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ELASTICITY OF VORTEX LATTICES

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The shear modulus G is calculated for a triangular vortex lattice in rotating helium. The energy of normal oscillations of an arbitrary stable simple lattice is determined. In the long wave limit it is the same as the energy of transverse sound in a body with a shear modulus G.

1. We consider in this paper the elastic properties of lattices of straight-line vortices in an ideal incompressible liquid. It was found earlier[1-3] that triangular and near-triangular lattices are stable against small perturbations. Such lattices could exist in rotating He II. The oscillation spectrum of the vortex lattices is known[1,3]. In the long-wave limit, the oscillations of a triangular lattice propagate with a velocity $s = \frac{1}{2}\sqrt{\Gamma\Omega/2\pi}$, where Γ is the circulation of the velocity around the vortex and Ω is the angular velocity of rotation. Stauffer[3] considered the influence of the normal component of helium on the damping of the oscillations. The energy in the rotating system was calculated for all simple lattices[4]. The energy of a small deformation of a triangular lattice will be determined below, and we shall show that the long-wave oscillations are similar to transverse sound in a solid.

2. We start with a small shear deformation of a triangular lattice. Such a deformation leads to a vortex lattice which is close to triangular and has a larger free energy. The entire analysis will be carried out in a rotating reference frame, and the free energy is $E_r = E - M\Omega$, where E is the kinetic energy of the liquid and M is the angular momentum.

The triangular lattice is stable, meaning that we can use elasticity theory[5] and describe the stiffness of this lattice by means of a shear modulus G. Owing to the high symmetry of the triangular lattice, this modulus should be isotropic. For the increment of the free energy as the result of the deformation we obtain

$$\delta E_r = G\xi^2/2,$$

where ξ is the deformation angle. We shall calculate below the modulus in accordance with the formula $G = \partial^2 E_r/\partial\xi^2$.

3. We introduce a complex coordinate z such that the position of each vortex can be described by a complex number $z_{mn} = 2m\,\omega_1 + 2n\,\omega_2$, where m and n are integers and ω_1 and ω_2 are complex quantities, which represent the half-periods of the lattice. For a triangular lattice $2\,\omega_2 = (1 + i\sqrt{3})\,\omega_1$. Let us assume that only the half-period ω_2 is changed by the deformation. Using the notation $\omega_2/\omega_1 = \tau = |\tau| e^{i\varphi}$, we can readily obtain $d\tau/d\xi = \sqrt{3/2}$ and

$$\frac{d\ln|\tau|}{d\xi} = \frac{\sqrt{3}}{4} \qquad \frac{d\varphi}{d\xi} = -\frac{3}{4}$$

The dependence of E_r on the parameter τ for a liquid density $\rho = 1$ and a velocity circulation around the vortex $\Gamma = 2\pi$ was obtained in[4]. It is described by the formula

$$\frac{\partial E_r}{\partial\ln|\tau|} = -2\,\mathrm{Im}(\alpha\omega_1\omega_2)N, \qquad \frac{\partial E_r}{\partial\varphi} = -2\,\mathrm{Re}(\alpha\omega_1\omega_2)N,$$

where N is the density of the vortices per unit area and α is a complex number, defined by the relation

$$\alpha = \Omega\frac{\bar{\omega}_1}{\omega_1} + \frac{\pi^2}{12\omega_1^2}\frac{\Theta_1^{\mathrm{III}}(0)}{\Theta_1^{\mathrm{I}}(0)}.$$

Here Θ_1 is the elliptic theta-function (roman numerals denote the order of the derivative).

