

QUASILINEAR RELAXATION OF AN ELECTRON BEAM IN AN INHOMOGENEOUS PLASMA

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The one-dimensional quasilinear relaxation of an electron beam in a plasma with small concentration inhomogeneities is investigated. It is shown that the set of quasilinear equations are appreciably modified if the effect is taken into account. In particular, quasilinear relaxation leads, in the presence of small concentration inhomogeneities, to the appearance of electrons with velocities appreciably exceeding the initial beam velocity. The role of regular concentration inhomogeneities in the problem of stationary injection of a beam into plasma half-space is also elucidated.

1. INTRODUCTION

THE problem of quasilinear relaxation of an electron beam in a homogeneous plasma was formulated already in the first papers on the quasilinear theory.<sup>[1, 2]</sup> The one-dimensional model of quasilinear relaxation was investigated in detail in the same papers. It was established that, within the framework of such a model, the relaxation process leads to the formation of a plateau on the electron distribution function.

Of greatest practical interest is the case when the initial velocity of the beam  $v_0$  greatly exceeds the thermal velocity of the plasma electrons  $v_T$ , and the initial velocity spread in the beam  $\Delta v_0$  is small compared with  $v_0$ . According to the results of<sup>[1, 2]</sup>, the electron distribution function in the final stage should have under these conditions the form of a "step" with a leading front located at the point  $v \approx v_0$  (Fig. 1), i.e., relaxation should not lead to the appearance of accelerated electrons (with velocity  $v > v_0$ ).

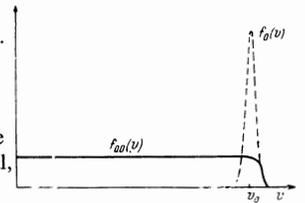
There are, however, a number of experiments (for example,<sup>[3]</sup>) in which, on the one hand, the conditions for the applicability of the one-dimensional model were apparently satisfied,<sup>1)</sup> and on the other hand accelerated electrons were observed. We shall show in this paper that this effect finds a natural explanation if account is taken of the role of the inhomogeneities of the plasma, no matter how insignificant they are.<sup>2)</sup>

We assume that the beam moves in the direction of a strong magnetic field parallel to the  $x$  axis. We assume further that the dependence of the plasma concentration on the coordinate  $x$  is of the form  $n(x) = n_0 + \Delta n(x)$ , where  $n_0$  is the average value of the concentration and  $\Delta n(x)$  is a small and time-independent deviation of the concentration from the mean value. It is immaterial in what follows whether the function  $\Delta n(x)$  is a random or a regular (periodic) function. It only matters that its

<sup>1)</sup>For example, the presence of a strong magnetic field (such that the electron cyclotron frequency greatly exceeds the electron plasma frequency) parallel to the beam axis.

<sup>2)</sup>We note that theoretical investigations devoted to the acceleration of electrons in three-dimensional relaxations have already been made (see [4]).

FIG. 1. Relaxation of electron beam with small initial velocity spread.  $f_0(v)$  - distribution function of beam electrons at the initial instant of time;  $f_\infty(v)$  - the same after the end of the relaxation process. The areas under the  $f_0(v)$  and  $f_\infty(v)$  curves should be equal, but for clarity this circumstance is not reflected in the figure.



spatial scale  $a$  is large compared with the characteristic wavelength of the Langmuir oscillations excited by the beam:

$$(1)$$

here  $\omega_p = (4\pi n_0 e^2/m)^{1/2}$  is the electron plasma frequency calculated from the mean value of the concentration. In real experiments, inhomogeneities of this type can result from the development of various low-frequency (for example, drift) instabilities, as well as a result of the imperfection of the method used to prepare to the plasma.

When inequality (1) is satisfied, the Langmuir oscillations of the plasma can be described as a superposition of quasiparticles (wave packets), the motion of which in phase space is described by the Hamiltonian equations (see [5, 6]):

$$\frac{dx}{dt} = \frac{\partial}{\partial k} \omega(k, x), \quad \frac{dk}{dt} = -\frac{\partial}{\partial x} \omega(k, x).$$

Here  $x$  and  $k$  are the coordinate and momentum (wave vector) of the quasiparticle and  $\omega(k, x)$  is its Hamiltonian function (frequency), which is determined from a solution of dispersion equation. The quasiparticle is characterized also by the magnitude of the phase velocity  $v_{ph} = \omega(k, x)/k$ , the interaction between a quasiparticle and an electron having a velocity  $v$  being possible only if  $v_{ph} = v$ . The momentum distribution of the quasiparticle is described by the spectral energy density  $W(k, x)$ .

In our case we have

$$\omega(k, x) \approx \omega_p \left( 1 + \frac{1}{2} \frac{\Delta n(x)}{n_0} \right) + \frac{3}{2} \frac{k^2 v_T^2}{\omega_p}, \quad (2)$$

where  $v_T$  is the thermal velocity of the plasma electrons, assumed small compared with  $v_0$ . As seen from this relation, the Hamiltonian of the quasiparticle coincides with the Hamiltonian of the ordinary nonrelativistic particle in a potential field, and the first term plays the role of the potential energy while the second plays the role of the kinetic energy.

Let us determine the concentration inhomogeneity  $\Delta n$  necessary in order for the inhomogeneities to begin to influence strongly the motion of the quasiparticles. To this end we note that the characteristic momentum  $k$  of the quasiparticles produced as the result of two-stream instability is equal to  $\omega_p/v_0$ , i.e., the "kinetic energy" of the quasiparticles is of the order of  $\omega_p v_T^2/v_0^2$ . It is obvious that the influence of the inhomogeneities of the concentration, i.e., of the term  $\Delta n(x)/n_0$  in formula (2), will be appreciable if  $\omega_p \Delta n/n_0 \gtrsim \omega_p v_T^2/v_0^2$ . We shall henceforth stipulate satisfaction of an even stronger inequality

$$\Delta n/n_0 \gg v_T^2/v_0^2 \quad (3)$$

(this simplifies the derivations).

