APPLICABILITY OF THE APPROXIMATION OF A MARKOV RANDOM PROCESS IN PROBLEMS RELATING TO THE PROPAGATION OF LIGHT IN A MEDIUM WITH RANDOM INHOMOGENEITIES

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The problem of light propagation in a medium with random inhomogeneities of the refractive index is considered in the approximation of a parabolic equation. The condition of a Markov process of light-wave propagation is formulated as the stipulation that the characteristics of the electromagnetic field are independent of the statistical characteristics of the refractive-index field in the region of space which is without significance in the sense of the causality principle. This condition allows us to write closed equations for the statistical characteristics of the light-wave field when the fluctuations of the index of refraction are of nongaussian character. The Markov property in the problem considered is due to the smallness of the ratio of the correlation range of the refractive-index field to the longitudinal correlation range (along the original direction of propagation of the wave) of the light-wave field, which is determined by the path length of the light. The first corrections to the mean field and the coherence function, caused by nonmarkov aspects of the process, are found.

1. FUNDAMENTAL EQUATIONS

LET us consider the propagation of a light wave in a medium with random inhomogeneities of the dielectric constant, which fills the half-plane (sic) $x > 0$. For short waves the electromagnetic field can be described in the approximation of the parabolic equation (quasi-optics)[1]

$$2ik\frac{\partial u(x,\rho)}{\partial x}+\Delta_\perp u+k^2\varepsilon(x,\rho)u(x,\rho)=0, \qquad (1)$$

which is to be solved with the initial condition

$$u(0,\rho)=u^0(\rho). \qquad (2)$$

In Eq. (1), ρ, $\Delta_\perp$, and $\epsilon(x,\rho)$ denote respectively the transverse coordinates, the Laplace operator with respect to the transverse coordinates, and the deviation of the dielectric constant from its mean value, which we shall take to be unity.

When there are no inhomogeneities of the dielectric constant (i.e., for $\epsilon \equiv 0$) the solution of Eq. (1) is of the form

$$u(x,\rho)=\int d\rho' g(x,\rho;0,\rho')u^0(\rho'), \qquad (3)$$

where

$$g(x,\rho;x',\rho')=\frac{k\theta(x-x')}{2\pi i(x-x')}\exp\left\{\frac{ik(\rho-\rho')^2}{2(x-x')}\right\} \qquad (4)$$

is the Green's function of the problem in question. For $x' \to x$, Eq. (4) goes over into the formula

$$g(x,\rho;x,\rho')={}^1\!/_2\delta(\rho-\rho'), \qquad (5)$$

being an extension of a well known formula of diffusion theory to the case of an imaginary diffusion coefficient. In the general case the solution of Eq. (1) is the function

$$u(x,\rho)=\int d\rho' G(x;\rho;0\ \rho')u^0(\rho'), \qquad (6)$$

where G is the Green's function of the problem now in question and satisfies the integral equation

$$G(x,\rho;x',\rho')=g(x,\rho;x',\rho')$$
$$\times\frac{ik}{2}\int_{x'}^{x}d\xi\int d\mathbf{r}\,g(x,\rho;\xi,\mathbf{r})\varepsilon(\xi,\mathbf{r})G(\xi,\mathbf{r};x',\rho'). \qquad (7)$$

In the derivation of (7) use is made of the fact that the function $G(x,\rho;x',\rho')$, like the function $g(x,\rho;x',\rho')$, is different from zero only in the region $x > x'$, i.e.,

$$G(x,\rho;x',\rho')\sim\theta(x-x')=\begin{cases}1 & \text{if } x>x'\\ {}^1\!/_2 & \text{if } x=x'.\\ 0 & \text{if } x<x'\end{cases} \qquad (8)$$

The Green's function of our present problem is a functional of the dielectric-constant field, and it is through it that the functional dependence of the light-wave field on ϵ is determined. Equation (7) can be solved by iterations, and we then get an expansion of the function G in a Taylor's series in the field ϵ. By varying each term of this expansion with respect to the field ϵ at the point $(\eta, \mathbf{r})$, one easily sees that

$$\frac{\delta G(x,\rho;x',\rho')}{\delta\varepsilon(\eta,\mathbf{r})}=\frac{ik}{2}G(x,\rho;\eta,\mathbf{r})G(\eta,\mathbf{r};x',\rho'). \qquad (9)$$

Consequently, for the functional derivative of the light-wave field with respect to the dielectric-constant field we get, by varying (6), the expression

$$\frac{\delta u(x,\rho)}{\delta\varepsilon(x',\rho')}=\frac{ik}{2}G(x,\rho;x',\rho')u(x',\rho'). \qquad (10)$$

Let us examine some consequences of (10).