We can now calculate $dE_r/d\xi = G\xi$:

$$\frac{dE_r}{d\xi} = \frac{\partial E_r}{\partial\ln|\tau|}\frac{\partial\ln|\tau|}{d\xi} + \frac{\partial E_r}{\partial\varphi}\frac{d\varphi}{d\xi} = -\sqrt{3}N\,\mathrm{Im}(\alpha\omega_1^2).$$

For g we get

$$G = \frac{d^2E_r}{d\xi^2} = -\frac{3}{2}N\,\mathrm{Im}\frac{d(\alpha\omega_1^2)}{d\tau}.$$

From the definition of α and from the differential equation for the theta-function[6] we obtain

$$G = \frac{\pi^3}{32}\left(\frac{\Theta_1^{\mathrm{V}}(0)}{\Theta_1^{\mathrm{I}}(0)} - \frac{(\Theta_1^{\mathrm{III}}(0))^2}{(\Theta_1^{\mathrm{I}}(0))^2}\right)N.$$

For the invariant g_2 of the elliptic Weizstrauss functions, we can obtain from the Laurent expansion of the zeta function

$$\zeta(z) = \frac{1}{z} - \frac{g_2z^3}{60} - \frac{g_3z^5}{140}\cdots$$

the expression

$$g_2 = 2\left(\frac{\pi}{2\omega_1}\right)^4\left(\frac{5}{3}\frac{(\Theta_1^{\mathrm{III}}(0))^2}{(\Theta_1^{\mathrm{I}}(0))^2} - \frac{\Theta_1^{\mathrm{V}}(0)}{\Theta_1^{\mathrm{I}}(0)}\right).$$

but for a triangular lattice $g_2 = 0$ and $\alpha = 0$, and when these relations are taken into account we obtain $G = \pi N/4$, or finally $G = \Omega/4$, since $\pi N = \Omega$.

The velocity s of long waves is known[1]: $s = \frac{1}{2}\sqrt{\Omega}$, meaning that $G = s^2$. For a liquid density $\rho \neq 1$ it would be necessary to write

$$G = \rho s^2.$$

This relation is typical of a solid, if s is interpreted as the velocity of the transverse sound. Here ρ the meaning of the density of the lattice (but not of the liquid, although they are equal).

4. We now find the free energy of small oscillations of a vortex lattice (not necessarily triangular). This energy does not depend on the time, as can be shown by direct calculations. The displacements c_{mn} of the vortices for normal oscillations are given by

$$c_{mn} = \varepsilon e^{-i\chi}\left(\cos\omega t - i\frac{\Omega - r}{\omega}\sin\omega t\right)\cos(m\varphi + n\psi),$$

where ϵ is a small number, $\omega = \sqrt{\Omega^2 - r^2}$, $re^{2i\chi} = B(\kappa)$ is the function defined[1] for each normal oscillation with

$$(\varphi\omega_2 - \psi\omega_1) / \pi = \varkappa = i\pi / \Omega\bar{\lambda},$$

λ is the wavelength on the complex plane. For $t = 0$ we have

$$c_{mn} = \varepsilon e^{-i\chi}\cos(m\varphi + n\psi), \quad \dot{c}_{mn} = -i\varepsilon e^{-i\chi}(\Omega - r)\cos(m\varphi + n\psi).$$

Let us find the energy as a function of ϵ. The velocity of the liquid at the point where the vortex is located, in the absence of a vortex, is $\dot{c}_{mn}$. To increase ϵ by $d\epsilon$ it is necessary to shift each vortex by a distance $e^{-i\chi}\cos(m\varphi + n\psi)d\epsilon$. This must be performed against the force $i\Gamma\dot{c}_{mn} = 2\pi\epsilon e^{-i\chi}(\Omega - r)\cos(m\varphi + n\psi)$, owing to motion of the liquid with an energy loss $2\pi\epsilon\, d\epsilon\, (\Omega - r)\cos^2(m\varphi + n\psi)$. The mean square of the cosine is $1/2$, so that the mean energy per vortex is $\pi\epsilon^2(\Omega - r)/2$.

In the long-wave limit, these oscillations are transverse. In the case of long waves of a triangular lattice, the expression $\pi(\pi\epsilon/2\lambda)^2$ is valid for the average energy since $|B(\kappa)| \infty \Omega - \Omega^2|\kappa|^2/2$.[1] This energy can be compared with the energy of standing waves of transverse sound in a solid with a shear modulus G. In the long-wave approximation they are equal.

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201