, never considered before. We have in mind the excitation of high-frequency electronic oscillations of hybrid frequencies

$$\omega_\pm^2 = \frac{1}{2}\{\Omega_e^2 + \omega_{Le}^2 \pm [(\Omega_e^2 + \omega_{Le}^2)^2 - 4\Omega_e^2\omega_{Le}^2\cos^2\theta]^{1/2}. \quad (1)$$

Here $\Omega_\alpha = e_\alpha B_0/m_\alpha e$ are the cyclotron frequencies of the particles of type α, $\omega_{L\alpha} = (4\pi N_\alpha e_\alpha^2/m_\alpha)^{1/2}$ are the Langmuir frequencies of particles of type α, and θ is the angle between the wave vector $\mathbf{k}$ and the direction of the magnetic field $\mathbf{B}_0(0, 0, B_0)$. It is assumed here that the angle θ is not too close to $\pi/2$ ($\cos\theta \gg (m_e/m_i)(1 + \omega_{Le}^2/\Omega_e^2)$).

Let us consider the region of the external high-frequency field frequencies

$$\omega_0 \approx (\omega_+ + \omega_-)/p, \quad p = 1, 2, 3, \dots . \quad (2)$$

The dispersion equation for the high frequency longitudinal oscillations in the frequency region of interest to us is of the form

$$\Big[1 + \delta\varepsilon_e(\omega, \mathbf{k}) + \sum_{n=-\infty}^{\infty} J_n^2(a_B)\delta\varepsilon_i(\omega + n\omega_0, \mathbf{k})\Big]\Big[1 + \delta\varepsilon_e(\omega - p\omega_0, \mathbf{k}) + \sum_{n=-\infty}^{\infty} J_n^2(a_B)\delta\varepsilon_i(\omega - p\omega_0 + n\omega_0, \mathbf{k})\Big] = \Big[\sum_{n=-\infty}^{\infty} J_n(a_B)J_{n+p}(a_B)\delta\varepsilon_i(\omega + n\omega_0, \mathbf{k})\Big]^2. \quad (3)$$

Here $\delta\epsilon_\alpha(\omega, \mathbf{k})$ is the partial contribution of the particles of type α to the dielectric constant of the plasma in the absence of the high-frequency field. The argument of the Bessel function $J_n(a_B)$ of order n is $a_B = \mathbf{k}\cdot\mathbf{r}_B$, where $\mathbf{r}_B$ is the vector of the amplitude of relative displacement of the particles in the constant magnetic field and the high-frequency electric field:

$$a_B^2 = \frac{k^2e^2E_0^2}{m_e^2\omega_0^4}f(\theta), \quad (4)$$

$$f(\theta) = \Big[\frac{E_{0z}}{E_0}\cos\theta + \frac{\omega_0^2}{(\omega_0^2 - \Omega_e^2)}\frac{E_{0y}}{E_0}\sin\theta\Big]^2 + \frac{E_{0x}^2}{E_0^2}\sin^2\theta\frac{\omega_0^2\Omega_e^2}{(\omega_0^2 - \Omega_e^2)^2}.$$

We represent the complex quantity $\delta\epsilon_\alpha(\omega, \mathbf{k})$ in the form of a sum of real and imaginary parts:

$$\delta\varepsilon_\alpha(\omega, \mathbf{k}) = \delta\varepsilon_\alpha'(\omega) + \delta\varepsilon_\alpha'(\omega, \mathbf{k}) + i\delta\varepsilon_\alpha''(\omega, \mathbf{k}). \quad (5)$$

We shall consider below the case of large phase velocities and oscillations, $\omega \gg k_z v_{Te}$, where $v_{Te} = (T/m_e)^{1/2}$ is the thermal velocity of the electrons. Under these conditions, the real parts of $\delta\epsilon_\alpha$ greatly exceed the imaginary parts. If we neglect in (3) the imaginary parts of the dielectric constants, the spatial dispersion, and the contribution of the ionic component, then this equation breaks up into two equations:

$$1 + \delta\varepsilon_e'(\omega) = 0, \quad (6)$$

$$1 + \delta\varepsilon_e'(\omega - p\omega_0) = 0 \quad (7)$$

with solutions $\omega = \omega_+$ and $\omega - p\psi_0 = -\omega_-$. From this we get the condition for the frequency of the high-frequency field:

$$p\omega_0 = \omega_+ + \omega_-, \quad p = 1, 2, 3, \dots . \quad (8)$$

Allowance for the contribution of the ionic component and of the dissipative effects in Eq. (3) causes the solutions to differ from those given above:

$$\omega = \omega_+ + \gamma + \delta/2 \equiv \tilde{\omega}_+, \quad (9)$$

$$p\omega_0 = \omega_+ + \omega_- + \delta \equiv \tilde{\omega}_+ + \tilde{\omega}_-, \quad (10)$$

where γ and δ are small corrections.

