WEAK TURBULENCE OF HELICONS IN AN ELECTRON-HOLE PLASMA

V. M. YAKOVENKO

Institute for Radio Engineering and Electronics, Ukrainian Academy of Sciences

Submitted February 21, 1969

Zh. Eksp. Teor. Fiz. 57, 574–579 (August, 1969)

A kinetic equation is derived which describes helicon interaction in an electron-hole plasma. The turbulence spectrum is found on the basis of dimension considerations. The effect of the helicons on the carrier drift velocity induced by a stationary electric field is considered.

1. It is well known that in a nonequilibrium electron-hole plasma in a magnetic field, there exist weakly damped electromagnetic oscillations, called helicons. These oscillations become unstable in the presence of a carrier drift, which can occur, for example, under the influence of a constant electric field. The initial stage of development of the instability is described by a linear theory[1,2]. The limitation of the wave amplitude and the nature of the stationary state in the plasma are usually determined by nonlinear effects of the interactions between the waves and charged particles. Such a plasma state, in which a large number of collective degrees of freedom is excited, is turbulent[3]. Since the thermal motion of the charged particles is unimportant in the propagation of helicons, it is natural to expect that the processes of decay and merging of the helicons will play the dominant role (it is known that helicons have a decay spectrum[4]).

In the present paper we consider a turbulent plasma that represents a system of interacting helicons. The interaction is assumed to be quite small (weak turbulence). This allows one to write down the equation for helicons following the method proposed in the paper by Galeev and Karpman[5]. It is assumed that the weak turbulence[8,9], like the usual hydrodynamic turbulence[6,7], is localized. On this basis we find the spectrum of helicon turbulence and evaluate the influence of helicons on the constant carrier drift velocity.

2. The following equations are used for the description of the wave interaction:

$$\operatorname{rot}\mathbf{H}=\frac{4\pi}{c}\mathbf{j},\quad \operatorname{rot}\mathbf{E}=-\frac{1}{c}\frac{\partial\mathbf{H}}{\partial t},\quad \operatorname{div}\mathbf{H}=0,\quad \mathbf{j}=\sum e_\alpha N_\alpha\mathbf{v}_\alpha,$$
$$m_\alpha\nu_\alpha\mathbf{v}_\alpha=e_\alpha\mathbf{E}+\frac{e_\alpha}{c}\{[\mathbf{v}_\alpha^0\mathbf{H}]+[\mathbf{v}_\alpha\mathbf{H}_0]+[\mathbf{v}_\alpha\mathbf{H}]\},\qquad \operatorname{div}\mathbf{v}_\alpha=0. \qquad (1)^*$$

Here **E** and **H** are the alternating fields; N_α, m_α, e_α, ν_α, and $\mathbf{v}_\alpha$ are respectively the concentration, effective mass, charge, effective collision frequency, and velocity of electrons (α = e) and holes (α = h); $\mathbf{H}_0$ is the constant magnetic field; $\mathbf{v}_\alpha^0$ is the constant velocity of the carrier drift ($\mathbf{v}_\alpha^0 = e_\alpha\mathbf{E}_0/m_\alpha\nu_\alpha$). In the equation of motion the inertial term is omitted, as the wave frequency is assumed to be small in comparison with the collision frequency.

It is convenient to rewrite the system of equations (1) in the following way:

$$\frac{\partial\mathbf{H}}{\partial t}=\frac{c(\mathbf{H}_0\nabla)\operatorname{rot}\mathbf{H}}{4\pi eN}-(\mathbf{u}_0\nabla)\mathbf{H}+\frac{\operatorname{rot}[\mathbf{jH}]}{eN}-\frac{c}{eN}\operatorname{rot}\sum_\alpha N_\alpha\nu_\alpha m_\alpha\mathbf{v}_\alpha, \qquad (2)$$

where

$$N=N_h-N_e,\qquad \mathbf{u}_0=\frac{N_h\mathbf{v}_h^0-N_e\mathbf{v}_e^0}{N}.$$

Following[4,5], we represent all the variable quantities appearing in (2) in the form:

