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QUASICLASSICAL SCATTERING IN A CENTRALLY SYMMETRIC FIELD

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The scattering amplitude is investigated in the quasiclassical approximation for the case when the limiting classical angle of scattering is not small. In the presence of one caustic a single formula is obtained which describes the cross section in both the classically accessible and inaccessible regions.

1. In a central field the quasiclassical wave function of scattering has a rather simple analytical form, which enables one to obtain interesting results in practice and to investigate in detail such fundamental questions as the range of applicability of the quasiclassical approximation and its connection with the Born approximation.

This program was carried out for weak even potentials in[1], where the quasiclassical radial solutions of the Schrödinger equation were investigated in the complex angular momentum plane, and the amplitude was found by the saddle-point method from the Faxén-Holtsmark formula.

For potentials U with simple poles it was shown that for $kaU/E \sim 1$ (k, E are the wave number and energy of the incident particle, a is the characteristic dimension of U), because of the proximity of a radial turning point and a pole of the potential the quasiclassical approximation is violated, and for its improvement it is necessary to solve the Schrödinger equation exactly in the neighborhood of the pole, after which the result found for $kaU/E \ll 1$ goes over into the Born approximation. For small potentials which do not possess the property of being even, for similar reasons the quasiclassical approximation as $U/E \to 0$ does not agree with the Born approximation. In this sense the case of strong fields ($U \sim E$) is simpler.

A somewhat different, in comparison with[1], method was used in[2] for construction of the quasiclassical wave function, namely, the method of "complex trajectories." However, the results of[2] are only valid far away from a caustic and for sufficiently strong fields in the sense indicated above. Nevertheless, the "trajectory method" with the caustics taken into consideration is more convenient for generalizations than the "saddle-point method" (for example, for a generalization to potentials where the Schrödinger equation is not separable), and it is rather descriptive both for construction of the wave function and for estimation of the approximation parameter. In fact, in its own meaning a caustic geometrically expresses the shielding properties of the potential, and one can regard its characteristic dimension as the parameter (l) of the effective barrier, which must be compared with the wavelength ($\lambda\!\!\!^{-}_0$) of the incident particles. (Of course, in the method of complex trajectories it is necessary to examine the caustic over the entire complex space.) Thus, for example, in the case of the potential α/r, where the caustic $r(1 - \cos\theta) = 2\alpha/E$, the size of the barrier will be given by $l = \alpha/E$, and $l/\lambda\!\!\!^{-}_0$ coincides with the parameter of the quasiclassical approximation (E is the energy of the incident particles).

In the present article the "method of complex trajectories" with caustics taken into account will be used to examine the problem of scattering in strong central fields, where the quasiclassical approximation is valid.

2. In the "method of complex trajectories" the wave function is constructed in the form of a sum[2]

$$\psi = \sum_n A_n \exp\{iS_n/\hbar\} \tag{1}$$

over all trajectories arriving at a given point **r** and satisfying certain conditions (determinacy, nonhomotopy[2]). In a central field one can confine one's attention to an investigation of the projections of these trajectories on the complex r plane, having written the equation of the trajectories in the form of an integral over a contour:

$$-\varepsilon\theta = \pi - \int_C^r \frac{\rho\, dr}{r^2 k(\rho, r)} \equiv \Theta(\rho, r). \tag{2}$$

Here

$$k(\rho, r) = (1 - \nu(r) - \rho^2/r^2)^{1/2}, \quad \nu(r) = U(r)/E, \tag{3}$$

$\epsilon = -1$ for $\mathbf{p}_0 \cdot \mathbf{r} \to -\infty$, the sign reverses when the trajectory passes through $\mathrm{Re}\,\theta = 0, \pi$; ρ is the impact parameter; the contour C is the desired projection.

Using (2), we transform the action for the scattering problem from an integral over the trajectory into an integral along the contour C:

$$p_0^{-1} S(r, \theta, \rho) = \int_C^r k(\rho, r)\, dr - \int_\rho^\infty (1 - \rho^2/r^2)^{1/2} dr + \rho(\pi/2 + \varepsilon\theta). \tag{4}$$

It is convenient to classify the entire set of contours which, of course, must satisfy the same requirements of the method as the trajectories (determinancy, nonhomotopy) in the following way.

First we distinguish a set of different contours, satisfying the conditions of the method and corresponding to a fixed value of ρ. Such a set will exist by virtue of the periodicity of $\Theta(\rho, r)$, which is defined in Eq. (2). It is determined by the position of the radial turning points in the r plane for a given value of ρ. Secondly, let us consider the set of contours which arise when the value of ρ is changed. A set may exist since several trajectories with different values of ρ may arrive at a given point (r, θ).

A resonance in a quasidiscrete level may serve as an example of a case when the first set exists. Here it is necessary to take into account all trajectories in which the particle which penetrates under the barrier with momentum $p_0\rho$ is moving inside the potential well prior to its emission in a fixed direction θ.