Carrying out the corresponding expansions in (3), we obtain

$$[\gamma + \delta/2 - i\gamma_0(\tilde{\omega}_+) + \beta(\omega_+)][\gamma - \delta/2 - i\gamma_0(-\tilde{\omega}_-) + \beta(-\omega_-)] = \alpha_p\gamma_0(\tilde{\omega}_+)\gamma_0(-\tilde{\omega}_-), \quad (11)$$

where

$$\beta(\omega) = \Big(\frac{\partial\delta\varepsilon_e'(\omega)}{\partial\omega}\Big)^{-1}\Big\{\sum_{n=-\infty}^{\infty} J_n^2(a_B)\delta\varepsilon_i'(\omega + n\omega_0) + \delta\varepsilon_e'(\omega, \mathbf{k})\Big\}, \quad (12)$$

$$\alpha_p = \Big[\delta\varepsilon_e''(\tilde{\omega}_+)\delta\varepsilon_e''(\tilde{\omega}_-)\Big]^{-1}\Big[\sum_{n=-\infty}^{\infty} J_n(a_B)J_{n+p}(a_B)\delta\varepsilon_i'(\omega_+ + n\omega_0)\Big]^2, \quad (13)$$

and $\gamma_0(\omega) = -\partial\delta\epsilon_e'/\partial\omega)^{-1}\delta\epsilon_e''(\omega)$ is the usual expression for the damping decrement in the absence of a high-frequency field.

Being interested in the stability of the oscillations, let us investigate that solution of (11), the imaginary

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part Im γ of which can reverse sign

$$\operatorname{Im}\gamma=\frac{1}{2}(\gamma_0(\tilde\omega_+)+\gamma_0(-\tilde\omega_-))+2^{-1/2}\Bigg(\frac{(\Delta'')^2-(\Delta')^2}{4}+\alpha_p\gamma_0(\tilde\omega_+)\gamma_0(-\tilde\omega_-)+\Bigg\{\Bigg[\frac{(\Delta'')^2-(\Delta')^2}{4}+\alpha_p\gamma_0(\tilde\omega_+)\gamma_0(-\tilde\omega_-)\Bigg]^2+\Big(\frac{\Delta'\Delta''}{2}\Big)^2\Bigg\}^{1/2}\Bigg)^{1/2}; \tag{14}$$

here

$$\Delta'=\delta+\beta(\omega_+)-\beta(-\omega_-), \tag{15}$$

$$\Delta''=\gamma_0(\tilde\omega_+)-\gamma_0(-\tilde\omega_-). \tag{16}$$

In the absence of the high-frequency field ($\alpha_p = 0$), we obtain from (14) the usual expressions for the damping decrements of the oscillations $\gamma_0(\tilde\omega_+)$ or $\gamma_0(-\tilde\omega_-)$.

An investigation of the sign of Im γ as a function of the frequency deviation Δ' shows that Im γ is positive, i.e., small perturbations in the plasma grow exponentially in time if the following condition is satisfied

$$0\leqslant(\Delta')^2<(\alpha_p-1)(\gamma_0(\tilde\omega_+)+\gamma_0(-\tilde\omega_-))^2. \tag{17}$$

The inequality (17) can be satisfied if $\alpha_p > 1$. The latter condition, which determines the threshold value of the intensity of the high frequency field, is entirely due to allowance for the dissipative effects.

From (17) we can easily determine the upper frequency of the HF field, starting with which the oscillations start to grow. In this case the instability begins with excitation of waves propagating almost along the magnetic field. If the Langmuir frequency ω_{Le} exceeds Ω_e, then the limiting value of the frequency is

$$\omega_0=(\Omega_e+\omega_{Le})^{1/2}+\beta(-\Omega_e)-\beta(\omega_{Le})+(\gamma_0(\tilde\omega_{Le})+\gamma_0(-\tilde\Omega_e))(\alpha_1,1)^{1/2}, \tag{18}$$

where $\tilde\omega_{Le} = \tilde\omega_+(\theta = 0)$ and $\tilde\Omega_e = \tilde\omega_-(\theta = 0)$. In the opposite case ($\omega_{Le} < \Omega_e$) it is necessary to interchange ω_{Le} and Ω_e in (18).