First we note that it follows from (10) that, owing to (8),

$$\delta u(x,\rho)/\delta\varepsilon(x',\rho')\sim\theta(x-x'),$$

a condition which expresses the causality principle: the light-wave field at any point depends only on the dielectric-constant field at preceding points (with respect to the x axis), and does not depend on the field ϵ at subsequent points.

Setting x = x′ in (10) and using the fact that G (x, ρ; x, ρ') and Eq. (7), we get the expression

$$\frac{\delta u(x,\rho)}{\delta\varepsilon(x,\rho')}=\frac{ik}{4}\delta(\rho-\rho')u(x,\rho), \tag{11}$$

which was first derived in [2].[1)]

2. THE APPROXIMATION OF A MARKOV RANDOM PROCESS

Let us now turn to the statistical description of the field of a light wave. The statistical characteristics of the field of the wave in a plane x = const are determined only by the statistical characteristics of the dielectric-constant field in the layer extending from the boundary of the "random" medium to the plane x = const. But the field of the light wave at the point (x, ρ) is also correlated with the field ϵ at points located further along the x axis, owing to the function dependence of u (x, ρ) on ϵ. It is clear, however, that this correlation will be determined by the ratio of the range of correlation of the field ϵ to the longitudinal correlation range of the light-wave field, which is determined by the length of the entire light path.[1] If this ratio is small, we can assume in first approximation that the average value of the product of the fields u, taken at points (ξ_i, ρ_i), and ϵ, taken at points (η_j, ρ_j), where $\eta_j > \xi_i$, breaks up into a product in which the fields are averaged separately, i.e.,

$$\left\langle \prod_{i,j} u(\xi_i,\rho_i)\varepsilon(\eta_j,\rho_j')\right\rangle = \left\langle \prod_i u(\xi_i,\rho_i)\right\rangle\left\langle \prod_j \varepsilon(\eta_j,\rho_j')\right\rangle \quad \text{if} \quad \eta_j>\xi_i. \tag{12}$$

This approximation means that the statistical characteristics of the light-wave field are independent of those of the dielectric-constant field in the region of space which is irrelevant in the sense of the causality principle. We note that the model of inhomogeneities with delta-correlations along the x axis (i.e., with zero range of correlation of the field ϵ along the x axis), which was considered in [2], is equivalent to this approximation. Since for this model we can write an equation for the characteristic functional of the light wave field which will for our present problem play the role of the Fokker-Planck equation, we can regard this approximation as the approximation corresponding to a Markov random process.

The condition for the Markov property, written in the form (12), allows us to derive very simply equations for the various moments of the field u in a plane x = const, without restricting ourselves to the condition that the field ϵ be of Gaussian type, which was essential in [2]. To do so we write (1) in the form of an integral equation

$$u(x,\rho)=u^0(\rho)\exp\left\{\frac{ik}{2}\int_0 d\xi\,\varepsilon(\xi,\rho)\right.$$
$$\times\frac{ik}{2}\int_0^x d\xi\exp\left\{\frac{ik}{2}\int_\xi^x d\eta\,\varepsilon(\eta,\rho)\right\}\Delta_\perp u(\xi,\rho). \tag{13}$$

Noting that, according to (12), the field $\epsilon(\eta, \rho)$ that appears in the second term on the right side of Eq. (13) does not correlate with the field u (ξ, ρ), since $\eta > \xi$, we can carry out an averaging in (13) and get an equation for the mean field:

$$\langle u(x,\rho)\rangle = u^0(\rho)\left\langle \exp\left\{\frac{ik}{2}\int_0^x d\xi\,\varepsilon(\xi,\rho)\right\}\right\rangle$$
$$\times\frac{ik}{2}\int_0 d\xi\left\langle\exp\left\{\frac{ik}{2}\int_\xi^x d\eta\,\varepsilon(\eta,\rho)\right\}\right\rangle\Delta_\perp\langle u(\xi,\rho)\rangle. \tag{14}$$