In this case there will be at least two branch points of $k(\rho, r)$ in the r plane: $r_1(\rho)$ and $r_2(\rho)$ are the turning points for particles in a quasidiscrete state. The set C_n which is involved in the construction of expression (1) will consist of an infinite number of contours which differ by the number of circuits around the points r_1 and r_2. In this connection it is necessary that

$$\operatorname{Im}\int_{r_1}^{r_2} k(\rho, r)\,dr = 0,$$

then the contribution to (1) from all C_n will be of the same order (in the one-dimensional case, such a problem is considered in[2]). The condition for a resonance has the form

$$\int_{r_1}^{r_2} k(\rho, r)\,dr = \lambda\!\!\!^-_0 \pi (n + 1/2), \quad \lambda\!\!\!^-_0 = \hbar/p_0 \tag{5}$$

and can be satisfied for certain complex ρ which obviously will correspond to poles of the scattering matrix.[1]

Trajectories with different values of ρ are of fundamental interest in the present article since the caustics are determined by precisely such a collection of trajectories.[3]

Now let us construct the scattering wave function in the approximation (1). The amplitude will have the form (1.9)[1)]

$$A(r, \theta, \rho) = \varphi(\rho^{-1} r^2 \sin\theta\, k(\rho, r)\, \Theta_\rho(\rho, r))^{-1/2}. \tag{6}$$

Here φ is constant along a trajectory and is determined by the boundary conditions and the matching conditions (1) at the boundary of applicability of the approximation.[3] Later on the case is considered when φ can be regarded as equal to unity,

$$\Theta_\rho(\rho, r) \equiv \frac{\partial}{\partial\rho}\Theta(\rho, r) = -\int_C^r \frac{dr}{rk(\rho, r)}\frac{d}{dr}g + \frac{g(r)}{rk(\rho, r)}$$
$$g(r) = \left(1 - \frac{r\nu'}{2(1-\nu)}\right)^{-1}. \tag{7}$$

Then $S_n = S(r, \theta, \rho_n)$ and $A_n = A(r, \theta, \rho_n)$, where the contours C_n and the ρ_n are determined from condition (2)

$$-\varepsilon\theta = \Theta(\rho_n, r). \tag{8}$$

From Eq. (6) it is seen that the approximation is violated on the line $\sin\theta = 0$ and on the caustics $r = r_k(\rho)$, $\theta = \theta_k(\rho)$, where

$$\Theta_\rho(\rho, r_k) = \Theta(\rho, r_k) + \varepsilon\theta_k = 0. \tag{9}$$

The approximation is not violated at the radial turning points $r_0(\rho)$. The turning points $r_0(0)$, through which the caustics pass, are an exception. In fact, assuming that $\nu'(r_0(0)) = 0$ and that the integral in (7) does not vanish for $\rho \to 0$ and $r \to r_0(0)$, from Eqs. (7) and (9) we obtain

$$r_k(\rho) \cong r_0(0) - \frac{\rho^2}{r^2\nu'}\bigg|_{r_0(0)}.$$

[1)]Equation (1.9) indicates a reference to formula (9) of the author's previous article [3]. For a central field, in the formulas of article [3] one can assume that $S_{\alpha\phi} = S_{\alpha\alpha} = p_0$, and all remaining partial derivatives of S containing differentiation with respect to α are equal to zero, and $\beta = \rho$.

If $\nu(r) < 1$ on the real axis, the $r_0(0)$ are complex and belong to the complex part of the caustic surface. For example, in the case of Coulomb attraction $r_0(0) = -|\alpha|/E$, but the caustic on the $(-r, \theta)$ plane has the same form as in a repulsive potential on the (r, θ) plane. It is convenient to use this characteristic point $(r_0(0))$ of the caustic in order to estimate the parameter of the quasiclassical approximation, since the caustic is a geometrical expression of the scattering properties of the potential. For example, for potentials of the type $\nu = \alpha(1 + r^2a^{-2})^{-n}$ the size of the effective barrier (l), which scatters particles just like a potential, is of order

$$l \sim |r_0(0) - ia| = a|1 - \sqrt{1 - \alpha^{1/n}}|. \tag{10}$$

and the condition for the quasiclassical nature of the problem has the form $l \gg \lambda\!\!\!^-_0$.

