ON THE NATURE OF THE SINGULARITIES IN THE GENERAL SOLUTION OF THE GRAVITATIONAL EQUATIONS

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A general solution of the gravitational equations which contain a fictitious singularity was investigated in ref [1]. A solution with a physical singularity was found which was deficient with respect to an arbitrary function which would make it a general solution. The present paper investigates the question of the way in which inclusion of the deficient function as a perturbation may, during evolution of the solution with a physical singularity, destroy it and yield a fictitious singularity.

1. INTRODUCTION

In this article we wish to return once more to the question of the character of the temporal singularity in the general solution of the equations of gravitation theory.

We recall first the main premises of[1], which we shall find useful in what follows.

1. In any space-time it is possible to construct a synchronous reference frame, in which the metric coefficients satisfy the conditions $g_{00} = -1$, $g_{0\alpha} = 0$.

2. The determinant of the metric tensor in the synchronous reference frame must vanish at a certain instant of time $t = t_0$ (in the past or in the future). Consequently, the appearance of singularities in the metric is inevitable (for an arbitrary distribution of matter).

3. The solution of Einstein's equations near this singularity contains a complete set of arbitrary functions of the spatial coordinates (eight physical functions in the presence of matter, plus one function due to the arbitrariness in the choice of the initial hypersurface that determines the synchronous reference system).

Thus, the solution near the singularity is general. The indicated singularity is in the general case not of the one-time type, but may be made so by suitable choice of the reference system.

4. The invariants of the curvature tensor near the singularity under consideration are regular, thus indicating that the singularity has a fictitious (coordinate) character[1)].

5. In a synchronous reference system, we obtained a solution with a physical singularity, containing seven arbitrary functions of the spatial coordinates. The metric of this solution has the following form:

$$\alpha, \beta = 1, 2, 3. \tag{1.1}$$

Near the singularity $t = 0$, the solution is obtained in the form of series in powers of t. The zeroth approximation is described by the metric (1.1), in which the vectors $\mathbf{l}^{(0)}$, $\mathbf{m}^{(0)}$, and $\mathbf{n}^{(0)}$ do not depend on the time. The exponents p_1, p_2, and p_3 are connected by the following two relations:

$$p_1 + p_2 + p_3 = p_1^2 + p_2^2 + p_3^2 = 1. \tag{1.2}$$

Thus, these exponents contain also one arbitrary function. One of the exponents must be negative (for concreteness we shall designate it by the index 1). It should also be recalled that in the chosen reference frame there is an arbitrariness that depends on the three transformations of the coordinates $x^\alpha = x^\alpha(\bar{x}^\beta)$.

The nine arbitrary functions $l_\alpha^{(0)}$, $m_\alpha^{(0)}$, and $n_\alpha^{(0)}$ are connected by three conditions that follow from the R_α^0 components of the Einstein equations. In addition, the vector $\mathbf{l}^{(0)}$ in the metric (1.1), preceding the factor t^{2p_1} ($p_1 < 0$), satisfies one additional condition

$$\mathbf{l}^{(0)} \operatorname{rot} \mathbf{l}^{(0)} = 0. \tag{1.3}$$

The coefficients in the expansion of the vectors $\mathbf{l}$, $\mathbf{m}$, and $\mathbf{n}$ in powers of t are expressed in a unique manner in terms of the zeroth-approximation function. Thus, this solution contains only three arbitrary functions (seven in a space with matter), i.e., one less than needed for a general solution.

The considerations advanced in Items 1 and 4, together with this result, argue in favor of the absence of a physical singularity in the general cosmological solution of Einstein's equations. On the other hand, however, it follows from the Penrose theorem[3] that there exists (under very natural assumptions) a singularity whose character, however, no one has succeeded in establishing and which, apparently, is so weak that it does not appear in the invariants of the curvature tensor[2)].

A natural way of clarifying the character of the singularity in the general solution is as follows: starting from the solution (1.1), we perturb it by including in the equations terms with $\mathbf{l} \operatorname{curl} \mathbf{l} \neq 0$ and

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[1)] A physical singularity is taken to be a singularity that appears in scalar quantities, with a direct physical meaning, such as the energy density of the matter, or in second-order invariants of the curvature tensor, which represent the eigenvalues of the matrix of the Riemann tensor in bivector space [2]. Of course, one cannot exclude the possible occurrence of singularities of a different nature, of the type of geodesic incompleteness, which we are not considering here (see [3]).

[2)] Oral communication by Carter at the Fifth International Conference on Gravitation and Relativity theory.

trace the evolution of this solution[3]. If there is no physical singularity in the general solution, then we should arrive at a solution of the type $p_1 = p_2 = 0$, $p_3 = 1$. Precisely such a set of exponents corresponds to the case of the general solution with a one-time fictitious singularity[4].