In the general case (14) cannot be reduced to a differential equation. If we assume that the field ϵ(x, ρ) is a Gaussian random field, delta-correlated with respect to x, then (14) reduces to the equation for the average field derived in [2]. We note that in this case it is not in general essential that the field ϵ(x, ρ) be Gaussian. All that is required is that the phase field

$$S(x,\rho)=\frac{k}{2}\int_0^x d\xi\,\varepsilon(\xi,\rho),$$

found in the first approximation of the method of smooth perturbations, be Gaussian; in virtue of the central limit theorem this can be true even though the field $\epsilon(\xi, \rho)$ is not itself Gaussian.

In analogy with this derivation of the equation for the mean field one can derive integral equations for arbitrary moments of the fields u and u* in a plane x = const. For this purpose we introduce the quantity

$$\Gamma_n(x;\rho_i;\rho_i')=\prod_{i=1}^n u(x,\rho_i)u^*(x,\rho_i'). \tag{15}$$

Using Eq. (1) for the field u (x, ρ) and the analogous equation for the complex conjugate field, we can easily derive for $\langle\Gamma_n\rangle$ the integral equation

$$\langle\Gamma_n(x;\rho_i;\rho_i')\rangle=\Gamma_n^0(\rho_i;\rho_i')\left\langle\exp\left\{\frac{ik}{2}\sum_{l=1}^n\int_0^x d\xi[\varepsilon(\xi,\rho_l)-\varepsilon(\xi,\rho_l')]\right\}\right\rangle+\frac{i}{2k}\int_0^x d\xi\left\langle\exp\left\{\frac{ik}{2}\sum_{m=1}^n\int_\xi^x d\eta[\varepsilon(\eta,\rho_m)-\varepsilon(\eta,\rho_m')]\right\}\right\rangle\sum_{l=1}^n(\Delta_{\rho_l}-\Delta_{\rho_l'})\langle\Gamma_n(\xi;\rho_l;\rho_l')\rangle, \tag{16}$$

which for the case of a Gaussian field ϵ(x, ρ) delta-correlated along the x axis agrees with the analogous equations obtained by variational differentiation of the equation for the joint characteristic functional of the fields u (x, ρ) and u*(x, ρ).

3. THE LIMITS OF APPLICABILITY OF THE MARKOV APPROXIMATION

We mentioned earlier that in the problem of the propagation of a light wave the Markov property is due to the fact that the ratio of the correlation range of the dielectric-constant field to the longitudinal correlation range of the light-wave field is small, the latter range being the length of the light path. Consequently, we can regard the approximation by a Markov random process as the first approximation of a method of steepest descent, relative to the small parameter defined by that ratio. In this section we consider the next term of the expansion, and verify that it indeed goes in powers of this parameter.

[1)]From the expression (10) it is also easy to obtain a representation of the field u(x, ρ) in terms of its variational derivative with respect to the field ϵ at points on the boundary of the "random" medium, namely

$$u(x,\rho)=\frac{2}{ik}\int d\rho'\frac{\delta u(x,\rho)}{\delta\varepsilon(0,\rho')}.$$

We shall treat the dielectric-constant field $\epsilon(x, \rho)$ as a Gaussian random field. For simplicity we confine ourselves to the case in which the incident light wave is plane, i.e., $u^0(\rho) \equiv u_0$. We shall take the random quantity $\epsilon(x, \rho)$ to be homogeneous and isotropic. In this case the field of the light wave will also be homogeneous and isotropic in a plane perpendicular to the x axis.

