EQUATION OF STATE OF NEUTRAL MATTER AND FLUCTUATIONS

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Submitted December 4, 1968

Zh. Eksp. Teor. Fiz. 56, 1968–1975 (June, 1969)

The equation of state of truly neutral matter consisting of neutral particles and equal amounts of charged particles and antiparticles is considered. The pressure may exceed one-third the energy density in the presence of particle and antiparticle repulsion; however, at high temperatures, it approaches this value asymptotically, owing to the dominating contribution of weakly interacting particles. The density fluctuations are characterized by a universal dependence on the specific entropy for any equation of state. The fluctuations of the electric charge correspond to the equipartition principle.

NEUTRAL matter (NM) is defined by the condition that all strictly conserved charges—baryons and leptons in addition to electrons—are equal to zero; thus NM is the excited state of the vacuum. An example of NM is the equilibrium electromagnetic radiation. In the early stages of evolution of the universe, matter was evidently neutral with high accuracy.

The thermodynamic properties of NM, consisting of weakly interacting particles, are easily calculated. The energy density ϵ is expressed in terms of the temperature T by formulas of the form $\epsilon = \sigma T^4$ for radiation and for neutrinos and by corresponding expressions for electron-positron $e^\pm$ pairs (we shall always give the temperature in energy units). However, at sufficiently high temperatures, hadrons pions, kaons, baryons and antibaryons, etc. should take part in the equilibrium. Their interaction cannot be regarded as small; we are a long way from the establishment of a theory of interaction in a dense medium of hadrons.

The general thermodynamic relations, as is well known, furnish one relation[1)] that is specific for NM:

$$p = -\varepsilon + Ts = -\varepsilon + s\frac{d\varepsilon}{ds}, \tag{1}$$

where p is the pressure and s the entropy density (referred to unit volume). We assume that as the result of the interaction of the particles and of the fact that the particles possess a rest mass, the interpolation formula $\epsilon = bT^k$ holds over any range of temperatures (even if this is bounded and small), so that $k = d\ln\epsilon/d\ln T$; it follows from Eq. (1) that

$$s = \frac{k}{k-1}bT^{k-1}, \quad p = \frac{1}{k-1}\varepsilon. \tag{2}$$

in this range of temperatures (see[1,2]). Here, weakly interacting particles with a rest mass of the order of $m \sim T/c^2$ make a contribution with an effective exponent $k' > 4$.

The conclusion can be drawn directly that in thermodynamic equilibrium, in which weakly interacting particles (for which $k' \geq 4$) take part, there cannot be a total $k < 4$ asymptotically; if strongly interacting particles give $k' < 4$, then the asymptote at high temperatures will be determined not by these particles, but by the weakly-interacting ones, the contribution of which becomes dominant at high temperatures, thanks to a higher exponent of the temperature.[2)]

We recall that the asymptote $p = \epsilon$ was obtained previously for cold matter in the limit of high baryon charge density, under the assumption that there exists a vector interaction that leads to mutual repulsion of the baryons, but differs by a finite radius of action from the electrostatic interaction.[4] The energy density of the interaction in this case depends on the density of the baryons as $\epsilon_i = an^2/2$; the other energy components, namely the rest mass nmc^2 and the Fermi energy $\sim n^{4/3}$, are smaller than the energy of interaction in the limit as $n \to \infty$. From $\epsilon = an^2/2$ it follows that

$$p = -\frac{dE}{dV} = -\frac{d(\varepsilon/n)}{d(1/n)} = \varepsilon. \tag{3}$$

In the theory of quantized fields, which is necessary for the consideration of antiparticles, the vector interaction leads to the result that the particle and the antiparticle are attracted to each other. Transferring verbatim the assumptions of reference[4] to NM, we find that there is no interaction. However, we assume that there is some other interaction which adds the term $aq^2/2$ to the energy density. Here q is the sum of the densities of the baryons and the antibaryons. We now make the calculation of the equation of state under two assumptions: 1) considering only the baryons and the antibaryons, 2) considering them in equilibrium with the weakly interacting particles. For simplicity, we neglect the rest mass of the baryons, assuming that the effects of interaction are essentially established at temperatures above mc^2. We also neglect the differences between Bose, Fermi, and classical statistics. In fact, in the expression for the energy density for rest mass and chemical potential equal to zero, we get in the three cases

$$\int (e^x-1)^{-1}x^3dx = \pi^4/15 = 6{,}4, \quad \int (e^x+1)^{-1}x^3dx = 7\pi^4/120 = 5.6,$$

[1)]This relation follows from $E = \mu N - pV + TsV$ with account of the fact that for NM, $\mu = 0$, $N \equiv O$), inasmuch as one must understand by N the difference in the number of particles and antiparticles.