The oscillation increment (14) assumes the maximum value at $\Delta' = 0$, i.e., at the frequencies

$$p\omega_0=\omega_++\omega_-+\beta(-\omega_-)-\beta(\omega_+). \tag{19}$$

At a given frequency ω_0 of the high-frequency field, expression (19) determines the value of the angle θ between the direction of the wave vector **k** and the direction of the magnetic field, at which the most rapidly increasing oscillations propagate; these oscillations have an increment

$$(\operatorname{Im}\gamma)_{max}=\frac{\gamma_0(\tilde\omega_+)+\gamma_0(-\tilde\omega_-)}{2}+\Bigg[\Big(\frac{\gamma_0(\tilde\omega_+)+\gamma_0(-\tilde\omega_-)}{2}\Big)^2+(\alpha_p-1)\gamma_0(\tilde\omega_+)\gamma_0(-\tilde\omega_-)\Bigg]^{1/2}. \tag{20}$$

We now determine the minimum value of the electric field intensity, starting with which the oscillations begin to increase. This value is determined from the inequality $\alpha_p > 1$, in which it is necessary to substitute explicit expressions for the quantities $\delta\epsilon'_\alpha(\omega)$ and $\delta\epsilon''_\alpha(\omega, \mathbf{k})$ for the cold magnetoactive plasma[3]:

$$\delta\varepsilon_e'(\omega)=-\frac{\omega_{Le}^2}{\omega^2}\Big(\frac{\omega^2-\Omega_e^2\cos^2\theta}{\omega^2-\Omega_e^2}\Big), \tag{21}$$

$$\delta\varepsilon_e''(\omega,\mathbf{k})=\frac{\omega}{k_zv_{Te}}\sqrt{\frac{\pi}{2}}\Bigg\{(kr_{De})^{-2}\exp\Big[-\frac{1}{2}\Big(\frac{\omega}{k_zv_{Te}}\Big)^2\Big]+\frac{1}{2}\frac{\omega_{Le}^2}{\Omega_e^2}\sin^2\theta\Bigg(\exp\Big[-\frac{1}{2}\Big(\frac{\omega+\Omega_e}{k_zv_{Te}}\Big)^2\Big]+\exp\Big[-\frac{1}{2}\Big(\frac{\omega-\Omega_e}{k_zv_{Te}}\Big)^2\Big]\Bigg)\Bigg\}-\frac{\nu}{\omega}\Big[1+2\frac{\omega_{Le}^2\Omega_e^2\sin^2\theta}{(\omega^2-\Omega_e^2)^2}\Big]\delta\varepsilon_e'(\omega). \tag{22}$$

The last term in (22) represents the contribution made to the imaginary part of the dielectric constant by the collisions between the electrons and ions, with effective frequency $\nu(\omega \gg \nu)$, whereas the first two terms describe the collisionless damping.

Recognizing that $a_B \ll 1$ near the threshold, we expand the Bessel functions in (13). It turns out here that the minimum threshold value of the field intensity occurs in the resonance region $\omega_0 \approx \omega_+ + \omega_-$ (p = 1). The condition for the growth of the oscillations has in this case the form

$$E_0^2>4(E^*)^2\frac{m_i^2\omega_+^4\omega_-^4\delta\varepsilon_e''(\tilde\omega_+)\delta\varepsilon_e''(-\tilde\omega_-)}{m_e^2\omega_{Le}^4(\omega_+^2-\omega_-^2)^2k^2r_{De}^2}, \tag{23}$$

where $E^* = m_e\omega_0^2r_{De}/e_ef(\theta)^{1/2}$ is the value of the external field at which the amplitude of the oscillations of the electron in external fields equals the Debye radius r_{De}. Neglecting the contribution made by the Landau damping at the frequency ω_+ and the cyclotron damping, we find that the right side of the inequality (23) is minimal for the wavelength

$$\lambda\!\!\!^{-}_0=\frac{1}{k_z}=\frac{\sqrt{2}v_{Te}}{\omega_-}\Big\{\ln\Big(\frac{\omega_{Le}^2\cos^2\theta}{\omega_-\nu\xi_-}\Big)\Big\}^{1/2}, \tag{24}$$

where

$$\xi_\pm=1+2\frac{\omega_{Le}^2\Omega_e^2\sin^2\theta}{(\omega_\pm^2-\Omega_e^2)^2}$$

and is equal to

$$E_0^2>8(E^*)^2\frac{m_i^2\omega_+^3\omega_-\nu^2\xi_+\xi_-\cos^2\theta}{m_e^2\omega_{Le}^2(\omega_+^2-\omega_-^2)^2}\ln\Big(\frac{\omega_{Le}^2\cos^2\theta}{\omega_-\nu\xi_-}\Big). \tag{25}$$