Let us consider in more detail how this condition appears in connection with the construction of the wave function. Far away from a caustic, where the average radius of curvature of the wave surface is much larger than the particle's wavelength, expression (1) is suitable for a description of the wave function provided the φ_n are known. But the φ_n are determined from the conditions for matching in the region of the caustic. If $l \gg \lambda_0$ one can assume that all caustic points are located far away from the singular points of the potential, and in the neighborhood of a caustic one can write the solution of the Schrödinger equation in the following form (1.33):

$$\psi - \sum_{n\neq 1,2} A_n \exp\{iS_n/\hbar\} \sim y^{1/3}(H^{(1)}_{1/3}(y) + H^{(2)}_{-1/3}(y)), \tag{11}$$

$$y = \frac{1}{3\lambda\!\!\!^-}\sqrt{(2\eta)^3\gamma}, \qquad \gamma = \sqrt{1 - \nu(r_k)}\,(r_k{}^3 k^3 \Theta_{\rho\rho})^{-1} \tag{12}$$

where η is the distance to the caustic along the normal to it.

In this case $\varphi_1 = \varphi_2 = 1$. If $l \lesssim \lambda\!\!\!^-_0$, in order to solve the Schrödinger equation close to a caustic it is necessary to take into account the nearby singularities of the potential. Here φ_n will be some function of ρ whose form is determined by the nature of the potential (see, for example,[1] for potentials with a simple pole). It is difficult to call this case quasiclassical in the proper sense of the word (the size of the barrier is of the order of the wavelength) especially since for $l \ll \lambda\!\!\!^-_0$ the results correspond to the Born approximation.[1]

In what follows we shall assume that $l \gg \lambda\!\!\!^-_0$; in particular, let us consider large potentials when both the classically accessible as well as the inaccessible regions of space may be of interest. One of the caustics serves as the boundary between these regions.

3. Let us consider (1) as $r \to \infty$, and let us investigate the form of the scattering amplitude with the caustics taken into consideration. From Eqs. (4) and (6) we obtain

$$p_0^{-1}S \xrightarrow[r\to\infty]{} r + 2\delta(\rho) + \varepsilon\rho\theta \equiv r + \lambda\!\!\!^-_0 s(\rho, \theta), \tag{13}$$

$$rA \xrightarrow[r\to\infty]{} (\rho^{-1}\sin\theta(-\varepsilon\Theta'(\rho)))^{-1/2} = a(\rho, \theta). \tag{14}$$

Here the following notation has been introduced:

$$2\delta(\rho) = \int_C^\infty k(\rho, r)\,dr - 2\int_\rho^\infty (1 - \rho^2/r^2)^{1/2} dr, \tag{15}$$

$$-\varepsilon\Theta(\rho) = \lim_{r\to\infty}\Theta(\rho, r) = 2\delta'(\rho).$$

Then the trajectories of the particles being scattered at an angle θ are determined from the equation

$$\theta = \Theta(\rho_n), \tag{16}$$

and far away from the caustics one can write the amplitude in the form

$$f = \sum_n f_n = \sum_n a(\rho_n \theta) \exp\{is(\rho_n)\},$$
$$2\lambda_0 s(\rho) = \int_c \frac{r\nu' dr}{k(\rho, r)}. \tag{17}$$

As $r \to \infty$ the caustics will asymptotically approach the trajectories of particles with impact parameter ρ_0, where $\Theta'(\rho_0) = 0$, and the scattering angle $\theta_0 = \Theta(\rho_0)$. In particular, a caustic will always exist for $\rho_0 \to \infty$ and $\theta_0 \to 0$ since in this connection $\Theta'(\rho) \to 0$. For example, for potentials which fall off at infinity like α/r^n, the equation for the trajectories at large distances ($r \gg \rho$, $\theta \ll 1$) has the form

$$-\varepsilon\theta = \alpha M_n / \rho^n + \rho / r,$$

and the equation of the caustic is given by

$$r_k = \frac{\rho^{n+1}}{\alpha n M_n}, \quad -\varepsilon\theta_k = \alpha \frac{(n+1)M_n}{\rho^n}.$$

For attractive potentials ($\alpha < 0$, $\epsilon = +1$) there is no trace of such a caustic in the real plane, but it approaches the real plane as $\rho \to \infty$.

From this example it is seen that as $\theta \to 0$ the nature of the singularities of the quasiclassical wave function (1) essentially depends on the form of the potential at large distances ($A \sim \theta^{-(1+(1/n))}$, $\theta \to 0$). A divergence of this type is caused, in the first place, by the "nonstandard nature" of the caustic at small angles and, in the second place, by the singularity of the Schrödinger equation at $\theta = 0$. We shall not investigate this "problem of small angles" here, confining our attention to finite angles (finite values of ρ in expression (16)) in accord with the well-known estimate[4]

$$|U'(\rho)|\rho^2 \gg \hbar v. \tag{18}$$

Not far from a caustic for which $\theta_0 \neq 0$, $a_{1,2} \sim (\theta - \theta_0)^{-1/4}$ since

$$\Theta(\rho) \cong \theta_0 + {}^1/_2(\rho - \rho_0)^2\Theta''(\rho_0), \quad \rho_{1,2} = \rho_0 \pm \sqrt{\frac{2(\theta - \theta_0)}{\Theta''(\rho_0)}}$$

and the classical cross section σ^0 diverges:

$$\sigma^0 = \sum_n |a_n|^2 \sim |\theta - \theta_0|^{-1/2}.$$

After improvement of the quasiclassical approximation, the cross section becomes finite, which one can easily verify by going to the limit $r_k \to \infty$ (1.44) in Eq. (11). In this connection

$$\frac{\eta}{r_k} \to \theta - \theta_0 \equiv \Delta\theta, \quad y \to y_0 = \frac{1}{3\lambda_0}\sqrt{\frac{(2\Delta\theta)^3}{\Theta''(\rho_0)}}.$$