[2)]This consideration is not applicable to a situation for collisions of particles of high energy in cosmic rays, where equilibrium with weakly interacting particles has not been established in the general case. [3]

$$\int e^{-x}x^3dx = 6,$$

which differ from one another by less than 10%.

We now construct an expression for the NM energy density, for NM consisting of relativistic baryons and antibaryons, characterizing the material by two parameters: q—the total density, Θ—one-third of the mean energy of a single particle. Correspondingly, the volume in the momentum space occupied by particles, and the density of particles in phase space, and consequently also, the entropy, are expressed in terms of Θ. As a result of calculations, it will be shown that Θ is the temperature but at the beginning, we shall consider Θ as a parameter and for this purpose we introduce a symbol other than T.

The desired expressions for ϵ and s have the form

$$\varepsilon = \frac{aq^2}{2} + 3\Theta q; \quad s = q\ln(re^4\Theta^3/q), \tag{4}$$

where r is a constant proportional to the statistical weight g of the baryons, $r = 8\pi g/(2\pi\hbar c)^3$ (as one can show by successive comparison of the results with known formulas for a = 0); the factor e^4 is separated from r for convenience of description of the formulas that follow below. We now consider Θ and q as two parameters connected by the condition of minimum ϵ for a given s or maximum s for a given ϵ. This condition has the form of a Jacobian $\partial(\epsilon, s)/\partial(q, \Theta) = 0$ and allows us to connect q and Θ with each other, reducing the number of independent parameters to one. After this we can in principle express the remaining parameter in terms of s and then get $\epsilon(s)$, which gives the complete description of the thermodynamic properties of the material. In practice, the Jacobian yields the transcendental equation

$$q = r\Theta^3 e^{-aq/\Theta} = \frac{g\Theta^3}{\pi^2\hbar^3c^3}e^{-aq/\Theta} \tag{5}$$

with an explicit physical meaning (aq is the potential energy of a single baryon or antibaryon in space with given baryon (antibaryon density q).

All the calculations can be done exactly if we introduce the parameter $\gamma = aq/\Theta$ and express all the quantities in terms of this parameter. In elementary fashion, we get

$$\Theta = e^{\gamma/2}\sqrt{\gamma/ra}, \quad q = e^{\gamma/2}\sqrt{\gamma^3/ra^3}, \quad s = e^{\gamma/2}\sqrt{\gamma^3/ra^3}(\gamma+4),$$

$$\varepsilon = e^{\gamma}\left(\frac{\gamma}{2}+3\right)\gamma^2/ra^2, \quad T = \frac{d\varepsilon}{ds} = \frac{d\varepsilon}{d\gamma}\Big/\frac{ds}{d\gamma} \equiv \Theta,$$

$$p = -\varepsilon + Ts = e^{\gamma}\left(\frac{\gamma}{2}+1\right)\gamma^2/ra^2, \quad \frac{p}{\varepsilon} = \frac{\gamma+2}{\gamma+6}. \tag{6}$$

Thus, as expected, the effective exponent of $d\ln\epsilon/d\ln T$ changes from four to two, and the ratio of the pressure to the energy density changes from $1/3$ to unity for change in temperature from 0 to ∞.

