DOPPLER BROADENING OF SPECTRAL LINES OF IONS IN STOCHASTIC VARYING FIELDS IN A PLASMA

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Doppler broadening of spectral lines of ions in stochastic varying fields in a plasma is analyzed theoretically. Relations are derived which define the shape of the spectral lines in those cases when displacement of the ions is due to drift in crossed electric fields, appearing on development of instability, and in a constant magnetic field. Expressions are also derived in the more general case when ion motion can be described by the Langevin equation, and the fields acting on the ions are electric fields with a finite correlation time between their amplitudes.

1. THE question of the Doppler broadening of ionic lines in a plasma, due to motion in constant electric fields and to thermal motion, was investigated by Frish and Kagan,[1] and the line shape of the spectral lines was determined for this case by Sena,[2], Fock,[3] and Kagan and Perel'.[4] Since in a non-equilibrium turbulent plasma there are excited intense stochastic oscillations and waves whose field intensities are quite appreciable, we can expect the velocities acquired by the ions in such fields to be large, and the shape of the spectral line to be determined by the motion of the ions in the stochastic alternating electric fields. This raises the question of determining under these conditions the contour of the spectral line.

In the case of steady-state stochastic fields, this problem is quite analogous in its method of solution to the problem of determining the shape of the spectral line of an atom executing Brownian motion in a dense gaseous medium. This question was investigated by Podgoretskiĭ and Stepanov[5] and by Dicke.[6] Naturally, the motion of ions in random alternating fields differs greatly from Brownian motion of an atom in a gaseous medium, and the parameters characterizing the contour of the spectral line should differ from those obtained in [5, 6]. The purpose of the present paper is to determine the contour of the spectral lines of plasma ions moving in stochastic alternating fields.

The energy acquired by the ions in high-frequency fields is much lower than in low-frequency fields, but taking into account the fact that the intensities of the high-frequency fields are in many cases much higher than the intensities of the low-frequency fields, it is necessary also to estimate the influence of the ion acceleration due to the high-frequency waves. It should be noted that measurements of the contours of the spectral lines of ions are important not only because they make it possible to determine the temperature or the ordered velocity of the ions, but also because their measurements apparently can determine the intensities of the fields of the high-frequency and low-frequency oscillations, the important parameter $n_{pl}kT$ (n_{pl}—plasma density), and also the coefficient of anomalous diffusion due to the low-frequency fields of the instabilities. The problem of determining the energy of fast ions by spectral methods is made difficult in many cases by the fact that the fast ions constitute a small fraction of the plasma. We note that this problem is analogous in many respects to the problem of the Stark effect in stochastically alternating fields.[7] 1)

As is well known,[11] the simplest method of determining the contour of a spectral line is the method based on the use of the correlation function $K(\tau)$, with which the radiation density $I(\omega)$ is connected by the relation[12]

$$I(\omega)=\frac{1}{\pi}\operatorname{Re}\int_0^\infty e^{i(\omega_0-\omega)\tau}K(\tau)\,d\tau. \tag{1}$$

In the case of the Doppler effect $K(\tau)$ is equal to

$$K(\tau)=\langle\exp\{ik_0[x(t+\tau)-x(t)]\}\rangle. \tag{2}$$

Here ω_0 is the unperturbed frequency of the line and k_0 is the corresponding wave number.

In the case considered by us, the ion displacements $x(t)$ are due to the action of alternating stochastic electric fields:

$$E(x,t)=\sum_k E_k\exp[i(kv-\omega_k)t+ikx_0],$$

where the phases $\varphi_k = kx_0$ are random quantities satisfying the condition $\langle\exp\{i(k+k')x_0\}\rangle = \delta_{k,-k'}$. We assume here that in a steady-state turbulence the displacement of the ions in the electric field, excited upon development of the instability of the waves, is small ($k\Delta x \ll 1$), and therefore the fields acting on the ions depend only on the time. For φ_k there is a uniform distribution law, and the spectrum of the wave numbers k is quite broad.

When the number of harmonics k is large, the displacement x obeys a Gaussian distribution. A simple method of calculating mean values of the type (2) has been used in [5]. Expanding the exponential (2) in powers of $k_0\Delta x$ and noting that the odd powers vanish in this expansion, and $\langle\Delta^{2n}x\rangle = (2n-1)!!\langle x^2\rangle$ for the even

1)As is well known, the question of the Stark broadening of lines in a high-temperature plasma in the absence of high-frequency stochastic fields was considered by Kogan [8] and by Kolb, Griem, and Chen [9], and nonstationary processes of pair collisions were considered by Vaĭnshteĭn and Sobel'man [10].

powers, the authors find that

$$\langle \exp[ik_0(x(t+\tau)-x(t))]\rangle = \exp[-{}^1\!/_2 k_0^2 \langle (x(t+\tau)-x(t))^2\rangle]. \tag{3}$$

Thus, calculations of the correlation function reduces to a determination of the mean-squared displacement $\langle \Delta x^2 \rangle$.