As a consequence of the interference of the waves incident on the caustic and reflected from it, the cross section oscillates in the accessible region and its value will significantly depend on the presence of the other waves ($n \neq 1, 2$) in expression (11). If there are several caustics in the classically accessible region, the curve of the cross section will be determined by a superposition of several oscillating functions of the type (11).

Let us clarify certain conditions for the existence of caustics in the case of an arbitrary potential for real trajectories. Then as $r \to \infty$, ρ_0 is determined from Eq. (7)

$$\int_{r_0(\rho_0)}^{\infty} \frac{dr}{rk(\rho_0, r)} \frac{d}{dr} g(r) = 0, \tag{19}$$

and then one can determine θ_0 from the relation

$$\theta_0 = -\varepsilon\Big(\pi - \int_{r_0(\rho_0)}^{\infty} \frac{\rho_0\, dr}{r^2 k(\rho_0, r)}\Big) = \varepsilon \int_{r_0(\rho_0)}^{\infty} \frac{\rho_0 dr}{rk(\rho_0, r)} \frac{\nu'(r)}{1 - \nu(r)}. \tag{20}$$

Since $k \geq 0$ it is obvious that the existence of an extremum of g(r) is necessary for the fulfillment of condition (19). In addition, due to the singularity in k^{-1} at $r = r_0$ and under the assumption that g(r) is regular, one can assume that Eq. (19) is satisfied near each extremum of g(r), as for example in the case

$$\int_{\rho_0}^{\infty} \frac{\sin r\, dr}{\sqrt{r^2 - \rho_0^2}} = \frac{\pi}{2} J_0(\rho_0) = 0, \quad \rho_{0m} \cong \pi(m - {}^1/_4).$$

Let us discuss the following three possible cases.

1. $\nu_0 < 1$ (ν_0 is the largest value of $\nu(r)$ for $r \geq 0$). The turning point $r_0(\rho)$, corresponding to real trajectories, varies from 0 to ∞; in this connection

$$g(r) \underset{r\to 0}{\longrightarrow} 1 + \frac{r\nu'(0)}{2(1-\nu(0))}, \quad g \underset{r\to\infty}{\longrightarrow} 1 + \frac{r\nu'(r)}{2}.$$

The extrema of g will be located between the points r_j and r_{j+1}, where $g(r_j) = 1$ or

$$r_j U'(r_j) = 0. \tag{21}$$

In any case condition (21) is satisfied at two points (0 and ∞) so that for $\nu_0 < 1$ there always exists at least one caustic (for finite values of ρ). In addition, caustics associated with the extrema of $\nu(r)$ are not possible. Their total number (without taking into consideration the caustic as $\rho_0 \to \infty$) is not smaller than[2)] n + 1, where n is the number of extrema of $\nu(r)$ at interior points.

2. $\nu_0 > 1$. Here the region (0, $r_0(0)$) is inaccessible and $g(r_0(0)) = 0$. The number of caustics (see case 1, Eq. (21)) is not less than n.

3. $\nu_0 = \nu(0) = 1$. Here everything asserted for case 2 is valid (since $g < 1$ as $r \to 0$). In addition, the following refinement is possible for monotonic potentials. If $\nu'(0) \neq 0$ (obviously $\nu'(0) < 0$) then from the expansions of g(r) near r = 0 and r = ∞ it follows that a caustic exists for $\nu''(0) < 0$

$$g \underset{r\to 0}{\longrightarrow} \frac{2}{3}\Big(1 - \frac{r\nu''(0)}{6\nu'(0)}\Big).$$

Similarly, for even potentials a caustic exists for $\nu^{(4)}(0) < 0$ since

$$g \underset{r\to 0}{\longrightarrow} \frac{1}{2}\Big(1 - \frac{r^2\nu''}{24\nu^{(4)}}\Big).$$

Thus, the caustics are determined by the extremal properties of the potential, and inside the regions under consideration their number does not depend on the energy (see Eq. (21)). Only upon a change from case 2 to case 1 is one caustic added; it is "squeezed out" of the trajectory for $\theta = \pi$ and creates a classically in-

[2)]Three extrema of g are possible between the neighboring points r_j, for example, in the case when $\nu(r)$ has three inflection points in this interval.

accessible region between $\theta = \pi$ and $\theta = \theta_{0\max}$ (the limiting value of the classical scattering angle). For very large energies $\theta_{0\max} \sim \nu_0 \ll 1$, and all real caustics come together in a cone $\theta \sim \nu_0$.