We now proceed to the consideration of the second assumption. We assume that we have in equilibrium with the baryons and the antibaryons, weakly interacting particles with zero rest mass and with a statistical weight f (carrying out the summation over all types of particles). For them, in the expression of classical statistics (the index w means weak)

$$p_w = \frac{\varepsilon_w}{3}, \quad \varepsilon_w = \frac{3f}{\pi^2\hbar^3c^3}T^4. \tag{7}$$

For the ratio of the total pressure p_t to the total energy density ϵ_t, we get

$$z = p_t/\varepsilon_t = \frac{p+p_w}{\varepsilon+\varepsilon_w} = \frac{1}{3} + \frac{2}{3}\,\frac{\gamma}{\gamma+6+6e^{\gamma}(f/g)}. \tag{8}$$

This ratio z is equal to $1/3$ both for T = 0 and for T = ∞. A maximum is achieved in the middle; thus, for example, at g = f the maximum z is achieved for $\gamma = 1.28$, $T = 2.14/\sqrt{ra}$ and amounts to $z_{max} = 0.39$.

The example given above illustrates the general situation; in neutral matter, the excess of $z = p_t/\epsilon_t$ over $1/3$ is not large and exists in a region of temperatures bounded both above and below even in the presence of repulsion of one type of particle or antiparticle.[3)]

On the other hand, account of the rest mass of the particles lowers the ratio p_t/ϵ_t in NM. As an example, we can cite the calculations of[6] which pertain to photons and $e^\pm$ pairs in equilibrium (minimum z is equal to $\sim 1/5$). This minimum is achieved for $T/mc^2 \approx 1/3$. Naturally, $z \sim 1/3$ for further increase in the temperature $T \to \infty$.

Recently, in connection with the discovery of families of "resonances" of strongly interacting particles (baryons and mesons), the question of the limiting equation of state for a mass spectrum that is unbounded from above has been posed.[7] It is easy to obtain asymptotic formulas by replacing the discrete rest masses with a continuous distribution with spectral density u(m) and mean statistical weight g(m). The spectral density y is defined so that the number of different types of particles with different angular momentum, charge, strangeness in a small range of masses from m_1 to m_2 is equal in the mean to $(m_2 - m_1)y(\sqrt{m_1m_2})$. Below, following the suggestion made by Kompaneets in a discussion of the work, we obtain

$$\varepsilon = \frac{\sigma}{2}T^4\int\psi\left(\frac{T}{mc^2}\right)y(m)g(m)dm, \quad \psi(x) = \int_0^\infty \frac{z^2\sqrt{z^2+1}\,dz}{\exp\{x^{-1}\sqrt{z^2+1}\}\pm 1}, \tag{9}$$

the constant σ being the same as in the formula for radiation.

Thus, for example, if the number of Regge trajectories is bounded and equal to R, and these trajectories are straight lines in the J-m^2 plane (angular momentum—mass squared), then $g(m) = 2J+1 \sim m^2$, $y(m) = R(dm/dJ)^{-1} \sim m$, so that

$$\varepsilon = \text{const}\cdot T^4\int\psi(T/mc^2)m^3dm = \text{const}\cdot T^8 \quad p = \varepsilon/7. \tag{10}$$

A qualitatively new situation arises only in the case of an exponential growth $g(m)y(m) \sim e^{mc^2/\tau}$, where τ is a constant with the dimensions of energy; in this case, we get

$$\varepsilon \sim T^5/(\tau - T) \to \infty, \quad p/\varepsilon \sim \frac{\tau - T}{\tau}\ln\frac{\tau}{\tau - T} \to 0 \text{ as } T \to \tau \tag{11}$$

and an upper temperature limit exists, equal to τ. However, these calculations are rather naive, since, at high density of the different particles, the interaction among them has not been taken into account. If in reality the strongly interacting particles are "compound," for example as in quark models, then this also means that it is impossible to use ex-

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3)For this reason, it is uncertain whether real cosmological solutions with a limiting fixed equation of state is applicable to a hot universe. [5]

pressions of the type given above (see the similar considerations applicable to cold matter[8]). The only more or less reliable conclusion is that in NM one must expect that $p \leq \epsilon/3$ asymptotically, although it is not excluded that this inequality is not violated in some temperature region.