We can now determine how the presence of concentration inhomogeneities leads to the appearance of accelerated electrons. To this end, it should be noted that the quasilinear diffusion coefficient in velocity space  $D(v)$  is proportional to  $W(k, x)|_k = \omega/v$ , i.e., the diffusion of the electrons with velocity  $v$  is determined completely by the energy density of the quasiparticles with phase velocity  $v_{ph} = v$ . Quasiparticles with phase velocity are produced only if

$$\partial f/\partial v|_{v=v_{ph}} > 0,$$

where  $f$  is the electron distribution function. As applied to our problem, this means that quasiparticles are produced only in the phase-velocity region  $v_{ph} < v_0$ . Since the momentum of a quasiparticle in a homogeneous plasma does not change after the quasiparticle is produced, its phase velocity likewise remains unchanged. Consequently, in a homogeneous plasma the diffusion coefficient  $D(v)$  differs from zero only when  $v < v_0$ , i.e., the relaxation proceeds in the direction of small velocities and no accelerated electrons appear.

In an inhomogeneous plasma, when condition (3) is satisfied, the situation changes greatly. Although, as before, the quasiparticle has at the instant of production a phase velocity  $v_{ph} < v_0$ , now this quantity no longer remains constant during the course of quasiparticle motion. Assume, for example, that the quasiparticle is produced at the point  $x_0$  (Fig. 2). At the instant of production, this quasiparticle has a momentum  $k \gtrsim \omega_p/v_0$  and moves to the right. During the course of the motion, it falls into the region of large values of the "potential energy," as the result of which its momen-

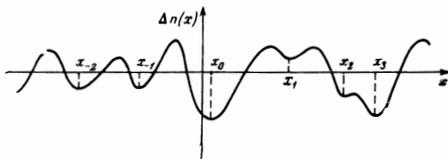


FIG. 2. Dependence of the plasma concentration on the coordinate  $x$ . The points  $x_i$  denote the minima of the function  $\Delta n(x)$ .

tum  $k$  decreases and vanishes in general at the turning point.<sup>3)</sup> This means that on moving to the turning point, the phase velocity of the quasiparticle ( $v_{ph} = \omega(k, x)/k \approx \omega_p/k$ ) increases from the initial value ( $\lesssim v_0$ ) to infinity. Therefore the quasilinear diffusion coefficient  $D(v)$  turns out to be different from zero not only when  $v < v_0$ , but also at arbitrarily large values of  $v$ , i.e., the relaxation goes both in the direction of  $v < v_0$  and in the direction of  $v > v_0$ .

It must be emphasized here that the influence of the concentration inhomogeneities on the relaxation process will be appreciable only if one more condition is satisfied, namely

$$a \ll \gamma^{-1} |\partial \omega / \partial k| \quad (4)$$

( $\gamma$  is the increment of the two-stream instability), for otherwise each particle will not have time to cover during the time of the quasilinear relaxation a distance sufficient to "sense" the inhomogeneity of the plasma. Bearing in mind the fact that  $|\partial \omega / \partial k| \sim v_T^2/v_0$  and  $\gamma \sim \omega_p n'/n_0$  (where  $n'$  is the beam-electron concentration), we obtain from (4) the condition

$$\frac{a \omega_p}{v_0} \ll \frac{n_0 v_T^2}{n' v_0^2}. \quad (5)$$

Therefore our analysis is valid only for sufficiently weak beams.

## 2. DERIVATION OF QUASILINEAR EQUATIONS FOR A WEAKLY INHOMOGENEOUS PLASMA

The evolution of the spectral function of the quasiparticles  $W(k, x, t)$  can be described with the aid of the Liouville equation:<sup>[5, 6]</sup>

$$\frac{\partial W}{\partial t} + \frac{\partial \omega}{\partial k} \frac{\partial W}{\partial x} - \frac{\partial \omega}{\partial x} \frac{\partial W}{\partial k} = 2\gamma W, \quad (6)$$

where  $\omega \equiv \omega(k, x)$  is the solution of the dispersion equation and  $\gamma \equiv \gamma(k, x, t)$  is the increment of the two-stream instability. Strictly speaking, it would be necessary to substitute in (6) the quantity  $N = W/\omega$ , called the "number of quasiparticles",<sup>[6]</sup> instead of the quantity  $W$ , but since  $\omega \approx \omega_p = \text{const}$  it follows that  $W \approx \text{const} \cdot N$ , and consequently  $W$  satisfies the same equation as  $N$ .

The two-stream instability increment  $\gamma$  is expressed in terms of the beam-electron distribution function  $f$ , which, as we shall show later, can be considered independent of  $x$ . In calculating the increment, we can neglect also the inhomogeneities of the plasma concentration, since they are small. Accordingly, we can use the following formula for  $\gamma$ :

$$\gamma(k, x, t) \approx \gamma(k, t) = \frac{\pi}{2} \omega_p \frac{n'}{n_0} \left( v^2 \frac{\partial f}{\partial v} \right) \Big|_{v=\omega_p/k}. \quad (7)$$

Here the distribution function of the beam electrons is assumed to be normalized to unity:

$$\int_0^{\infty} f dv = 1.$$

In solving Eq. (6), it is necessary to take into account the condition (4), which means that the energy of each wave packet changes "slowly" in the time scale  $a |\partial \omega / \partial k|^{-1}$ . The latter circumstance makes it possible

<sup>3)</sup> A turning point must exist if condition (3) is satisfied.