Naturally, we could in this case obtain also some other solution containing an unknown type of singularity. In general form, obviously, the posed problem cannot be solved, since the method of small perturbations is not applicable in this case. Any term l curl l which is arbitrarily small at a certain instant of time, becomes in the course of time of the same order as the principal terms in the Einstein equations. We shall therefore carry out the investigation with several particular cases of the metric (1.1), and trace, to the extent that it is possible, the further evolution of the solution (1.1) after turning on perturbations of the type $\mathbf{l}\,\mathrm{curl}\,\mathbf{l} \neq 0$. We shall consider the case of empty space since, as shown in[1], the inclusion of matter does not change essentially the character of the singularity.

2. INVESTIGATION OF THE PARTICULAR CASE WHEN l curl l ≠ 0

We consider the particular case of the metric (1.1), when only $\mathbf{l}\,\mathrm{curl}\,\mathbf{l} \neq 0$[5], and all the remaining scalar products of the type m curl l, m curl m, etc. are equal to zero. We write the metric in the form

$$-ds^2 = -dt^2 + a^2(dx + \eta dy)^2 + b^2dy^2 + c^2dz^2, \qquad (2.1)$$

where the functions a, b, c, and η depend only on t and z. The metric (2.1) corresponds to the following choice of vectors:

$$\mathbf{l}(1, \eta, 0), \quad \mathbf{m}(0, 1, 0), \quad \mathbf{n}(0, 0, 1). \qquad (2.2)$$

Here $\mathbf{l}\cdot\mathbf{m}\times\mathbf{n} = 1$ and the determinant of the metric tensor is equal to

$$-g = a^2b^2c^2(\mathbf{l}[\mathbf{mn}])^2 = a^2b^2c^2. \qquad (2.3)$$

All the scalar products of the fundamental vectors and of their curls vanish, with the exception of l curl l, which equals[6]

$$\mathbf{l}\,\mathrm{rot}\,\mathbf{l} = -\eta'. \qquad (2.4)$$

The components of the Ricci tensor are:

$$R_0^0 = \frac{\ddot a}{a} + \frac{\ddot b}{b} + \frac{\ddot c}{c} + \frac{a^2\dot\eta^2}{2b^2}, \qquad (2.5)$$

$$R_3^0 = \frac{c}{ab}\left[\frac{a^3}{2bc}\dot\eta\eta' + b\left(\frac{\dot a'}{c}\right) + a\left(\frac{\dot b'}{c}\right)\right], \qquad (2.6)$$

$$R_3^3 = \frac{1}{abc}\left[(ab\dot c)^{\cdot} - \frac{a^3}{2bc}\eta'^2 - b\left(\frac{a'}{c}\right)' - a\left(\frac{b'}{c}\right)'\right] \qquad (2.7)$$

$$R_l^l = R_1^1 + \eta R_1^2 = \frac{1}{abc}\left[(\dot abc)^{\cdot} + \frac{a^3}{2bc}\eta'^2 - \left(\frac{a'b}{c}\right)' - \frac{a^3c}{2b}\dot\eta^2\right], \qquad (2.8)$$

$$R_m^m = R_2^2 - \eta R_1^2 = \frac{1}{abc}\left[(a\dot bc)^{\cdot} - \frac{a^3}{2bc}\eta'^2 - \left(\frac{ab'}{c}\right)' + \frac{a^3c}{2b}\dot\eta^2\right], \qquad (2.9)$$

$$R_l^m = R_1^2 = \frac{1}{2abc}\left[\left(\frac{a^3c}{b}\dot\eta\right)^{\cdot} - \left(\frac{a^3}{bc}\eta'\right)'\right]. \qquad (2.10)$$

In empty space, Einstein's equations are obtained by equating to zero the written-out components of the Ricci tensor. An exact solution of these equations can be easily obtained in the case when $\eta = \eta(z)$, and the functions a, b, and c depend only on the time. It then follows from (2.10) that

$$\eta' = \mathrm{const} = \lambda \qquad (2.11)$$

and the solution can be written in the form

$$a^2 = \frac{2|p_1|}{\lambda\,\mathrm{ch}\,[2p_1\tau + \ln(\lambda a_0^2/4|p_1|)]} \qquad (2.12)$$

$$b^2 = \frac{\lambda b_0^2a_0^2}{2|p_1|}e^{2(p_2+p_1)\tau}\,\mathrm{ch}\,[2p_1\tau + \ln(\lambda a_0^2/4|p_1|)], \qquad (2.13)$$

$$c^2 = \frac{\lambda c_0^2a_0^2}{2|p_1|}e^{2(p_3+p_1)\tau}\,\mathrm{ch}\,[2p_1\tau + \ln(\lambda a_0^2/4|p_1|)], \qquad (2.14)$$