2. Let us calculate the mean-square ion displacement due to the drift in crossed electric fields of low-frequency waves excited during the development of instabilities, and in a constant magnetic field. Since in this case the ion velocity is

$$v = \frac{c}{H_0}\sum_k E_k \exp[i(k_z v_0 - \omega_k)t + i\varphi_k]; \tag{4}$$

we get

$$x(t+\tau)-x(t) = \frac{c}{H_0}\sum_k E_k \frac{\exp\{i(k_z v_0-\omega_k)\tau + i\hat{\varphi}_k\} - \exp\{i\hat{\varphi}_k\}}{i(k_z v_0 - \omega_k)}, \tag{5}$$

where $\hat{\varphi}_k = k_z z_0(t) - \omega_k t$. Using (5) and performing the corresponding calculations, assuming the random phases to be equally probable, we obtain

$$\langle [x(t+\tau)-x(t)]^2\rangle = \frac{2c^2}{H_0^2}\sum_k |E_k|^2 \frac{1-\cos\alpha_k\tau}{\alpha_k^2}, \quad \alpha_k = k_z v_0 - \omega_k. \tag{6}$$

Using the formula for the mean-square displacement (6), we calculate with the aid of relations (3) and (1) the ion spectral-line shape determined by the Doppler effect in stochastic alternating fields. It should be noted that mathematically this problem is analogous to the problem of determining the spectrum of oscillations with a fluctuating frequency, considered in a large number of papers (see, for example, [13, 14]) and most thoroughly by Rytov.[12]

Let us consider first the displacement of thermal particles $k_z v_0 \to 0$. For small τ ($\tau \ll 1/\bar{\Omega}$, where $\bar{\Omega}$ is the average frequency in the spectrum) we then obtain

$$\langle [x(t+\tau)-x(t)]^2\rangle = \frac{c^2}{H_0^2}\tau^2 \sum_k |E_k|^2. \tag{7}$$

For large τ ($\tau \gg 1/\Omega$) we get

$$\langle [x(t+\tau)-x(t)]^2\rangle = \frac{2c^2}{\pi H_0^2}\int \frac{d\alpha}{|d\omega/dk_z|}|E_k|^2 \frac{1-\cos\alpha\tau}{\alpha^2}$$

$$= \frac{2c^2}{\pi H_0^2}\tau \int_{-\infty}^{\infty} dx' \left(\frac{|E_k|^2}{|d\omega/dk_z|}\right)\Bigg|_{\omega=-x'/\tau} \frac{1-\cos x'}{x'^2}.$$

Hence

$$\langle \Delta x^2\rangle = \frac{c^2}{H_0^2}\tau\left(\frac{|E_k|^2}{|d\omega/dk_z|}\right)\Bigg|_{\omega=-x'/\tau} = \frac{c^2}{H_0^2}\tau |E_\omega|^2\Big|_{\omega=0}. \tag{8}$$

Great interest attaches to the calculation of $\langle \Delta x^2\rangle$ for resonant ions, since the Doppler shift for them is maximal. For these particles $k_z v_0 \approx \omega_k$ and $\alpha_k \to 0$. Calculations similar to those given above, together with the relation

$$\lim_{\tau\to\infty}\frac{1}{\tau}\int_{-\infty}^{\infty} f(x)\frac{1-\cos x\tau}{x^2}dx = \pi f(0), \tag{9}$$

yield

$$\langle \Delta x^2\rangle = \frac{2c^2}{H_0^2}\tau\left(\frac{|E_k|^2}{|v_0 - d\omega/dk_z|}\right)\Bigg|_{\omega=k_z v_0}$$

Knowing the mean-square displacement of the ions, we can determine the shape of the spectral line.

For thermal particles at $\tau \to \infty$ the function $K(\tau)$, according to (2) and (3), is equal to

$$K(\tau) = \exp\left[-\frac{k_0^2}{2}\langle \Delta x^2\rangle\right] = \exp\left[-\frac{k_0^2}{2}D\tau\right], \tag{10}$$

where

$$D = \frac{c^2}{H_0^2}|E_\omega|^2\Big|_{\omega=0}.$$

For resonant particles we have

$$K(\tau) = \exp\left[-\frac{k_0^2}{2}D_{\text{pes}}\tau\right], \tag{11}$$

where

$$D_{\text{res}} = \frac{2c^2}{H_0^2}\left(\frac{|E_k|^2}{|v_0 - d\omega/dk_z|}\right)\Bigg|_{\omega=k_z v_0}.$$

Substituting (10) and (11) in (1) we obtain the spectral density $I(\omega)$ of the radiation distribution, which determines the shape of the spectral line; in our case we have a resonance line shape:

$$I(\omega) = \frac{1}{2\pi}\frac{k_0^2 D}{(\omega-\omega_0)^2 + {}^1\!/_4 k_0^4 D^2}.$$