to characterize the rate of energy growth of the wave packet with the aid of a certain average increment  $\gamma$ . In order to find this quantity, we change over in (6) from the variables  $t, x, k$  to the variables  $t, x, \omega$ . As the result we get

$$\frac{\partial W}{\partial t} + \frac{\partial \omega}{\partial k} \frac{\partial W}{\partial x} = 2\gamma W,$$

where  $W, \partial\omega/\partial k$ , and  $\gamma$  are now regarded as functions of the new variables. Taking the inequality (4) into account, we seek the solution of this equation in form of a series in powers of  $\gamma$ . Since it is obvious that  $\partial/\partial t \sim \gamma$ , we obtain  $\partial W^{(0)}/\partial x = 0$  in the zeroth approximation in  $\gamma$ , i.e.,  $W^{(0)} = W^{(0)}(\omega, t)$ . The dependence of  $W^{(0)}$  on the time can be obtained from the condition that the equation

$$\frac{\partial W^{(0)}}{\partial x} = \frac{2\gamma W^{(0)} - \partial W^{(0)}/\partial t}{\partial \omega/\partial k}, \quad (8)$$

which determines the first correction to  $W$ , have a solution.

As will be shown later, the main role in the relaxation process is played by the quasiparticles that execute finite motion near the bottoms of the potential wells. For such quasiparticles, the condition for the solvability of (8) is

$$\oint \frac{2\gamma W^{(0)} - \partial W^{(0)}/\partial t}{\partial \omega/\partial k} dx = 0. \quad (8')$$

The integral is taken here over the trajectory of the quasiparticle between the turning points at  $\omega = \text{const}$ . Recognizing that  $W^{(0)}$  does not depend on  $x$ , we can transform (8') into

$$\partial W^{(0)}/\partial t = 2\bar{\gamma} W^{(0)},$$

where  $\bar{\gamma}$  is the sought average increment:

$$\bar{\gamma} = \left[ \oint v \frac{dx}{\partial \omega/\partial k} \right] \left[ \oint \frac{dx}{\partial \omega/\partial k} \right]^{-1}$$

An expression of this type can be obtained also by another method based on the use of the quasiclassical quantization rules.<sup>[7, 8]</sup>

We note that since the beam contains only particles with  $v > 0$ , we get  $\gamma = 0$  when  $k < 0$ , and consequently

$$\bar{\gamma} = \frac{1}{2} \left[ \int_{\xi^{(1)}(\omega)}^{\xi^{(2)}(\omega)} \gamma \frac{dx}{\partial \omega/\partial k} \right] \left[ \int_{\xi^{(1)}(\omega)}^{\xi^{(2)}(\omega)} \frac{dx}{\partial \omega/\partial k} \right]^{-1}, \quad (9)$$

where  $\xi^{(1)}$  and  $\xi^{(2)}$  are the left and right turning points.

To obtain an explicit expression for  $\gamma$ , let us consider in greater detail an individual potential well, the bottom of which is located at the point  $x = x_i$  (the index  $i$  denotes the number of the potential well). Near this point, i.e., when  $|x - x_i| \ll a$ , the following expansion is valid

$$\Delta n(x) = \Delta n_i + |\Delta n_i| (x - x_i)^2 / 2a_i^2.$$

We have introduced here the notation

$$\Delta n_i = \Delta n \Big|_{x=x_i}, \quad \frac{1}{a_i^2} = \frac{1}{|\Delta n_i|} \frac{\partial^2 \Delta n}{\partial x^2} \Big|_{x=x_i}$$

thus, the Hamiltonian function for the quasiparticles moving at the bottom of the  $i$ -th potential well is given by

$$\omega = \text{const} + \omega_F \frac{|\Delta n_i|}{2n_0} \frac{(x - x_i)^2}{2a_i^2} + \frac{3}{2} \frac{k^2 v_T^2}{\omega}. \quad (10)$$

The use of (10) makes it possible to carry out the integration in (9). It is convenient here to go over to a new integration variable  $v$ , which is connected with  $x$  by the relation

$$v = \omega_p / k(x),$$

where  $k(x)$  is determined from the condition  $\omega(k, x) = \text{const}$ . The physical meaning of the quantity  $v$  is obvious: this is simply the phase velocity of the quasiparticle at the point  $x$ . Simple calculations show that

$$(x - x_i)^2 = \frac{a_i^2 v_T^2}{\epsilon_i} \left( \frac{1}{u^2} - \frac{1}{v^2} \right), \quad \frac{\partial \omega}{\partial k} = 3 \frac{v_T^2}{v}, \quad (11)$$

where  $\epsilon_i = |\Delta n_i| / 6n_0$  and  $u = \omega_p / k(x_i)$ . The quantity  $u$ , which is the phase velocity of the quasiparticle at the bottom of the potential well, is uniquely connected with the frequency  $\omega$ . This makes it possible to express the spectral function  $W$  and the increment  $\gamma$  not in terms of  $\omega$  but in terms of  $u$ , as we shall do from now on.

Substituting (11) in the integrals of (9), we get

$$\bar{\gamma}(u, t) = \frac{\omega_p}{2} \frac{n'}{n_0} \int_u^\infty \frac{\partial f(v, t)}{\partial v} \left( \frac{1}{u^2} - \frac{1}{v^2} \right)^{-1/2} dv \quad (12)$$

(we have used expression (7) for  $\gamma$ ).

It is very important that the last result does not depend on the parameters  $a_i$  and  $\Delta n_i$ , which determine the form of the potential well, and is therefore applicable to each of them. We note further that in case when the beam-particle velocity spread  $\Delta v$  is small compared with the average beam velocity  $v_0$ , the maximum value of  $\bar{\gamma}$  is of the order of magnitude  $\omega_p(n'/n_0) \times (v_0/\Delta v)^{3/2}$ , whereas in a homogeneous plasma the corresponding estimate gives a much larger value, viz.,  $\omega_p(n'/n_0)(v_0/\Delta v)^2$ . For a smeared-out beam with  $\Delta v \sim v_0$ , the characteristic increment is  $\bar{\gamma} = \omega_p n' / n_0$ , which coincides with the corresponding estimate for the homogeneous plasma.