In conclusion, we note that the problem of the contribution of weak interaction to the equation of state remains unresolved to date. The standard Hamiltonian of universal weak interaction gives a contribution to the scattering of neutrinos by electrons: the corresponding term can, according to Fierz, be represented as

$$G(\psi_e O\bar{\psi}_\nu)(\psi_\nu O\bar{\psi}_e) \rightarrow G(\psi_e O'\bar{\psi}_e)(\psi_\nu O'\bar{\psi}_\nu). \tag{12}$$

The contribution to the energy density can be written

$$\varepsilon_{iw} = G(n_{e^-} - n_{e^+})(n_\nu - n_{\bar{\nu}}). \tag{13}$$

We note that in this form it is more convenient to consider the coherent scattering of neutrinos by the electrons of a macroscopic body: the potential energy of a single neutrino located inside a body is equal to

$$\Delta E = Gn_{e^-} = 6\cdot 10^{23} G\rho Z/A \sim 10^{-25}\rho Z/A \,\mathrm{erg} \sim 10^{-13}\rho Z/A\,\mathrm{eV}, \tag{14}$$

where ρ is the density of the body, Z the nuclear charge, and A the atomic weight. It is then easy to find the refractive index ξ for the transition of a neutrino from a vacuum to the volume of the body, $\xi = 1 = \Delta E/E$ (private communication from V. B. Belyaev and B. N. Zakhar'ev).

In NM, $\epsilon_{iw} \equiv 0$, obviously. For a leptonic charge different from zero, $n_\nu - n_{\bar{\nu}} \neq 0$. At high temperature, and at a leptonic charge different from zero, $n_e^- - n_e^+ \neq 0$ also, while the electric charge can be compensated by muons or pions. If we take into account only the energy of free particles $(\sim n^{4/3})$ and an interaction of first order in $G(\sim n^2)$, then it is seen that one can always construct a state with negative energy for any sign of G and arbitrary given mean density of the leptonic charge; in particular, this holds for zero charge (as a consequence of local inhomogeneity. However, ϵ_{iw} becomes of the order of the energy of free particles precisely for such a density for which, as is seen from the dimensionality, terms of the type $G^2n^{8/3}$, $G^3n^{10/3}$, G^4n^4, ... r each the same order of magnitude. As when the "weak" interaction becomes "strong," we cannot compute anything with certainty, just as we cannot compute the "strong" interaction of hadrons at high hadron density.

Gravitational interaction is long-range action and should be considered separately; it is not clear how one can distinguish the gravitational interaction of a separate pair of particles against this background. Therefore, we shall not consider the contribution of this interaction to the equation of state, limiting ourselves to those densities $\leq 10^{95}$ g/cm^3. Attempts to advance farther are discussed in[9].

We return to the problem of equilibrium fluctuations of the NM density. For noninteracting particles, $\delta N \sim \sqrt{N} = \sqrt{nV}$, where N is the mean number of particles in the given volume and δN is the mean fluctuation of this number; more precisely, $\delta N = \sqrt{\langle(\bar{N} - N)^2\rangle}$. Inasmuch as $s = 4n$ with excellent accuracy, it follows that on going to macroscopic quantities we can write, using the dimensionless entropy s,

$$(\delta\rho/\rho)_V \approx \frac{2}{\sqrt{Vs}} = \frac{2}{\sqrt{S}}, \tag{15}$$

where S is the total entropy of the considered volume V.

It is curious that a similar expression holds for any power law for the dependence of ϵ on T or on s, and not only for $\epsilon = \mathrm{const}\cdot T^4 = \mathrm{const}\cdot s^{4/3}$, which is characteristic of noninteracting particles. In fact, the amplitude of long longitudinal sound waves in NM is described by the Rayleigh-Jeans approximation:

$$E_\varkappa = \frac{1}{2}\rho u_\varkappa^2 = \frac{1}{2}\varepsilon\frac{u_\varkappa^2}{c^2} = T, \quad \frac{u_\varkappa}{c} = \sqrt{\frac{2T}{\varepsilon}}, \quad \varkappa \rightarrow 0, \tag{16}$$

where E_κ, u_κ are the spectral energy density and the mass velocity, κ the sound-wave vector, and c the speed of light. The amplitude of oscillations of the particle number density (or entropy), the amplitude of the mass density and the amplitude of the velocity are connected (the index κ denotes the amplitude of the Fourier expansion with wave vector κ):

$$\left(\frac{\delta n}{n}\right)_\varkappa = \left(\frac{\delta s}{s}\right)_\varkappa = \frac{k-1}{k}\left(\frac{\delta\rho}{\rho}\right)_\varkappa = \frac{u_\varkappa}{c'}, \tag{17}$$

where c' is the sound velocity, $c' = \sqrt{dp/d\rho} = c\sqrt{1/(k-1)}$, with k the exponent in the law $\epsilon = aT^k$. We then obtain

$$\left(\frac{\delta\rho}{\rho}\right)_\varkappa = \sqrt{\frac{2kT}{(k-1)\varepsilon}}. \tag{18}$$