$$p_1 + p_2 + p_3 = p_1^2 + p_2^2 + p_3^2 = 1. \qquad (2.15)$$

Here a_0, b_0, c_0 are integration constants, which we shall set equal to unity for brevity. The variable τ is expressed in terms of the time t by the relation

$$d\tau = dt/abc. \qquad (2.16)$$

The singularity in the solution takes place at $\tau = \pm\infty$. Let us trace the variation of the asymptotic form of the solution when τ ranges from $+\infty$ to $-\infty$. Leaving out the exponentially small additions, we obtain (we assume that $p_1 < p_2 < p_3$):

$$\text{I. } \tau \to +\infty,\ t = e^\tau: \quad a^2 = e^{2p_1\tau} = t^{2p_1}, \quad b^2 = e^{2p_2\tau} = t^{2p_2}, \quad c^2 = e^{2p_3\tau} = t^{2p_3}, \qquad (2.17)$$

$$\text{II. } \tau \to -\infty, \quad t = \frac{\lambda}{4|p_1|(1+2p_1)}e^{(1+2p_1)\tau}:$$

$$a^2 = \left(\frac{4p_1}{\lambda}\right)^2 e^{-2p_1\tau} \sim \left(\frac{t}{\lambda}\right)^{-2p_1/(1+2p_1)}\lambda^{-2},$$
$$b^2 = \left(\frac{\lambda}{4p_1}\right)^2 e^{2(p_2+2p_1)\tau} \sim \left(\frac{t}{\lambda}\right)^{(2p_2+4p_1)/(1+2p_1)}\lambda^2, \qquad (2.18)$$
$$c^2 = \left(\frac{\lambda}{4p_1}\right)^2 e^{2(p_3+2p_1)\tau} \sim \left(\frac{t}{\lambda}\right)^{(2p_3+4p_1)/(1+2p_1)}\lambda^2.$$

Thus, starting at large t with a metric of the Kasner type $a^2, b^2, c^2 \sim t^{2p_1}, t^{2p_2}, t^{2p_3}$, we arrive as $t \to 0$ again at a metric of the same type, but with different exponents:

$$p_1' = -\frac{p_1}{1+2p_1}, \quad p_2' = \frac{p_2+2p_1}{1+2p_1}, \quad p_3' = \frac{p_3+2p_1}{1+2p_1}. \qquad (2.19)$$

These exponents satisfy, naturally, the condition (2.15).

We recall the parametric representation of the exponents p:

$$p_1 = \frac{-u}{1+u+u^2}, \quad p_2 = \frac{1+u}{1+u+u^2}, \quad p_3 = \frac{u(1+u)}{1+u+u^2}. \qquad (2.20)$$

The chosen order of the exponents, $p_1 < p_2 < p_3$, corresponds to a parameter $u > 1$. The set of expon-

[3]Such a procedure, which is possible for the cosmological problem, cannot be used to solve, generally speaking, the question of the character of the singularity arising when the conditions of the Penrose theorem are satisfied, since it is not clear to what extent these conditions are necessary for the general solution of the gravitation equations.

[4]We recall that in the synchronous reference system the singularity cannot vanish at all.

[5]In this section we present results obtained by one of the authors (Khalatnikov) with E. M. Lifshitz earlier in 1962, but not published before.

[6]The prime denotes the derivative with respect to z, and the dots the derivative with respect to t.

ents (2.19), corresponding to the asymptotic form as $t \to 0$, is obtained from (2.20) by replacing u with $u - 1$ and interchanging p_1 and p_2. Thus, whereas at $t \to \infty$ we had a power-law asymptotic form corresponding to the parameter $u > 1$ with negative p_1 (which corresponds in turn to a broadening of the scales along the x axis), as $t \to 0$ we have again a power-law asymptotic form with negative p_2' and with $u' = u - 1$ (broadening of the scales along the y axis, contraction along the other two axes). The transition from the asymptotic form (2.17) to (2.18) occurs in a region of τ close to $\tau^{I}_{cr} = -(1/2p_1)\ln(\lambda/4|p_1|)$ (at τ^{I}_{cr} the function a^2 has a maximum), or in synchronous time

$$t_{cr}{}^{I} \sim \lambda^{-1/2p_1}. \tag{2.21}$$

In the general case, when all the functions depend on the variables t and z, it is impossible to solve the equations (2.5)—(2.10) exactly, but it is possible to investigate them qualitatively. A simple analysis shows that for $t \gg t^{I}_{cr}$ and for a specified function $\eta' = \lambda(z)$ that depends in an arbitrary manner on z, all terms with coordinate derivatives turn out to be small in the indicated equations, and can be neglected. The terms $a^2\eta^2/2b^2$ and $a^3c\eta^2/2b$ in Eqs. (2.5), (2.8), and (2.9) likewise turn out to be small. When $t \sim t^{I}_{cr}$, only the term $a^3\lambda^2/2bc$ becomes appreciable in the equations; this term ensures the transition from the asymptotic form (2.17) to the asymptotic form (2.18). In region II, which is described by (2.18), the role of the terms with coordinate derivatives diminishes, and the role of the terms containing η increases gradually. An estimate shows that these terms become of the order of the main terms in the equations at $t \sim t^{II}_{cr}$, where