In the derivation of (12) we used the expansion (10), and this imposes definite limitations on the conditions for its applicability. As seen from (11), the distance from the turning points  $\xi_1^{(1,2)}$  to the bottom of the potential well is expressed in terms of the phase velocity  $u$  of the quasiparticle at the bottom of the well:

$$(\xi_i^{(1,2)} - x_i)^2 = a_i^2 v_T^2 / \epsilon_i u^2.$$

Since the expansion (10) is valid only when  $|x - x_i| \ll a_i$ , this yields the following limitation on the permissible values of  $u$ :

$$u \gg u_0 = v_T / \sqrt{\epsilon_i} \sim v_T \sqrt{n_0 / \Delta n}. \quad (13)$$

Since the condition (3) leads to the inequality  $u_0 \ll v_0$ , formula (12) can be used only so long as the quasilinear relaxation does not lead to the appearance of electrons with velocity of the order of  $u_0$ , i.e., essentially to the very end of the relaxation process ( $u_0 \ll v_0$ !). The limitation (13) is therefore not essential.

We now consider the quasilinear equation for the beam electrons:

$$\frac{\partial f}{\partial t} + v \frac{\partial f}{\partial x} = \frac{\partial}{\partial v} D \frac{\partial f}{\partial v} \quad (14)$$

where

$$D \equiv D(v, x, t) = \frac{4\pi^2 e^2}{m^2 v} W(k, x, t) \Big|_{k=\omega_p/v}$$

is the quasilinear diffusion coefficient. Averaging (14) with respect to  $x$ , we get

$$\frac{\partial \bar{f}}{\partial t} = \frac{\partial}{\partial v} \bar{D} \frac{\partial \bar{f}}{\partial v}$$

(the bar denotes averaging with respect to  $x$ ).

In analogy with the procedure used above for Eq. (6), it can be shown that if

$$a/v_0 \ll \tau, \quad (15)$$

where  $\tau$  is the characteristic time of the quasilinear relaxation, the function  $f$  does not depend on the coordinate  $x$ :  $f \approx \bar{f}$ . But since  $\tau \gtrsim \omega_p^{-1} n/n'$ , the condition (15) is certainly satisfied if inequality (5) holds. Consequently, the quasilinear equation can be represented in the form

$$\frac{\partial \bar{f}}{\partial t} = \frac{\partial}{\partial v} \bar{D} \frac{\partial \bar{f}}{\partial v},$$

where

$$\bar{D} = \lim_{L \rightarrow \infty} \frac{1}{2L} \int_{-L}^L D(x, v, t) dx$$

is the average diffusion coefficient.

It was noted above that two-stream instability leads to the production of only those quasiparticles that execute oscillations at the very bottoms of the potential wells. Therefore the expression for  $\bar{D}$  can be represented in the form

$$\bar{D} = \lim_{L \rightarrow \infty} \frac{1}{2L} \sum_i \int_{x_i - \xi}^{x_i + \xi} D(x, v, t) dx,$$

where the summation is over those wells that fall in the interval  $(-L, L)$ , and the quantity  $\xi$  is chosen from the condition  $|\xi^{(1,2)}| \ll \xi \ll a$ . To find the  $i$ -th term of this sum, it is necessary, as can be readily seen, to calculate the integral

$$I_i = \int_{-\xi}^{\xi} dy [W_i(k, y, t)|_{k=\omega_p/v}] = \int_{-\xi}^{\xi} dy W_i\left(\frac{\omega_p}{v}, y, t\right),$$

where we have introduced a new variable  $y = x - x_i$ . The index  $i$  of the function  $W$  denotes that this function corresponds to the  $i$ -th potential well. Since  $W_i = 0$  when  $|y| > \xi$ , we can replace the integration limits in this formula by infinity. Further, we can use the fact that  $W_i(\omega_p/v, y, t) \approx W_i^{(0)}[u(y, v), t]$ , where the function  $u(y, v)$  is determined by formula (11), and change over from integration with respect to  $y$  to summation over  $u$ . As the result we get

$$I_i = 2 \frac{a_i v_T v}{\sqrt{\epsilon_i}} \int_0^v \frac{W_i^{(0)}(u, t) du}{u^2 \sqrt{v^2 - u^2}}$$

i.e.,

$$\bar{D} = \frac{8\pi^2 e^2}{m^2} v_T \int_0^v \frac{\bar{W}(u, t) du}{u^2 \sqrt{v^2 - u^2}} \quad (16)$$

where

$$\bar{W} = \lim_{L \rightarrow \infty} \frac{1}{2L} \sum_i \frac{a_i W_i^{(0)}(u, t)}{\sqrt{\epsilon_i}}$$

We emphasize that the integral in formula (16) con-

verges, since the function  $W^{(0)}(u, t)$  and consequently also  $\bar{W}(u, t)$  vanishes when  $u \lesssim v_T/\sqrt{\epsilon}$ , where  $\epsilon$  is the characteristic value of the quantity  $\epsilon_i$ .