$$\mathbf{H}=\sum C_\mathbf{k}(t)\mathbf{h}_\mathbf{k}e^{i(\mathbf{kr}-\omega t)},\qquad \mathbf{v}=\sum C_\mathbf{k}(t)v_\mathbf{k}e^{i(\mathbf{kr}-\omega t)}, \qquad (3)$$

$$C_\mathbf{k}=C_\mathbf{k}^*,\qquad \mathbf{h}_\mathbf{k}=\mathbf{h}_\mathbf{k}^*,\qquad \omega(\mathbf{k})=-\omega(-\mathbf{k}), \qquad (3a)$$

where $C_\mathbf{k}$ is the slowly varying amplitude of the harmonic with wave vector **k** and frequency ω; $\mathbf{h}_\mathbf{k}$ is a solution of Eq. (2) with the dissipative and nonlinear terms neglected,

$$\omega-k_zu_0=\pm\frac{ckk_zH_0}{4\pi eN},\qquad \mathbf{v}_{\mathbf{k}\alpha}=\frac{\mathbf{h}_\mathbf{k}(k_zv_\alpha^0-\omega)}{k_zH_0},$$
$$\mathbf{j}_k=\frac{\mathbf{h}_k eN}{k_zH_0}(k_zu_0-\omega),\qquad z\parallel H_0. \qquad (4)$$

If dissipation is not included, the system of equations (1) has an integral of motion (energy conservation law)

$$\int\frac{H^2}{8\pi}dv=\text{const}=\sum_k|C_\mathbf{k}|^2\frac{|\mathbf{h}_\mathbf{k}|^2}{8\pi}. \qquad (5)$$

Assuming that $|\mathbf{h}_\mathbf{k}|^2/8\pi = |\omega - k_z\mu_0|$ is the energy of the helicon, the square of the amplitude modulus, $|C_\mathbf{k}|^2$, can be interpreted as the number of helicons in **k** space.

Inserting expression (3) into (2) and integrating over $d^3\mathbf{r}$, we obtain the equation for the amplitudes

$$\frac{\partial C_\mathbf{k}}{\partial t}=\gamma_kC_\mathbf{k}-i\sum V_{\mathbf{kk'k''}}C_{\mathbf{k'}}C_{\mathbf{k''}}e^{i(\omega-\omega'-\omega'')t}, \qquad (6)$$

$$\gamma_k=-\frac{c^2k^2}{4\pi e^2N^2}\sum m_\alpha N_\alpha\nu_\alpha+\frac{4\pi(k_zu_0-\omega)}{k_zH_0^2}\times\frac{N_eN_hv_0\sum m_\alpha\nu_\alpha}{N^2},\qquad v_0=v_h^0+|v_e^0|, \qquad (7)$$

$$V_{\mathbf{kk'k''}}=\frac{2\pi ieN}{cH_0^2k_z}\left(\frac{\omega''}{k_z''}-\frac{\omega'}{k_z'}\right)\sqrt{8\pi|(\omega-k_zu_0)(\omega'-k_z'u_0)(\omega''-k_z''u_0)|}\times\frac{(k_zu_0-\omega)}{|k_zu_0-\omega|}(\mathbf{a}_\mathbf{k}^*[\mathbf{a}_{\mathbf{k'}}\mathbf{a}_{\mathbf{k''}}]), \qquad (8)$$

$$a_x=\frac{\sqrt{k_y^2+k_z^2}}{k\sqrt{2}},\qquad a_z=\frac{a_xcH_0k_z(k_z^2-k^2)}{cH_0k_xk_z^2-4\pi ieNk_y(k_zu_0-\omega)},$$
$$a_y=a_x\frac{cH_0k_yk_z^2+4\pi ieNk_x(k_zu_0-\omega)}{cH_0k_xk_z^2-4\pi ieNk_y(k_zu_0-\omega)}. \qquad (9)$$

*$[\nu_\alpha\mathbf{H}]\equiv\nu_\alpha\times\mathbf{H}$.