In order to construct the scattering amplitude associated with a large number of caustics, it is necessary to know the function $\Theta(\rho)$ in the complex ρ plane. In the classically accessible region it is sufficient to know the form of the curve $\Theta(\rho)$ for $\rho \geq 0$. As an example, let us consider the case illustrated in Fig. 1 a ($\nu_0 < 1$). At large distances the (r, θ) plane will be divided by three caustics with the directions θ_{0m} (not considering the "zero" corresponding to the minimum of $\Theta(\rho)$ as $\rho \to \infty$) (see Fig. 1b). The pair of caustics at the "nodal points" merge together, where

$$\Theta_\rho(\rho, r) = \Theta_{\rho\rho}(\rho, r) = \Theta(\rho, r) + \varepsilon\theta = 0.$$

Excluding the region of small angles determined by inequality (18), we will have at least four regions in which, taking Eqs. (11) and (17) into consideration, different formulas are required in order to write down the amplitude.

Instead of this it is convenient to use an approximate formula (see, for example, Eq. (1.35)) of the form

$$f - \sum_{n\neq 1,2} f_n = \exp\left\{i\left(\frac{s_1+s_2}{2} + \arg a_1 - \frac{\pi}{12}\right)\right\}\sqrt{\frac{\pi}{2}Y} \qquad (22)$$

$$\times(|a_1|H^{(2)}_{-1/3}(Y) + |a_2|H^{(1)}_{1/3}(Y)) \equiv F_{12}, \quad Y = (s_2 - s_1)/2.$$

In the classically accessible region $Y > 0$, $\arg a_1 = 0$, $\arg a_2 = -\pi/2$, the index 1 refers to the wave which is incident on a caustic ($-\epsilon\Theta'(\rho) > 0$), and the index 2 refers to the reflected wave ($-\epsilon\Theta'(\rho) < 0$). At large distances from a caustic ($Y \gg 1$) $F_{12} \to f_1 + f_2$. Near a caustic

$$Y \cong y_0, \quad a_{1,2} \cong \sqrt{\mp\,\varepsilon(2\Theta''\Delta\theta)^{-1/2}\rho_0}\left(1 \pm \frac{1}{2\rho_0}\sqrt{\frac{2\Delta\theta}{\Theta''}}\right)$$

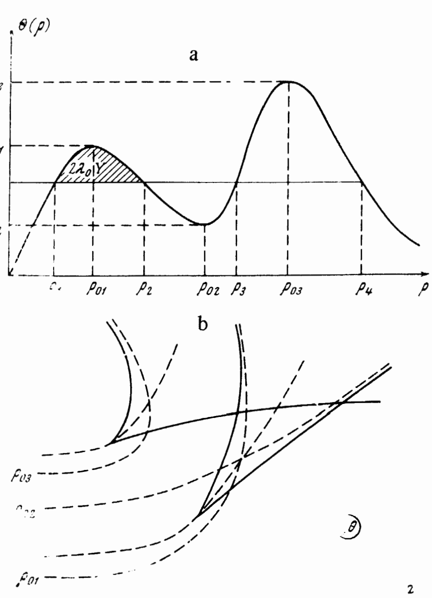

FIG. 1. Scattering by a potential which allows three real caustics: Fig. 1a shows the dependence of the scattering angle on the impact parameter, Fig. 1b shows a schematic diagram of the caustics (the solid lines) and certain trajectories (dashed lines) in the r, θ plane.

and F_{12} coincides with the amplitude obtained from expression (11).

At the caustic $Y = -i\,\mathrm{Im}\,s_1$ (the index 1 refers to the passing wave) $\mathrm{Im}\,s_1 > 0$, $a_2 = a_1^*$ and F_{12} is given by

$$F_{12} = a_1 \exp\{i\,\mathrm{Re}\,s_1\}\sqrt{\frac{2\,\mathrm{Im}\,s_1}{\pi}}K_{1/3}(\mathrm{Im}\,s_1). \qquad (23)$$

At large distances from the caustic ($\mathrm{Im}\,s_1 \gg 1$)

$$F_{12} \to f_1 = a_1 e^{is_1}.$$

Using (22) one can write down the scattering amplitude for the case under consideration with the aid of two formulas (instead of four):

1. $f = f_1 + F_{32} + f_4$, $\theta \lesssim \theta_{02}$,
2. $f = F_{12} + F_{34}$, $\theta > \theta_{02}$.

For angles between θ_{02} and θ_{01}, both formulas have the same form far away from a caustic: $f = \sum_1^4 f_n$.