It remains still to find the equation for  $W$ . This can be done by recognizing that  $\bar{\gamma}(u, t)$  does not depend on the number of the potential well, so that

$$\frac{\partial \bar{W}(u, t)}{\partial t} = 2\bar{\gamma}(u, t) \bar{W}(u, t).$$

### 3. INVESTIGATION OF NONLINEAR EQUATIONS

We first write down the system of nonlinear equations in the compact form

$$\frac{\partial \bar{f}}{\partial t} = \frac{\partial}{\partial v} \bar{D} \frac{\partial \bar{f}}{\partial v}, \quad (17)$$

$$\bar{D} = \frac{8\pi^2 e^2}{m^2} v_T \int_0^v \frac{\bar{W}}{u^2 \sqrt{v^2 - u^2}} du, \quad (18)$$

$$\frac{\partial \bar{W}}{\partial t} = 2\bar{\gamma} \bar{W}, \quad (19)$$

$$\bar{\gamma} = \frac{1}{2} \frac{\omega_p}{\omega_p} \frac{n'}{n} u \int_u^\infty \frac{v \partial \bar{f} / \partial v}{\sqrt{v^2 - u^2}} dv. \quad (20)$$

The system (17)–(20) has an energy integral

$$\frac{d}{dt} \left[ \int_0^\infty \frac{mn'v^2}{2} \bar{f} dv + 2\pi\omega_p v_T \int_0^\infty \frac{\bar{W}}{u^3} du \right] = 0, \quad (21)$$

where the first term represents the energy of the beam particles, and the second the quasiparticle energy. It is interesting to note that the concrete form of the function  $\Delta n(x)$  does not enter into Eqs. (17)–(21), and in this sense these equations are just as general as the quasilinear equations for a homogeneous plasma. It is furthermore important that although the beam produces only quasiparticles with phase velocity  $u < v_0$ , the diffusion coefficient  $\bar{D}$  differs from zero also when  $v > v_0$ , i.e., the quasilinear relaxation proceeds both in the direction  $v < v_0$  and in the direction  $v > v_0$ . The reason for this phenomenon was explained in the Introduction.

We now change over in (17)–(21) to new variables by making the substitutions

$$\bar{f} \rightarrow \frac{\bar{f}}{v_0}, \quad \bar{W} \rightarrow \frac{n' m v_0^4}{2\pi\omega_p v_T} \bar{W}, \quad t \rightarrow \frac{n}{n' \omega_p} t, \quad v \rightarrow v_0 v, \quad u \rightarrow v_0 u,$$

where the quantities  $\bar{f}$ ,  $\bar{W}$ ,  $t$ ,  $v$ , and  $u$  on the right-hand side are already dimensionless. With the aid of such a substitution, Eqs. (17)–(21) are transformed into

$$\frac{\partial \bar{f}}{\partial t} = \frac{\partial}{\partial v} \left[ \int_0^v \frac{\bar{W} du}{u^2 \sqrt{v^2 - u^2}} \right] \frac{\partial \bar{f}}{\partial v}, \quad (22)$$

$$\frac{\partial \bar{W}}{\partial t} = \bar{W} u \int_u^\infty \frac{v \partial \bar{f} / \partial v}{\sqrt{v^2 - u^2}} dv, \quad (23)$$

$$\frac{d}{dt} \left[ \frac{1}{2} \int_0^\infty v^2 \bar{f} dv + \int_0^\infty \frac{\bar{W}}{u^3} du \right] = 0. \quad (24)$$

We shall assume that only thermal noise exists in the plasma at the initial instant of time. Since the energy of this noise is small compared with the beam energy, it follows from (24) that

$$\int_0^\infty v^2 \bar{f} dv + 2 \int_0^\infty \frac{\bar{W}}{u^3} du = 1 \quad (25)$$

(at the initial instant of time  $\int_0^\infty v^2 \bar{f} dv = 1$ ).

The solution of Eqs. (22) and (23) can be represented in an easy-to-understand form only for the initial stage

of the quasilinear relaxation, so long as the velocity spread in the beam is small compared with the initial velocity. Introducing in place of  $u$  and  $v$  the variables  $\Delta u = u - 1$  and  $\Delta v = v - 1$ , and using the condition  $|\Delta v|, |\Delta u| \ll 1$ , we can write equations (22), (23), (25) in the form

$$\frac{\partial \bar{f}}{\partial t} = \frac{1}{\sqrt{2}} \frac{\partial}{\partial \Delta v} \left[ \int_{-\infty}^{\Delta v} \frac{\bar{W} d \Delta u}{\sqrt{\Delta v - \Delta u}} \right] \frac{\partial \bar{f}}{\partial \Delta v}. \quad (26)$$

$$\frac{\partial \bar{W}}{\partial t} = \frac{1}{\sqrt{2}} \bar{W} \int_{\Delta u}^{\infty} \frac{\partial \bar{f} / \partial \Delta v}{\sqrt{\Delta v - \Delta u}} d \Delta v \quad (27)$$

$$\int_{-\infty}^{+\infty} \Delta v \bar{f} d \Delta v + \int_{-\infty}^{+\infty} \bar{W} d \Delta u = 0. \quad (28)$$

Equations (26)–(28) admit of the self-similar solution

$$\bar{f}(\Delta v, t) = g(\xi) t^{-2/3}, \quad \bar{W}(\Delta u, t) = h(\eta), \quad (29)$$

where  $\xi = \Delta v / t^{2/3}$  and  $\eta = \Delta u / t^{2/3}$ . Substituting (29) and (26)–(28), we get

$$\frac{d}{d\xi} (\xi g) = -\frac{3}{2\sqrt{2}} \frac{d}{d\xi} \left\{ \left[ \int_{-\infty}^{\xi} \frac{h d\eta}{\sqrt{\xi - \eta}} \right] \frac{dg}{d\xi} \right\}, \quad (30)$$

$$\eta \frac{dh}{d\eta} = -\frac{3}{2\sqrt{2}} \left[ \int_{\eta}^{\infty} \frac{dg/d\xi}{\sqrt{\xi - \eta}} d\xi \right] h, \quad (31)$$

$$\int_{-\infty}^{+\infty} \xi g d\xi + \int_{-\infty}^{+\infty} h d\eta = 0. \quad (32)$$

In terms of the self-similar variables, we can also write down the conditions for the conservation of the number of particles in the beam:

$$\int_{-\infty}^{+\infty} g d\xi = 1. \quad (33)$$

The boundary conditions for Eqs. (30) and (31) are of the form  $g \rightarrow 0$  as  $|\xi| \rightarrow \infty$  and  $h \rightarrow h_T$  as  $\eta \rightarrow -\infty$  (where  $h_T$  is the dimensionless energy of the thermal noise).