Here γ_k is the growth rate (or attenuation rate), and $V_{kk'k''}$ is the matrix element for helicon scattering.

It is necessary to note that two types of helicons exist in a plasma in a constant electric field; fast ($\omega > k_z\mu_0$) and slow ($\omega < k_z\mu_0$). As is clear from formula (7), only slow helicons can be amplified in a constant field. Thus in the following we shall be concerned with the interaction of slow helicons. We assume that the phases of the oscillation amplitudes, corresponding to various k, are distributed completely randomly (just such a state is turbulent). Using this assumption we obtain the kinetic equation in the form[5]

$$\frac{\partial n_k}{\partial t} = 2\gamma_k n_k + \left(\frac{\partial n}{\partial t}\right)_{st};$$

$$\left(\frac{\partial n_k}{\partial t}\right)_{st} = 4\pi\sum_{k',k''} |V_{kk'k''}|^2 (n_{k'}n_{k''} - n_k n_{k'} - n_k n_{k''})\,\delta_{k,k'+k''}\delta(\omega-\omega'-\omega'')$$

$$+ 8\pi\sum_{k',k''} |V_{k,-k',k''}|^2 (n_{k'}n_{k''} - n_k n_{k'} + n_k n_{k''})\,\delta_{k+k',k''}\delta(\omega''-\omega'-\omega). \tag{10}$$

Here $n_k = |C_k|^2$ is the number of helicons with $\omega = k_z\mu_0 - cH_0 k k_z/4\pi eN$,

$$\gamma_k = -\alpha k^2 + \beta k, \quad \alpha = -\frac{c^2}{4\pi e^2 N^2}\sum m_\alpha N_\alpha \nu_\alpha,$$
$$\beta = \frac{cN_e N_n}{eN^2 H_0} v_0 \sum m_\alpha \nu_\alpha. \tag{11}$$

The summation runs over positive k ($\omega > 0$).

Equation (10) is subject to the energy conservation law

$$\frac{\partial}{\partial t}\int |\omega - k_z u_0| n_k d^3\mathbf{k} = 2\int \gamma_k |\omega - k_z u_0| n_k d^3\mathbf{k}. \tag{12}$$

The size of the system, d, is assumed to be sufficiently large that boundary effects are unimportant. This condition is fulfilled for $d \gg d_{min} = 2\alpha/\beta \sim cH_0/2\pi j_0$. Thus for $j_0 \sim 10^5$ A/cm^2 and $H_0 \sim 10^3$ Oe we obtain $d_{min} \sim 1/20\pi$ cm.

3. It is impossible, of course, to find the solution of (10) in a general form. Certain qualitative statements concerning the relation $n_k = n_k(k)$ can be made using the hypothesis of the localization of the weak turbulence, i.e., the assumption that the interaction is strong only if the space scaling lengths involved are of the same order[6-9].

As is shown in[8,9], the weak turbulence determined by wave interaction processes has properties that are quite analogous to those of the usual hydrodynamic turbulence[6,7]. It also appears that in the case of weak turbulence, a domain of wave vectors can exist (an inertial domain or a domain of complete equilibrium) in which the spectrum of turbulence is determined only by the flux of energy in the direction of the larger k.

This result was obtained from the analysis of the exact solution of the kinetic equation describing the interaction of plasmons[8] or the interaction of capillary waves[9]. This proves the localization of the weak turbulence in this case.

Although the localization of weak turbulence has not proved in general, the dimensional considerations developed in[8,9] for weak turbulence are apparently correct for helicons.

We first consider the problem of the evolution in time of a packet of helicons in the absence of a constant electric field. The attenuation of helicons is proportional to k^2. This means that the dissipative term in the kinetic equation is important only at large k. The range of influence of the dissipative term is to be determined from the condition $\alpha k_1^2 n(K_1) \sim (\partial n/\partial t)_{st}$.