Taking it into account that $s = k\epsilon/(k-1)T$, we get, transforming from the Fourier amplitude to the average over the volume

$$\left(\frac{\delta\rho}{\rho}\right)_V = \sqrt{V}\frac{k}{k-1}\sqrt{2}\sqrt{s} \cong \sqrt{sV} \cong \sqrt{S}. \tag{19}$$

For $k = 4$, the coefficient $\sqrt{2}k/(k-1) = 1.89$ does not differ much from the coefficient 2 obtained in elementary fashion for non-independent particles.

In NM, one can also consider the fluctuations of the charge density, since the material is neutral only in the mean. For leptonic charge, in the approximation of noninteracting particles (we recall that n has the dimensions cm^{-3}),

$$\delta(n_\nu - n_{\bar{\nu}})_V = \sqrt{\frac{n_\nu + n_{\bar{\nu}}}{V}}. \tag{20}$$

For the electric charge, the long-range action essentially decreases the fluctuations in the volume which is large in comparison with a cube of the Debye radius D. The Vlasov dispersion law[10] $\omega = \sqrt{\omega_0^2 + \kappa^2 c'^2}$, where ω_0 is the plasma frequency, leads, at long wavelengths, to the result

$$\left.\frac{\delta(n_+ - n_-)}{n_+ + n_-}\right|_\varkappa = \frac{\varkappa u_\varkappa}{\omega} \approx \frac{\varkappa u_\varkappa}{\omega_0} \quad \text{for } \varkappa c' \ll \omega_0. \tag{21}$$

If the spectral density of the amplitude for long wavelengths, i.e., for small wave vectors κ, is not constant, and depends on some power of κ, $\delta_\kappa \sim \kappa^x$, then the dependence of the quadratic deviation from the considered volume V also turns out to be different, not proportional to $1/\sqrt{V}$.[4)] To find $\overline{\delta^2}$, it is necessary to compute the integral

4) When $x \geq 1$, it is necessary to define properly the method of averaging over the volume (see [11,12]).

$$\delta^2|_V \sim \int_0^{V^{-1/3}} (\delta_\varkappa)^2 \varkappa^2 \, d\varkappa = \int_0^{V^{-1/3}} \varkappa^{2x} \varkappa^2 \, d\varkappa = V^{(-1-2x/3)}. \qquad (22)$$

In the case of fluctuations of the electric change, we have the exponent $x = 1$ and get (collecting the remaining factors, e_0 is the electron charge)

$$\delta (n_+ - n_-)|_V \approx V^{-5/6} \sqrt{T / e_0^2}. \qquad (23)$$

The illustrative meaning of the result is made clear if we find the charge $\overline{Z}$ and the electrostatic energy U of the volume V; the latter turns out to be independent of the volume V and of the electron charge e_0:

$$\overline{Z} = Ve_0\delta(n_+ - n_-) = V^{-1/6}\sqrt{T}, \quad \overline{\varphi} = \frac{\overline{Z}}{R} = \overline{Z}V^{-1/3}, \quad \overline{U} = \overline{Z}\overline{\varphi} = T. \quad (24)$$

It can be shown that the volume charge is one normal coordinate that satisfies the statistical law of energy distribution.

Finally, the interaction is certainly short-range for the baryon charge. Such an interaction does not change the asymptotic law ($V^{-1/2}$) of fluctuations in large volumes.