To solve the system (30)–(32), we used the method proposed in [9]. This method consists of using the small parameter  $\Lambda^{-1}$ , where  $\Lambda$  is the logarithmic ratio of the final energy of the quasiparticles to their initial energy. Since this ratio is very large in cases of practical interest, the quantity  $\Lambda$  is not sensitive to the final energy of the quasiparticle; roughly speaking,  $\Lambda$  is equal to the Coulomb logarithm.

In order to use the parameter  $\Lambda^{-1}$  in explicit fashion, we write down the solution of (31) in the form

$$h = h_T e^{\Gamma(\eta)},$$

where

$$\Gamma(\eta) = -\frac{3}{2\sqrt{2}} \int_{-\infty}^{\eta} \frac{d\eta'}{\eta'} \int_{\eta'}^{\infty} \frac{dg/d\xi}{\sqrt{\xi - \eta'}} d\xi.$$

The function  $h(\eta)$  reaches a maximum at the point  $\eta = \eta_0 < 0$ , which is determined from the condition

$$\Gamma'(\eta_0) = 0. \quad (34)$$

Introducing the notation  $h(\eta_0) = h_0$ , we get

$$\Gamma(\eta_0) = \ln(h_0 / h_T) \approx \Lambda \gg 1. \quad (35)$$

It is obvious that near the point  $\eta = \eta_0$

$$h(\eta) = h_0 \exp [1/2 \Gamma''(\eta_0) (\eta - \eta_0)^2],$$

with  $\Gamma''(\eta_0) < 0$ . Estimating the quantity  $\Gamma''(\eta_0)$  by means of the formula

$$|\Gamma''(\eta_0)| \sim \Gamma(\eta_0) / \eta_0^2 = \Lambda / \eta_0^2, \quad (36)$$

we find that the function  $h(\eta)$  is significantly different from zero only in a very narrow region of values of  $\eta$ :<sup>4)</sup>

$$|(\eta - \eta_0) / \eta_0| \lesssim \Lambda^{-1/2} \ll 1. \quad (37)$$

It can therefore be assumed that

$$h = A \delta(\eta - \eta_0),$$

where  $A$  is a certain constant. Using this formula, we can determine  $g$  from (30):

$$g = \begin{cases} 0, & \xi < \eta_0, \\ B \exp \left\{ -\frac{4\sqrt{2}}{15A} (\xi - \eta_0)^{3/2} \left( \xi + \frac{2}{3} \eta_0 \right) \right\}, & \xi > \eta_0 \end{cases}, \quad (38)$$

where  $B$  is an arbitrary constant.

The problem now consists of finding  $A$ ,  $B$ , and  $\eta_0$ . To this end it is necessary to use the conditions (33), (34), and (35).<sup>5)</sup> Introducing the notation

$$\frac{4\sqrt{2}}{15A} |\eta_0|^{3/2} = p, \quad B |\eta_0| = q, \quad \frac{3B}{2\sqrt{2}\Lambda |\eta_0|^{1/2}} = r,$$

we can transform the foregoing conditions into

$$q \int_{-1}^{\infty} \exp \left[ -p(z+1)^{3/2} \left( z - \frac{2}{3} \right) \right] dz = 1,$$

$$\int_{-1}^{\infty} z \exp \left[ -p(z+1)^{3/2} \left( z - \frac{2}{3} \right) \right] dz = 0,$$

$$1 = r \left\{ \pi - \frac{5}{2} \int_{-1}^{\infty} \sqrt{-z(z+1)} (\pi - 2 \operatorname{arctg} \sqrt{-\frac{z+1}{z}}) \right.$$

$$\times \exp \left[ -p(z+1)^{3/2} \left( z - \frac{2}{3} \right) \right] dz - 5 \int_0^{\infty} \sqrt{z(z+1)} \ln(\sqrt{z+1} - \sqrt{z}) \right.$$

$$\left. \times \exp \left[ -p(z+1)^{3/2} \left( z - \frac{2}{3} \right) \right] dz \right\}$$

and thus determine  $p$ ,  $q$ , and  $r$ . A simple investigation shows that this system of equations has a unique solution, with  $p$ ,  $q$ , and  $r$  positive and of the order of unity. Consequently  $A \sim \Lambda^{-5/3}$ ,  $B \sim \Lambda^{2/3}$ , and  $|\eta_0| \sim \Lambda^{-2/3}$ . The form of the function  $g(\xi)$  is illustrated qualitatively in Fig. 3.

Since  $\xi = \Delta v / t^{2/3}$ , it follows from the solution of (38) that the dimensionless beam width  $\Delta v$  increases with

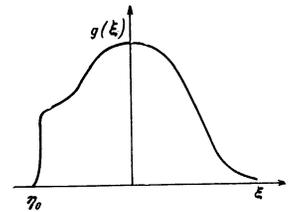


FIG. 3. The function  $g(\xi)$ .

<sup>4)</sup> A more detailed investigation, too complicated to report here, shows that actually this region is even narrower:  $|\eta - \eta_0| / \eta_0 \lesssim \Lambda^{-1}$ . The inaccuracy of formula (37) is connected with the fact that the estimate (36) is too crude.