We now go from summation to integration in the kinetic equation. Then the main term of the kinetic equation is of the order

$$\left(\frac{\partial n}{\partial t}\right)_{st} \sim \frac{V^2}{\omega_k} k^3 n_k^2, \quad \omega \sim k^2, \quad V \sim k^3;$$

the term $\partial n/\partial t$ is of the order n/τ, where τ is the characteristic time of the "nonlinear" attenuation. Clearly the term $\partial n/\partial t$ is important at small k. Thus the entire wave vector space can be broken up into three domains; energy-containing domain ($k \sim k_0$, $\partial n/\partial t \sim V^2\omega_k^{-1}k^3 n_k^2$), attenuation domain ($k \sim k_1$), and intermediate (inertial) domain ($k_0 < k < k_1$, $(\partial n/\partial t)_{st} \sim 0$).

Let the packet fill the energy-containing domain, $k \sim k_0$, at the initial time. The helicons "diffuse," as a result of interaction, to the intermediate domain and from there to the attenuation domain, where the dissipation of energy (the absorption of helicons) occurs. A helicon energy flux p toward larger k results, and in the inertial domain the spectral density n_k is determined by the value of the flux only. This interval is a domain of complete equilibrium. The energy flux p, i.e., the amount of energy flowing out of the energy-containing domain, is

$$p = \int_{k\sim k_0} \omega_k \frac{\partial n_k}{\partial t} d^3\mathbf{k} \sim V^2 n_{k_0}^2 k_0^6 = \text{const}, \tag{13}$$

where n_{k_0} is the density of particles in the energy-containing domain. The form of p should be the same in the inertial domain as well, so that we obtain the energy spectrum

$$n_k \sim \frac{p^{1/2}}{k^3 V} \simeq \frac{p^{1/2}}{k^6}, \quad \varepsilon_k = \omega_k n_k k^2 \sim p^{1/2} k^{-2}. \tag{14}$$

We now estimate k_1. The order of magnitude of the dissipative term is $\gamma n_{k_1} \sim \alpha p^{1/2} k_1^{-4}$, and the order of $(\partial n/\partial t)_{st} \sim p k_1^{-5}$; hence $k_1 \sim p^{1/2}\alpha^{-1}$.

In the energy-containing domain, $\partial n_{k_0}/\partial t \sim V^2 k_0^3 n_{k_0}^2/\omega k_0$ holds. From this condition we obtain $n_k \sim n_{k_0}\tau/t$; i.e., the number of helicons is inversely proportional to the time. The characteristic time of nonlinear damping, $\tau \sim 1/n_{k_0}k_0^7$, can be determined from the condition $\epsilon/\tau \sim p$, where $\epsilon \sim \omega_{k_0} k_0^3 n_{k_0}$ is the total energy of the wave packet. The criterion for existence of the inertial domain, $k_1 \gg k_0$, can be reduced to the condition $1/\tau \gg \alpha k_0^2$, i.e., in the domain k_0, the nonlinear damping $1/\tau$ should be larger than the "viscous" damping αk_0^2. The size of k_1 (the width of the inertial domain) is determined by the quantity of helicons in the energy-containing domain (i.e., by the initial energy of the wave packet) and by the collisions of the carriers. As the number of helicons decreases, the flux of helicon energy diminishes and the inertial domain narrows. The kinetic equation itself is valid for $\omega_{k_0} \gg 1/\tau$. It should be noted that in the inertial domain the kinetic equation has a solution $n_k = \text{const.}/\omega$ (the Rayleigh-Jeans distribution). However, it turns out then that the dissipative term is smaller than the collision term $(\partial n/\partial t)_{st}$ for arbitrarily large

k. This leads to the divergence of the integral determining the total energy, i.e., the Rayleigh-Jeans distribution cannot be realized in this case.