It must be recalled that for an arbitrary initial distribution of the total NM density and of the charges, the time of establishment of statistical equilibrium distributions of the fluctuations of the different types is significantly different. For density fluctuations, this time is equal to the damping time of long acoustic waves; for fluctuations of leptonic and baryonic charges, it is equal to the diffusion time of the corresponding particles, for the electric charge, the time is determined by the conductivity of the material. In principle, to consider all possible types of deviation from equilibrium, one would have to consider the damping of transverse waves, or, what amount to the same thing, vortical motions and the damping of the electric current and the magnetic field associated with it. However, these questions go far beyond the framework of the predominantly thermodynamic consideration of the properties of NM in the given paper.

We turn in conclusion to the problem of the neutrality of the matter of the universe. It is known that the baryonic charge (in a charged, nonsymmetric, homogeneous model of the universe) does not exceed 10^{-9}–10^{-8} per particle and, consequently, can in reality be considered small. In fact, the mean density of protons in the universe lies between 2×10^{-5} and 10^{-6} cm^{-3}, the density of the relict quanta at 2.7° is equal to 400 cm^{-3}.

A small baryon impurity ($\sim 10^{-9}$) in the presence of a vector field, which brings about repulsion, could lead to a dependence $p = \epsilon$, but only at very high density. We assume that for baryons $\epsilon = an^2 + mc^2 n$ with the constant a such that the two terms are equal at $\rho = 10^{16}$ g/cm^3 and $n = 3 \times 10^{39}$ cm^{-3}. By assuming $p = \epsilon/3$ for the remaining particles, we find that the transition to the law $p = \epsilon$ takes place at $n_t = 10^{27} \times 3 \times 10^{39}$, $n = 10^{18} \times 10^{39}$, and $\rho = 10^{36} \times 10^{16} = 10^{52}$ g/cm^3, where n_t is the total density of all sorts of particles, in contrast with n, which is the density of the baryons (more accurately, n is the density of the baryon charge), with $n_t \sim 10^9 n$.

The density of electric charge is undoubtedly equal to zero (a nonzero density is incompatible with a homogeneous and isotropic universe; see, for example,[13] for a closed world, and[11,12] for an open world).

Finally, we turn to the leptonic charges and assume that there exist two different, strictly conserved quantities q_e and q_μ. To describe the universe, we refer these quantities to the density of the quanta, $q_e = (e^- + \nu_e - e^+ - \overline{\nu}_e)/\gamma$. It is known that the presence of $|q_e|$ of the order of several units would have changed the composition of the initial matter materially.[14] For $q_e > 0$, the equilibrium between neutrons and protons shifts in the direction of the neutrons, which increases the constant of He^4 in prestellar matter by more than 30%. For $q_e < 0$, the expected helium content is reduced.

The situation with regard to the similar quantity q_μ, the muon charge, is more complicated. After the disappearance of the charged muons during the course of cooling, a nonzero muon charge leads only to the appearance of a surplus of degenerate muonic neutrinos or antineutrinos. If $|q_\mu| > 10^6$, the disappearance of μ is delayed over the period of nuclear reactions and directly affects the ratio e^+/e^- and thereby the reaction of mutual conversion of neutrons and protons. However, such large values are certainly excluded. Even for $|q_\mu| > 10^4$, the density of muon neutrinos would have been significantly larger than the density of ordinary matter at the present time: such $|q_\mu| > 10^4$ are excluded by considerations of the gravitational effect of neutrinos on expansion in our epoch and on the growth of the universe.[15]

Finally, according to a remark by Shvartsman, any particles influencing the rate of expansion in the period of nuclear reactions change the composition of prestellar matter such as to increase in He^4 content. This imposes the limit $|q_\mu| \lesssim 20$. However, one possibility of compensating $|q_\mu| \neq 0$ remains, that of taking $q_e < 0$. In this case, one can obtain any given He^4 content, including 30% of He^4, even if $|q_\mu| > 20$. Such a compensation is very artificial; nevertheless, it cannot be rejected merely on the basis of esthetic considerations. Direct measurements or even specific arguments refuting the hypothesis of compensation of q_μ or other particles at the expense of q_e are highly desirable.

I take this opportunity to thank B. L. Ioffe, A. S. Kompaneets, L. B. Okun', A. D. Sakharov, E. L. Feĭnberg and V. F. Shvartsman for discussion and advice.

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Translated by R. T. Byer
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