<sup>5)</sup> As to condition (32), it is the consequence of (34) when  $\Lambda \gg 1$ .

time like  $\Delta v \sim (t/\Lambda)^{2/3}$ , and becomes of the order of unity when  $t \sim \Lambda$  ( $t$  is the dimensionless time). The dimensionless quasiparticle energy

$$\int_{-\infty}^{+\infty} \overline{W} d\Delta u = t^{1/2} \int_{-\infty}^{+\infty} h(\eta) d\eta \sim \frac{1}{\Lambda} \left( \frac{t}{\Lambda} \right)^{2/3} \quad (39)$$

also increases with time. But even at the instant  $t \sim \Lambda$ , when the beam is already smeared out by an amount on the order of unity, the quasiparticle energy remains small compared with the initial beam energy, which equals  $1/2$  in terms of the dimensionless variables. This means that the number of accelerated electrons (with  $v > v_0$ ) appearing during the process of quasilinear relaxation is approximately equal to the number of decelerated electrons (with  $v < v_0$ ). In this sense, the obtained solution differs radically from the corresponding solution in a homogeneous plasma.<sup>[9]</sup>

#### 4. REFINEMENT OF CONDITIONS FOR THE APPLICABILITY OF THE THEORY

We have assumed above that the condition for the applicability of the Liouville equations (6) is

$$a\omega_p / v_0 \gg 1,$$

i.e., the wavelength of the quasiparticle should be small compared with the characteristic dimension of the potential well. In fact, however, the applicability condition turns out to be much more stringent. The reason lies in the following. When the quasiparticle moves, its phase velocity changes in a wide range (from a certain minimal value at the bottom of the well to infinity at the turning point), so that the quasiparticle interacts with the beam electrons only in a small section  $\Delta x$  of the potential well. It follows immediately from relation (11) that in the case of a narrow beam ( $\Delta v/v_0 \ll 1$ )<sup>6)</sup>

$$\Delta x \sim a \frac{v_T}{v_0 \sqrt{\varepsilon}} \left( \frac{\Delta v}{v_0} \right)^{1/2} \ll a.$$

On the other hand, in the calculation of the quantity  $\gamma$  that enters in the right side of the Liouville equation, we have used formula (7), which is valid when the interval  $\Delta x$  spans a large number of wavelengths:  $k\Delta x \sim \omega_p \Delta x / v_0 \gg 1$ . It is therefore necessary to use in place of (1) the stronger condition

$$\frac{a\omega_p}{v_0} \frac{v_T}{v_0 \sqrt{\varepsilon}} \left( \frac{\Delta v}{v_0} \right)^{1/2} \gg 1. \quad (40)$$

For a similar reason it is necessary to replace  $a$  in the inequality (5) by  $\Delta x$ . Taking furthermore account of the fact that in the case of a narrow beam  $\gamma \sim \omega_p (n'/n) \times (v_0/\Delta v)^2$ , we obtain in place of (5) the condition

$$\frac{a\omega_p}{v_0} \frac{n'}{n} \frac{v_0}{v_T \sqrt{\varepsilon}} \left( \frac{v_0}{\Delta v} \right)^{3/2} \gg 1. \quad (41)$$

It follows from (40) and (41) that  $a$  should satisfy the inequalities

$$\frac{v_0}{\omega_p} \frac{v_0}{v_T} \left( \frac{v_0}{\Delta v} \right)^{1/2} \ll \frac{a}{\sqrt{\varepsilon}} \ll \frac{v_0}{\omega_p} \frac{n}{n'} \frac{v_T}{v_0} \left( \frac{\Delta v}{v_0} \right)^{3/2}.$$

It is clear that if we let  $\Delta v$  approach 0 in these inequalities, then these inequalities no longer hold. In

other words, the very first stage of the quasilinear relaxation cannot be described by (17) and (18). But if the conditions

$$\frac{v_0}{\omega_p} \frac{v_0}{v_T} \ll \frac{a}{\sqrt{\varepsilon}} \ll \frac{v_0}{\omega_p} \frac{n}{n'} \frac{v_T}{v_0} \quad (42)$$

are satisfied, then (17) and (18) become valid already at very small values of  $\Delta v$ :

$$\frac{\Delta v}{v_0} \gg \max \left\{ \varepsilon \left( \frac{v_0}{a\omega_p} \right)^2 \frac{v_0^2}{v_T^2}; \varepsilon^{-1/2} \left( \frac{a\omega_p}{v_T} \frac{n'}{n_0} \right)^{2/3} \right\} \ll 1$$

and describe essentially the entire relaxation process. We can therefore assume the inequalities (42) to be indeed the conditions for the applicability of our theory.

In the derivation of (17) and (18) we have assumed that the amplitude of the concentration inhomogeneities is bounded from below

$$\varepsilon = \Delta n / n_0 \gg v_T^2 / v_0^2. \quad (43)$$

It turns out that in the case when the inhomogeneities are random, it is necessary to stipulate that this quantity be bounded also from above. Indeed, if the inhomogeneities are random, then the instability increment  $\bar{\gamma}$  experiences a change from one potential well to the other, by an amount  $\Delta \bar{\gamma} \sim \Delta n / n_0 \bar{\gamma}$  (owing to the fact that the value of the concentration at the bottom of the wells change in random fashion from well to well). On the other hand, in Sec. 2 we have neglected this scatter. The latter is valid only if  $\Delta \bar{\gamma} \tau \ll 1$ . Recognizing that  $\tau \sim \Lambda / \bar{\gamma}$ , we obtain the following limitation on the ratio  $\Delta n / n_0$ :

$$\varepsilon = \Delta n / n_0 \ll \Lambda^{-1}. \quad (44)$$

Since  $\Lambda$  is not too large a quantity (on the order of 10), this condition is not too stringent. If it is still not satisfied, then the relaxation process will proceed in the manner described above, but only some of the potential wells (those with maximum increment) will contribute to the diffusion coefficient, and not all of them.

It must be emphasized that condition (44) does not hold at all in the case of periodic inhomogeneities.