Let us consider the corrections to the mean field of the light wave. Averaging Eq. (1), we get

$$\frac{\partial}{\partial x}\langle u(x)\rangle = \frac{ik}{2}\langle \varepsilon(x, \rho) u(x, \rho)\rangle. \tag{17}$$

To find the mean value of the quantity on the right side of (17), we apply the formula[3]

$$\langle f(\xi) R[f]\rangle = \int d\eta \langle f(\xi) f(\eta)\rangle \langle \delta R[f]/\delta f(\eta)\rangle, \tag{18}$$

which holds for a Gaussian random field $f(\xi)$ with average value zero and a functional R of f (ξ denotes all the arguments of the field f).

Since the field $u(x, \rho)$ is a functional of the field ϵ, by using (18) and the expression (10) for the variational derivative of the light-wave field with respect to the field ϵ, we can write (17) in the form

$$\frac{\partial}{\partial x}\langle u(x)\rangle = -\frac{k^2}{4}\int_0^x d\xi \int d\rho' \Phi^\varepsilon(x-\xi; \rho-\rho')\langle G(x, \rho; \xi, \rho') u(\xi, \rho')\rangle, \tag{19}$$

where $\Phi^\epsilon(x-\xi; \rho-\rho') = \langle \epsilon(x, \rho)\epsilon(\xi, \rho')\rangle$ is the correlation function of the dielectric-constant field, and the Green's function G is defined by (7). Because the function $\Phi^\epsilon(x, \rho)$ is "sharp" in the neighborhood of the point x = 0 we can expand the function whose average is indicated in the right member of (19) in a power series in $x-\xi$. Let us confine ourselves to the linear term in the expansion. Then, using the fact that

$$G(x, \rho; x', \rho') - g(x, \rho; x', \rho') \approx {}^1/_2 ik(x-x')\delta(\rho-\rho')\varepsilon(x, \rho)$$

for $x' \to x$, which follows from (7), and also the fact that

$$u(x, \rho) - u(x', \rho) \approx (x-x')\left\{\frac{i}{2k}\Delta_\perp u(x, \rho) + \frac{ik}{2}\varepsilon(x, \rho) u(x, \rho)\right\},$$

which follows from the equation of motion (1) of the light wave, we easily get the expression

$$\langle G(x, \rho; \xi, \rho') u(\xi, \rho')\rangle \approx$$
$$\approx \{g(x, \rho; \xi, \rho') - {}^1/_{16}k^2(x-\xi)\delta(\rho-\rho')F_0(x, 0)\}\langle u(x)\rangle, \tag{20}$$

where

$$F_k(x, 0) = \int_0^x d\xi\, \xi^k \Phi^\varepsilon(\xi, 0).$$

In the derivation of (20) one uses the expression for the correlation of the field $\epsilon(x, \rho)$ with the field $u(x, \rho')$, which is described by (11).

Consequently, the solution of (19) in the indicated approximation is the function

$$\langle u(x)\rangle = u_0 \exp\left\{-\frac{k^2}{4}\int_0^x d\xi \int_0^\xi d\eta \int d\rho\, \Phi^\varepsilon(\eta, \rho) g(\eta, \rho) \times \frac{k^4}{64}\int_0^x d\xi F_1(\xi, 0) F_0(\xi, 0)\right\}. \tag{21}$$

The first term in the exponent, as is pointed out in [2], can be simplified by using the fact that the function $\Phi^\epsilon(\eta, \rho)$ is "sharp," which allows us to rewrite (21) in the form

$$\langle u(x)\rangle = u_0 \exp\left\{-\frac{k^2}{8}\int_0^x d\xi F_0(\xi, 0) + \frac{k^4}{64}\int_0^x d\xi F_1(\xi, 0) F_0(\xi, 0)\right\}. \tag{22}$$

For an estimate of the expressions involved in (22) we use a model in which the longitudinal correlation function is of the form

$$\Phi^\varepsilon(x, 0) = \sigma_\varepsilon^2 \exp\{-|x|/l_0\}, \tag{23}$$

where l_0 defines the range of the correlation, and σ_ϵ^2 is the dispersion of the fluctuations of the dielectric-constant field, which characterizes the "intensity" of the fluctuations of the field ϵ. Substituting (23) in (22), we find that for $x \gg l_0$

$$\langle u(x)\rangle = u_0 \exp\{-{}^1/_8 k^2\sigma_\varepsilon^2 l_0 x + {}^1/_{64} k^4 \sigma_\varepsilon^4 l_0^3 x\}. \tag{24}$$