4. We are now going to consider the influence of the constant electric field. As is clear from (11), the generation of waves occurs only at small k, i.e., in the energy-containing domain. It is clear that in the stationary state the outflow of helicons from that domain should be compensated by the generation of helicons. Thus in the energy-containing domain, $k \sim k_0$, the following condition is satisfied:

$$\beta k_0 n_{k_0} \sim V^2 k_0^3 n_{k_0}^2 / |\omega - k_z u_0|. \quad (15)$$

The energy flux is

$$p = \int \beta k |\omega - k_z u_0| n_k d^3\mathbf{k} \sim V^2 k^6 n_k^2$$

and the turbulence spectrum in the inertial domain is determined by (14), as before. The dissipation of energy occurs at $k \sim k_1 \sim p^{1/2}\alpha^{-1}$. The order of n_k in the energy—containing domain can be estimated from the condition (15):

$$\beta k_0 \sim k_0^7 n_{k_0}, \quad n_{k_0} \sim \beta / k_0^6. \quad (16)$$

Hence $p = \beta^2$; i.e., the energy flux is determined by the strength of the constant electric field. So, in the entire wave vector space, $k_0 < k < k_1$, the following condition is fulfilled:

$$\int \gamma_k |\omega - k_z u_0| n_k d^3\mathbf{k} = \int\limits_{k=k_0} \beta k |\omega - k_z u_0| n_k d^3\mathbf{k} - \int\limits_{k=k_1} \alpha k^2 |\omega - k_z u_0| h_k d^3\mathbf{k} = 0,$$

i.e., the total energy of the system is conserved. The above formulas are true under the condition ωk_0 $\beta k_0 \gg \alpha k_0^2$.

The drift velocity of the carriers is involved in the damping and in the dispersion relation. We now estimate the influence of the helicons on the constant carrier drift velocity (the electric field is fixed). To do this we use the system of equations (1), from which we determine v_z, the velocity of electrons with nonlinear terms taken into account, and then we perform the volume averaging. We obtain the result,

$$v_\alpha^{(1)} = v_\alpha^{(0)} + \frac{1}{H_0^2} \sum_k |C_k|^2 \left(\frac{\omega}{k_z} - v_\alpha^{(0)}\right) (|h_x|^2 + |h_y|^2). \quad (17)$$

We assume for simplicity that the mobility of the electrons is far higher than that of the holes ($m_e \nu_e \ll m_k \nu_k$). Then, using relation (15), we obtain

$$v_e^{(1)} \approx \frac{eE_0}{m_e \nu_e}\left(1 - \frac{N_h}{N}\frac{\nu_h}{\omega_h}\right); \quad \omega_h = \frac{eH_0}{m_h c}. \quad (18)$$

Thus there occurs an increase in the electron collision frequency, $\nu_{\text{eff}} \approx \nu_e(1 + N_k \nu_k / N\omega_k)$, and a decrease in the drift velocity of the electron conductivity.

The author thanks V. E. Zakharov for valuable advice.

[1] A. Bers and A. L. McWhorter, Phys. Rev. Letters 15, 755 (1965).

[2] V. I. Veselago, M. A. Glushkov and A. A. Rukhadze, Fiz. Tverd. Tela 8, 24 (1966) [Sov. Phys. Solid State 8, 18 (1966)].

[3] B. B. Kadomtsev, Voprosy Teorii Plazmy, 4, Atomizdat, 1964, p. 188 [Plasma Turbulence, London, New York, Academy Press, 1969].

[4] R. N. Sudan, A. Cavaliere and M. N. Rosenbluth, Phys. Rev. 158, 387 (1967).

[5] A. A. Galeev and V. I. Karpman, Zh. Eksp. Teor. Fiz. 44, 598 (1963) [Sov. Phys.-JETP 17, 403 (1963)].

[6] A. N. Kolmogorov, Dokl. Akad. Nauk SSSR 30, 299 (1941).

[7] A. M. Obukhov, Izv. Akad. Nauk SSSR, seriya geogr. i geof. 5, 1941.

[8] V. E. Zakharov, Zh. Eksp. Teor. Fiz. 51, 688 (1966) [Sov. Phys.-JETP 24, 455 (1967)].

[9] V. E. Zakharov and N. N. Filonenko, Prikl. Mat. i Teor. Fiz. 5, 62 (1967).

Translated by W. Zielke
66