An interesting feature of the solution obtained in Secs. 2 and 3 is that the quasiparticles generated by the beam execute oscillations at the very bottoms of the potential wells, in regions of width of the order of  $av_T/v_0\sqrt{\varepsilon} \ll a$ . The energy transferred from the beam to the quasiparticles is localized precisely in these regions, and its exchange density  $U$  can be estimated from the formula

$$U \sim \frac{mn'v_0^2 v_0 \sqrt{\varepsilon}}{\Lambda v_T}$$

(see (39)). Since  $U$  can greatly exceed the beam energy density  $mn'v_0^2/2$ , it is of interest to estimate the characteristic time  $\tau_{nl}$  of the nonlinear processes. Using the results of [10-12], it is easy to show that

$$\frac{\tau_{nl}}{\tau} \sim \left( \frac{v_0}{v_T} \right)^3 \frac{mn'v_0^2}{\Lambda U} \sim \left( \frac{v_0}{v_T} \right)^2 \frac{1}{\sqrt{\varepsilon}} \gg 1,$$

i.e., the influence of the linear processes can be neglected. It can thus be assumed that the theory proposed in this paper is valid if inequalities (42), (43), and (44) are satisfied. The last inequality pertains only to a plasma with random inhomogeneities.

<sup>6)</sup>For clarity, we transform here to dimensional variables.

5. DISCUSSION OF RESULTS

Let us examine the extent to which it is realistic to expect satisfaction of the conditions (42)–(44) in concrete experiments. In experiments described in [13] there were observed in the “Tokamak” streams of runaway electrons with energy  $\mathcal{E}$  on the order of  $10^5$  eV and relative concentration  $n'/n_0$  on the order of  $10^{-5}$ – $10^{-6}$ . The plasma electron temperature  $T$  was  $10^2$  eV, and the electron density was  $10^{13}$  cm $^{-3}$ . The inequalities (42)–(44) then take the form

$$4 \text{ cm}, < a/\gamma\epsilon < 4(10^2 - 10^3) \text{ cm}, 10^{-3} < \epsilon < 10^{-1}.$$

Assuming somewhat arbitrarily  $\epsilon \sim 10^{-2}$ , we find that the characteristic scale of the inhomogeneities should lie in the range between 0.4 and  $4 \times (10-100)$  cm. This seems quite realistic. In the experiments of the type described in [3], dealing with beam injection in a plasma, the following values of the beam and plasma parameters are standard:  $\mathcal{E} \sim 10^3$  eV,  $n'/n_0 \sim 10^{-3}$ ,  $n_0 \sim 10^{12}$  cm $^{-3}$ , and  $T \sim 10$  eV, i.e., the inequalities (42)–(44) take the form

$$0.4 \text{ cm} < a/\gamma\bar{\epsilon} < 4 \text{ cm}, 10^{-2} < \epsilon < 10^{-1}.$$

These conditions are also quite realistic.

However, one remark is called for here. For a theoretical description of the experiments of the type described in [3] it is necessary to consider the stationary boundary-value problem involving the injection of a beam into a plasma half-space. In the case of a homogeneous plasma, the stationary character of the solution is ensured by the fact that the generation of the quasiparticles in the case of two-stream instability is compensated by the fact that the quasiparticles drift away in the beam direction. [14] On the other hand, if the plasma is inhomogeneous and inequality (43) is satisfied, then the quasiparticles are “trapped” in the potential wells and the instability is not offset by the drift of the quasiparticles.<sup>7)</sup> Therefore there are no stationary quasilinear solutions in the boundary-value problem in the presence of concentration inhomogeneities satisfying the condition (43).

On the other hand, if the plasma density changes monotonically with the coordinate, then stationary solutions are possible, since the quasiparticles produced by the beam are not trapped in the volume of the plasma (Fig. 4). In the case when the plasma concentration increases in the direction of beam motion (Fig. 4a), a quasiparticle produced, say, at the point  $x_0$  experiences a reflection at a certain point  $x_1$  and subsequently goes off to the wall or is absorbed by the plasma electrons. When the quasiparticle moves from  $x_0$  to  $x_1$ , its phase velocity increases to infinity. The same pertains also to quasiparticles produced at other points. Therefore the quasilinear relaxation must be accompanied by the appearance of accelerated electrons. On the other hand, if the plasma density decreases in the direction of the beam motion (Fig. 4b), then the phase velocity of the produced quasiparticles decreases, and no accelerated electrons appear.

The dependence of the appearance of accelerated

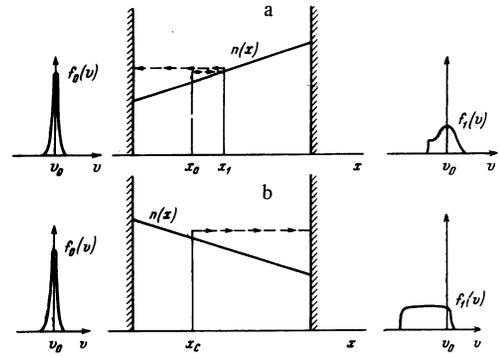


FIG. 4. Quasilinear relaxation of electron beam in a plasma with regularly varying concentration  $n(x)$ . The cross hatching denotes the boundaries of the plasma. The beam is injected into the plasma from the left. The dashed lines show schematically the character of motion of the quasiparticles. On the left and on the right are shown the distribution function of the beam electrons on entering the plasma ( $f_0(v)$ ) and on leaving the plasma ( $f_1(v)$ ).

electrons on the sign of the concentration gradient can be used for an experimental verification of the concepts developed in this paper.

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Translated by J. G. Adashko

<sup>7)</sup>This pertains in particular to potential wells near the plasma boundary, where the beam is not yet spread out in energy and the increment is maximal.