In the classically inaccessible range of angles, the amplitude is constructed in similar fashion according to the shape of the curve $\mathrm{Re}\,\Theta(\rho)$ above the line $\mathrm{Im}\,\Theta(\rho) = 0$ in the complex ρ plane, more precisely, above the lines $\rho = \rho_n(\theta)$ where $\rho_n(\theta)$ denotes the complex roots of (16). There may be several such lines depending on the number of periods of the function $\Theta(\rho)$ (or on the number of roots $k(\rho, r_0(\rho)) = 0$). The intersection of the lines on the Riemann surface $\Theta(\rho)$ gives the caustic. From here it follows, in particular, that such lines will depart into the complex plane from the points ρ_{0m} for all real caustics θ_{0m}. For large potentials when $\theta_{0\max}$ is large, the complex branches originating from real caustics (the "caustic" branches) will obviously give the major contribution to the cross section. In this case the scattering amplitude in the inaccessible region may be obtained by analytic continuation of the amplitude for the accessible region, which is written with the aid of (22). However, the physical poles (resonances) given by Eq. (5) may become important at such energies ($E \sim U$); then one should add resonance terms to the amplitude.[1,4] For small values of $\theta_{0\max}$, in order to construct the amplitude in the inaccessible region it may turn out to be necessary to take other branches of $\rho_n(\theta)$ into consideration besides the "caustic" branches, and also it may be necessary to take nonphysical poles into account.[1] Here more specific information about the potentials is needed.

As an example let us consider the potential $\nu = \alpha(1 + r^2a^{-2})^{-1}$. In this case with the aid of elliptic integrals one can write the function $\Theta(\rho)$ in the form

$$-\varepsilon\Theta(\rho) = \pi - 2\varphi_0(\rho),$$
$$\varphi_0(\rho) = {}^1\!/_2\pi\Lambda(\xi\varkappa) = E(\varkappa)F(\xi\varkappa') + K(\varkappa)E(\xi\varkappa') - K(\varkappa)F(\xi\varkappa'). \quad (24)$$

Here
$$\sin\xi = \frac{\sqrt{1-\alpha^{1/2}\varkappa}}{\varkappa'}, \quad \varkappa = \frac{\sqrt{b-b_2} - \sqrt{b-b_1}}{\sqrt{b-b_2} + \sqrt{b-b_1}},$$
$$\varkappa' = \sqrt{1-\varkappa^2},\ b = \rho^2/a^2,\ b_{1,2} = \rho_{1,2}{}^2/a^2 = -(1 \mp \sqrt{\alpha})^2.$$

We obtain the function $\Theta(\rho)$ over the entire complex ρ plane by analytic continuation of (24) from the real axis. In particular, on that sheet where the contour B arrives (see Fig. 2), φ_0 will have the form

$$\varphi_0 = \int_{r_0(\rho)}^{\infty} \frac{\rho\, d\rho}{r^2 k(\rho, r)} = \frac{\pi}{2}\Lambda(\xi\varkappa) - \frac{\pi}{2} - i(K'E(\xi\varkappa') - E'F(\xi\varkappa')), \quad (25)$$

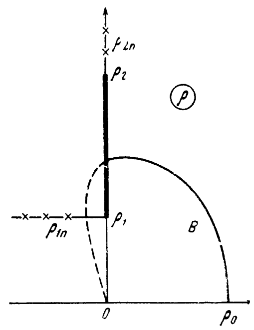

FIG. 2

where

$$r_2(\rho) \approx ia \text{ for } |\rho| \gg a|\alpha|^{1/4}.$$

On the real axis of ρ for $|\alpha| \ll 1$ the function $\Theta(\rho)$ is given by

$$\Theta(\rho) \cong {}^1/_2\pi a\sqrt{b}(b+1)^{-3}, \quad (b+1) \gg |\alpha|^{1/2}$$

and has a single maximum at $\rho_0 \cong a/\sqrt{2}$ as $\alpha \to 1$ and $\rho_0 \to 0$. The "caustic" branch B (see Fig. 2) which determines the amplitude in the inaccessible region goes from ρ_0 to the cut (ρ_1, ρ_2). Beyond the cut, on the second sheet of the Riemann surface, for $|b+1| \gg |\alpha|^{1/2}$ and $|\alpha| \ll 1$ we have

$$\Theta(\rho) \cong 2 \arccos(\rho / ia),$$

and in the vicinity of the cut $\Theta(\rho) \sim \alpha^{1/4}$. Associated with an increase of α, the line $\operatorname{Im}\Theta(\rho) = 0$ on the second sheet departs from the imaginary axis and arrives at the point $\rho = 0$ at an angle $\varphi > \pi/2$, more precisely

$$\operatorname{tg}\varphi = -\frac{\sqrt{\alpha}K'(\sqrt{\alpha}) - E'(\sqrt{\alpha})}{(1-\alpha)K(\sqrt{\alpha}) - E(\sqrt{\alpha})}.$$

Let us consider in more detail the case $1 - \sqrt{\alpha} \equiv \beta \ll 1$ when the entire "caustic" branch is located in the vicinity of $\rho = 0$. For $|b| \ll 1$ from Eq. (24) we obtain

$$\varphi_0(\rho) = \frac{\pi}{4} + \frac{i}{4}\ln\frac{z_1}{z_2} + \frac{z_1 + z_2}{16}\ln\frac{64}{e^2 z_1 z_2} + O(|z^2 \ln z|). \quad (26)$$

Here $z_{1,2} = \sqrt{b} \pm i\beta$; Eq. (25) gives the same result except for the substitutions $z_2 \to z_2'$ and $z_1 \to z_1' e^{-2\pi i}$.