Using the fact that the quantity

$$\frac{k^2}{2}\int_0^x d\xi F_0(\xi, 0) = \langle S^2(x, \rho)\rangle = \sigma_S^2$$

determines the dispersion of the phase of the wave, as found in first approximation of the method of smooth perturbations, we can rewrite (24) in the form

$$\langle u(x)\rangle = u_0 \exp\{-{}^1/_2\sigma_S^2 + {}^1/_4\sigma_S^4 l_0/x\}. \tag{25}$$

Generally speaking, the quantity σ_S^2 can be very large, but in this region the mean field is very small, and therefore the region is of no interest. The region of interest is that where $\langle u(x)\rangle \sim u_0$, which corresponds to values $\sigma_S^2 \sim 1$; consequently the correction caused by nonmarkovian aspects of the process is determined by the parameter l_0/x, in accordance with what was said above. Obviously this estimate does not depend on the form of the correlation function of the dielectric-constant field.

Special attention must be given to the case in which the fluctuations of the dielectric-constant field are due to turbulent motions of the medium. Here the mean light-wave field is determined by the large-scale fluctuations of ϵ and therefore is of no interest. On the other hand the quantity

$$\Gamma(x, \rho) = \langle u(x, \rho_1) u^*(x, \rho_2)\rangle \quad (\rho = \rho_1 - \rho_2),$$

called the "coherence function," will be determined by the small-scale fluctuations of the field of ϵ. Let us see what sort of parameter determines the Markov character of the process of propagation of a light wave in a turbulent medium. We can derive the equation for the coherence function by starting from Eq. (1). Using the property of homogeneity, we can write the equation for Γ in the form

$$\frac{\partial}{\partial x}\Gamma(x, \rho) = \frac{ik}{2}\{\langle \varepsilon(x, \rho_1) u(x, \rho_1) u^*(x, \rho_2)\rangle - \langle \varepsilon(x, \rho_2) u(x, \rho_1) u^*(x, \rho_2)\rangle\}. \tag{26}$$

Proceeding further, as in the case of the mean field, we easily see that the solution of the equation for Γ, including the first correction, is of the form

$$\Gamma(x, \rho) = \Gamma_0 \exp\left\{-\frac{k^2}{4}\int_0^x d\xi F_0(\xi, \rho) + \frac{k^4}{64}\int_0^x d\xi F_0(\xi, \rho) F_1(\xi, \rho)\right\};$$
$$F_k(x, \rho) = \int_0^x d\xi\, \xi^k [\Phi^\varepsilon(\xi, 0) - \Phi^\varepsilon(\xi, \rho)]. \tag{27}$$

We emphasize that for the derivation of the solution (27) it is essential that the field of the light wave is homogeneous.

The first term in the exponent is determined by the structure of the phase function $D_S(x, \rho)$, as found in the first approximation of the method of smooth perturbations, and for the structure function of the dielectric-constant field we can use the expression[1]

$$D_\varepsilon(\mathbf{r}) \approx C_\varepsilon^2 r^{2/3} \quad \text{if} \quad r \gg \lambda_0, \tag{28}$$

where the quantity λ_0 fixes the internal scale of the turbulence, and the constant C_ϵ^2 characterizes the "intensity" of the fluctuations of the field ϵ. When we make all the calculations we verify that the second term in the exponent in (27) is of the order of magnitude of $D_S^2(\rho/x)^{1/3}$. The quantity D_S—the structure function of the phase—can take very large values. In the region of interest, however, where $\Gamma \sim \Gamma_0$, we have $D_S \sim 1$, and the correction for nonmarkovian aspects of the field will be determined by the parameter $(\rho/x)^{1/3}$, which agrees with what we said earlier, since because of the homogeneity of the dielectric-constant field the longitudinal correlation range is the same as the transverse range, and in a turbulent medium the latter is indeed determined by the quantity ρ. We note that an indication that in the case of a turbulent medium it is this parameter that determines the Markov nature of light-wave propagation was given by the estimates derived in [2] on the basis of a study of the simplifications which can be made in each order of the theory of smooth perturbations.

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