Thus, the half-width of the spectral line, for thermal particles, is equal to

$$\Delta\omega = \frac{c^2 k_0^2}{2H_0^2}|E_\omega|^2\Big|_{\omega=0} \tag{12}$$

or

$$\frac{\Delta\omega}{\omega_0} = \frac{ck_0}{2H_0^2}|E_\omega|^2\Big|_{\omega=0},$$

and for resonant particles

$$\frac{(\Delta\omega)_{\text{res}}}{\omega_0} = \frac{ck_0}{2H_0^2}\left(\frac{|E_k|^2}{|v_0 - d\omega/dk_z|}\right)\Bigg|_{\omega=k_z v_0} \tag{13}$$

For short times ($\tau \ll 1/\bar{\Omega}$) $\langle \Delta x^2\rangle \sim \tau^2$, and the shape of the spectral line becomes Gaussian

$$I = \frac{1}{\sqrt{2\pi\overline{\omega^2}}}\exp\left[-\frac{(\omega-\omega_0)^2}{\overline{\omega^2}}\right], \quad \overline{\omega^2} = \frac{k_0^2c^2}{H_0^2}\sum_k |E_k|^2. \tag{14}$$

3. We have determined the spectral line shape for the case when the displacement of the ions is due to their drift in crossed electric fields of waves excited in the development of instabilities, and in a constant magnetic field. In the more general case, when the motion of the ions does not reduce to a simple drift in crossed electric and magnetic fields, the mean-square displacement $\langle \Delta x^2\rangle$, and consequently the spectral line shape, can be calculated by integrating the Langevin equation, which in our case is given by

$$\frac{dv}{dt} + \beta v = -\frac{e}{m}\sum_k E_k(t)\exp\{i(kv_0-\omega_k)t + ikx_0\}, \tag{15}$$

where β is the collision frequency. The random initial phases satisfy the condition $\langle \exp\{i(k+k')x_0\}\rangle = \delta_{k,-k'}$, and the random amplitudes satisfy the condition

$$E_k(\tau)E_k^*(\tau') = |E_k|^2\exp[-\nu(\tau-\tau')]^{2)}.$$

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2)Here $1/\nu$ is the correlation time. Thus, unlike the preceding section, we take into account the collisions and the finite correlation time of the Fourier harmonics of the electric field of the oscillations.

Then, integrating (15) and suitably averaging, we obtain

$$\frac{d}{d\tau}\frac{\langle(x-x_0)^2\rangle}{2}=\frac{e^2}{m^2\beta^2}\sum_k |E_k|^2\left\{e^{-\nu\tau}\left[\frac{\alpha_k\sin\alpha_k\tau-\nu\cos\alpha_k\tau}{\alpha_k^2+\nu^2}\right.\right.$$

$$\left.\left.-\frac{\alpha_k\sin\alpha_k\tau+(\beta-\nu)\cos\alpha_k\tau}{\alpha_k^2+(\beta-\nu)^2}\right]+\frac{\nu}{\alpha_k^2+\nu^2}\right\},\quad \alpha_k=kv_0-\omega_k. \tag{16}$$

At a finite correlation time $\nu \neq 0$ and at large times $\beta\tau \gg 1$ and $\nu\tau \gg 1$, we obtain for nonresonant particles ($v_0 \to 0$, $\alpha_k \approx -\omega_k$)

$$\langle\Delta x^2\rangle=\langle(x-x_0)^2\rangle=\frac{e^2}{m^2\beta^2}\frac{\tau}{2\pi}\int_{\omega^*}^{\infty}\frac{d\omega}{|d\omega/dk|}\frac{\nu}{\omega_k^2+\nu^2}|E_k|^2, \tag{17}$$

where ω^* is the minimum frequency in the spectrum of the oscillations excited upon development of the instabilities.

The relation $\omega = \omega(k)$ is determined from the dispersion equations, and $1/\nu$ is determined mainly from experiment.

For resonant particles $\alpha_k = kv_0 - \omega_k \to 0$, and using (16) we obtain after integrating with respect to ω

$$\langle\Delta x^2\rangle=\frac{e^2}{m^2\beta^2}\tau\left(\frac{|E_k|^2}{|v_0-d\omega/dk|}\right)\Big|_{\omega_k=kv_0}$$

As expected, the displacement of the resonant ions is determined by the intensity of the electric field of the waves whose phase velocity is close to ion velocity. Since the field intensities of such waves are quite large in a number of experiments, the Doppler broadenings should be quite appreciable. We note that measurement of the quantity $\langle \Delta x^2 \rangle$ makes it possible to obtain also estimates for the coefficient of the anomalous diffusion due to the instabilities.

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219