From Eq. (26) it follows that the line $\operatorname{Im}\Theta(\rho) = 0$ has the form (for $|b|^{1/2} \gg \beta$)

$$\ln\frac{8}{et} - \frac{2\beta}{t^2} = \chi \operatorname{ctg}\chi, \quad \sqrt{b} = te^{i\chi}.$$

Hence with logarithmic accuracy we have

$$\rho_0 \cong a\sqrt{2\beta}\left(\ln\frac{8}{e^2\sqrt{2\beta}}\right)^{-1/2}, \quad \theta_0 \cong \pi/2 - \frac{t}{2}\frac{\chi}{\sin\chi}\bigg|_{\chi=0} = \pi/2 - \rho_0/2a.$$

For $|b|^{1/2} \ll \beta$ one finds

$$\Theta(\rho) \cong \pi - \frac{t}{\beta}\left(1 - \frac{\beta}{2}\ln\frac{8}{e\beta}\right).$$

Thus, the scattering amplitude for the potential $\nu = \alpha(1 + r^2 a^{-2})^{-1}$ will be determined by only one caustic in the classically accessible range of angles and by one "caustic" branch in the inaccessible region, and may be approximated by the single function (22) for both regions.

Finally, let us find the resonances (poles of the scattering matrix[1]) and clarify the conditions under which it is not necessary to take the unphysical series into account. Integration in Eq. (5) gives

$$i\alpha^{-1/4}\pi\left(n + \frac{1}{2}\right)\frac{\lambda_0}{a} = \sqrt{\alpha}\,\varkappa K' - \varkappa^{-1/2}E' + \quad (27)$$

$$+ \frac{\pi}{2}\Lambda(\eta\varkappa')\left[\frac{(1-\sqrt{\alpha}\varkappa)(\varkappa - \sqrt{\alpha})}{\varkappa\sqrt{\alpha}}\right]^{1/2}, \quad \sin\eta = \frac{\alpha^{1/4}}{\sqrt{\varkappa}}$$

From here, for $a\alpha^{3/4} \gg \lambda_0$ we obtain the first poles of the physical (ρ_{1n}) and unphysical (ρ_{2n}) series:

$$\rho_{1n} = \rho_1 - (2n+1)\frac{\lambda_0}{a}\alpha^{-1/4}, \quad \rho_{2n} = \rho_2 + i(2n+1)\frac{\lambda_0}{a}\alpha^{-1/4}.$$

For $\alpha \sim 1$, when the "caustic" branch is located significantly below ρ_{2n} (see Fig. 2), the contribution of the poles of the unphysical series to the amplitude is exponentially small in comparison with the trajectory ρ_n. In this connection, the contribution of the poles of the physical series may be of the same order as that of the trajectories. As the value of α decreases, ρ_{2n} and ρ_n come together, and the condition under which the poles no longer need to be taken into account has the form

$$\lambda_0 \operatorname{Im}[s(\rho_{2n}, \theta) - s(\rho_n, \theta)] \cong \lambda_0 \operatorname{Im}(\rho_{2n} - \rho_n)^2 s_{\rho\rho} \gg 1. \quad (28)$$

Here $s_\rho(\rho_n, \theta) = \epsilon(\theta - \Theta(\rho_n)) = 0$, and $s_{\rho\rho}(\rho_n, \theta) = -\epsilon\Theta'(\rho_n)$. Using the quantity

$$\frac{d\Theta(\rho)}{d\varkappa} = \alpha^{1/4}\frac{(\varkappa')^2(1-\sqrt{\alpha}\varkappa)K - (\varkappa^2 + 1 - 2\varkappa\sqrt{\alpha})E}{\varkappa(\varkappa')^2(1-\sqrt{\alpha}\varkappa)^{1/2}(\sqrt{\alpha}-\varkappa)^{1/2}}, \quad (29)$$

from Eq. (28) we find, in particular, that one need not consider ρ_{2n} provided $a\alpha^{3/4} \gg \lambda_0$.