If the plasma dimension L exceeds $\lambda\!\!\!^{-}_0$, then the instability begins with a buildup of oscillations of wavelength $\lambda\!\!\!^{-}_0$. The threshold value of the field is then determined by the right side of the inequality (25). If $\lambda\!\!\!^{-}_0$ exceeds the dimensions of the system, then the threshold value of the field intensity is determined by the collisionless damping, and excitation of oscillations begins with wavelengths of the order of the system dimension L. In this case the condition for the instability of the oscillations is obtained from (23), where L should be substituted for the wavelength.

The largest oscillation increment (20) is obtained in the limit when $\alpha_p \gg 1$:

$$(\operatorname{Im}\gamma)^2_{max}=\frac{\Omega_e^3\omega_{Le}^3|\cos\theta|\sin^2\theta}{4[(\Omega_e^2-\omega_{Le}^2)^2+4\Omega_e^2\omega_{Le}^2\sin^2\theta]}\times\Big(\sum_{n=-\infty}^{\infty}J_n(a_B)J_{n+p}(a_B)\frac{\omega_{Li}^2}{(n\omega_0+\omega_+)^2}\Big)^2 \tag{26}$$

In the case of resonance at the overtones of the external frequency $p\omega_0$, the increment decreases with increasing p. For p = 1 and $a_B \ll 1$ we obtain

$$(\operatorname{Im}\gamma)^2_{max}=\frac{a_B^2\omega_{Li}^4\operatorname{tg}^2\theta}{16|\Omega_e\cos\theta|\omega_{Le}}. \tag{27}$$

When the angle θ approaches $\pi/2$, the increment (27) increases. However, as indicated above, in the direct vicinity of the angle $\pi/2$ the formulas obtained above are no longer valid. Namely, in this case the frequency ω_- tends to zero and it becomes important to take into account the motion of the ions in the spectrum of the lower hybrid resonance. For angles θ close to $\pi/2$ ($\cos^2\theta \ll m_e m_i^{-1}(1 + \omega_{Le}^2/\Omega_e^2)$) we have

$$\omega_-^2 = (\Omega_i^2 + \omega_{Li}^2) / (1 + \omega_{Le}^2 / \Omega_e^2) \tag{28}$$

and in place of (27) we obtain ($\theta = \pi/2$)

$$(\mathrm{Im}\,\gamma)^2_{max} = \frac{a_B^2 \omega_{Li}^8}{16\Omega_i(\Omega_i^2 + \omega_{Li}^2)^{5/2}}. \tag{29}$$

Formula (29) is valid if $(\mathrm{Im}\,\gamma)_{max}$ is much smaller than the frequency of the lower hybrid resonance. For example, if the condition $\omega_{Li}^2 \gg \Omega_i^2$ is satisfied, formula (29) holds when

$$E_0^2 \ll \frac{16 m_e^2 \omega_0^4 \Omega_i}{k^2 e^2 f(\pi/2)\omega_{Li}} \left(\frac{\Omega_e^2}{\Omega_e^2 + \omega_{Le}^2}\right). \tag{30}$$

If the inverse inequality is satisfied, the imaginary part of γ can become of the order of the frequency (28), and even appreciably larger. In the latter case, the theory of parametric resonance developed in[1] for strong external high frequency fields becomes valid.

Thus, in the presence of a high-frequency field, a plasma situated in a constant magnetic field turns out to be unstable against the buildup of high-frequency oscillations if the overtones of the external frequency are given by $p\omega_0 \approx \omega_+ + \omega_-$. Greatest interest attaches to the resonances with $p = 1$ and $p = 2$. These resonances, having the largest increment, indicate plasma instability both at external-field frequencies ω_0 larger than the frequency of the upper hybrid resonance, $\omega_0 \approx \omega_+ + \omega_-$, and at frequencies lower than the frequency of this resonance, $\omega_0 \approx (\omega_+ + \omega_-)/2$. It is precisely the latter which causes the instability, not observed in[1], in the frequency region $\Omega_e > \omega_0 > \omega_{Le}$. Finally, we note that when the overtone of the external frequency approaches the difference of the hybrid frequencies, no buildup of high frequency oscillations takes place.

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