4. Let us consider in more detail the case of one real caustic, corresponding to the limiting classical angle of scattering. With the aid of (22), one can describe the cross section over the entire range of angles of practical interest by the single formula:

$$\sigma(Y) = {}^1/_2\pi Y[c_1^2 p_1^2 + {}^1/_3 c_2^2 p_2^2], \quad (30)$$

where

$$c_{1,2} = |a_1| \mp |a_2|, \quad p_{1,2} = J_{-1/3}(Y) \mp J_{-2/3}(Y).$$

In the accessible region, not far from the caustic ($c_1 \approx 0$), the cross section oscillates, achieving maxima at the points $Y_+ \cong 0.7, 3.9, \ldots$, and minima at $Y_- \cong 2.3, 5.5, \ldots$; in this connection $\sigma(0.7)/\sigma(3.9) \cong 1.6$, and $\sigma(0.7)/\sigma(0) \cong 2.4$.

At large distances ($Y \gg 1$) from the caustic

$$\sigma = |a_1|^2 + |a_2|^2 + 2|a_1 a_2|\sin 2Y\left(1 + \frac{\cos 2Y}{Y}\right) + O(Y^{-2}),$$

i.e., the cross section oscillates between the smooth curves c_1^2 and c_2^2 around the classical cross section σ^0 (see Fig. 3). In the region of small angles all three curves come together since $|a| \to \infty$ as $\rho \to \infty$. For the case $\nu'(0) = 0$ the contribution to the cross section coming from small impact parameters reduces to a constant $|a| = |\Theta'(0)|^{-1}$, where

$$-\varepsilon\Theta'(0) = 2\int_0^\infty \frac{\nu(0) - \nu(r)}{\sqrt{(1-\nu(0))(1-\nu(r))}}\frac{dr}{r^2}. \quad (31)$$

For $\nu'(0) \neq 0$ and $\rho \to 0$, the quantity $\Theta(\rho) \sim |\rho \ln \rho|$ and $|a| \sim |\ln \rho|^{-1}$.

For potentials $\nu \sim r^{-n}$ the first extrema from the

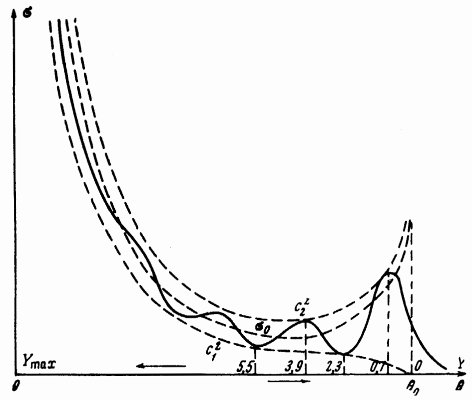

FIG. 3. Shape of the curve $\sigma(\theta)$ according to the approximate formula in the case of one caustic location at $\theta = \theta_0$.

side of small angles are determined with quasiclassical accuracy from the condition

$$\theta^2 = \frac{2(n+1)}{n^{3/2}} \frac{\lambda_0 |\Theta'(0)|}{\cos 2Y} + O\left[\left(\frac{\lambda_0}{a}\right)^{(3n+1)/2n}\right] \qquad (32)$$

and are located at $\theta \sim \sqrt{\lambda_0/a}$. For $\nu \sim r^{-n}$ condition (18) gives a value $\theta \gg (\lambda_0/a)^{n/(n-1)}$.

The position and total number of extrema depend on the values of Y and $Y_{max} = Y(0)$. Geometrically, Y is determined by the area of the peak of $\Theta(\rho)$ above the line $\Theta(\rho_1) = \Theta(\rho_2) = \theta$ (see Fig. 1 a):

$$-2\varepsilon\lambda_0 Y = \int_{\rho_1}^{\rho_2} (\Theta(\rho') - \theta)\, d\rho'. \qquad (33)$$

For attractive potentials ($\epsilon = +1$) $\rho_1 > \rho_2$. From Eqs. (33) and (15) we obtain the following result for Y_{max} as $\theta \to 0$:

$$2\lambda_0 Y_{max} = \int_0^\infty \Theta(\rho)\, d\rho = -2\varepsilon \int_0^\infty \left(1 - \sqrt{1 - \nu(r)}\right) dr. \qquad (34)$$

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[1] A. Z. Patashinskiĭ, V. L. Pokrovskiĭ, and I. M. Khalatnikov, Zh. Eksp. Teor. Fiz. 45, 989 (1963) [Sov. Phys.-JETP 18, 683 (1964)].

[2] V. P. Maslov, Dokl. Akad. Nauk SSSR 151, 306 (1963) [Sov. Phys.-Doklady 8, 666 (1964)].

[3] S. V. Khudyakov, Zh. Eksp. Teor. Fiz. 56, 938 (1969) [Sov. Phys.-JETP 29, 507 (1969)].

[4] L. D. Landau and E. M. Lifshitz, Kvantovaya mekhanika (Quantum Mechanics), Fizmatgiz, M. 1963.

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