$$t_{cr}{}^{II} \sim \lambda^{-p_3/(p_2+3p_1)}. \tag{2.22}$$

In order to trace the further evolution of the solution, it is necessary to solve the problem at $t \sim t^{II}_{cr}$, i.e., to find the general solution of the equations in the synchronous system, when the metric depends only on the time. In this case the solution can always be reduced to the form

$$-ds^2 = -dt^2 + t^{2p_1''}dx^2 + t^{2p_2''}dy^2 + t^{2p_3''}dz^2.$$

From this we get, by a linear transformation in the x, y plane and by changing the scale along the z axis,

$$-ds^2 = -dt^2 + (a_1^2t^{2p_1''} + a_2^2t^{2p''_2})\,dx^2 + (b_1^2t^{2p_1''} + b_2^2t^{2p_2''})\,dy^2 + 2\,(a_1b_1t^{2p_1''} + a_2b_2t^{2p_2''})\,dx\,dy + c_3^2t^{2p_3''}dz^2.$$

The functions a, b, c, and η are then obtained from the relations

$$a^2 = g_{11}, \quad a^2\eta = g_{12}, \quad a^2\eta^2 + b^2 = g_{22}, \quad c^2 = g_{33}.$$

We thus get

$$a^2 = a_1^2\,t^{2p_1''} + a_2^2\,t^{2p_2''},$$
$$b^2 = \frac{(a_1b_2 - b_1a_2)^2\,t^{2p_1''+2p_2''}}{a_1^2t^{2p_1''} + a_2^2t^{2p_2''}}$$
$$c^2 = c_3^2t^{2p_3''}, \tag{2.23}$$
$$\eta = \frac{a_1b_1t^{2p_1''} + a_2b_2t^{2p_2''}}{a_1^2t^{2p_1''} + a_2^2t^{2p_2''}}$$

This solution must be made continuous at the point t^{II}_{cr} with the solution (2.18) in the region II. We note that regardless of the signs of the new exponents p'', the function b^2 always decreases at $t \to 0$. Since this function increased in the region II, the point t^{II}_{cr} is extremal for this function. The continuity conditions yield

$$p_3'' = \frac{p_3 + 2p_1}{1 + 2p_1}, \quad p_2'' = \frac{p_2 + 2p_1}{1 + 2p_1}, \quad p_1'' = \frac{-p_1}{1 + 2p_1}$$

and in the region III, where $t \ll t^{II}_{cr}$, we obtain the following asymptotic form:

$$a^2 \sim \left(\frac{t}{\lambda}\right)^{(2p_2+4p_1)/(1+2p_1)}, \quad b^2 \sim \left(\frac{t}{\lambda}\right)^{-2p_1/(1+2p_1)}$$
$$c^2 \sim \left(\frac{t}{\lambda}\right)^{(2p_3+4p_1)/(1+2p_1)}\lambda^2. \tag{2.24}$$

Thus, in the region III, the axes x and y are again interchanged, and now the expansion (negative exponent of a^2) is along the x axis. The final transition from the region I to the region III corresponds to replacement of the parameter u by $u - 1$. The region III can serve as the starting point for the next stage, as a result of which we arrive in the region V, where we have a power-law asymptotic form with a new value of the parameter $u' = u - 2$. This process will continue until we separate the entire integer part of u and the resultant value of the parameter satisfies the condition $u < 1$. Then, in the next stage, the functions a^2 and b^2 begin to decrease, whereas the function c^2 begins to increase without limit (the corresponding exponent is $p_3 = u(u - 1)/(u^2 - u + 1) < 0$). The presence of only one diagonal product l curl l cannot stop this unlimited increase.

3. INVESTIGATION OF THE MORE GENERAL CASE l curl l ≠ 0, m curl m ≠ 0, n curl n ≠ 0.

It is of interest to consider the case when all three scalar products l curl l, m curl m, n curl n do not vanish. Since in this case there is no longer a preferred direction along which an asymptotic form with negative exponent can be established, we should expect a certain qualitatively new behavior of the metric.

We write the metric in the form

$$-ds^2 = -dt^2 + (a^2l_\alpha l_\beta + b^2m_\alpha m_\beta + c^2n_\alpha n_\beta)\,dx^\alpha dx^\beta, \tag{3.1}$$

where a, b, and c depend only on the time, and the vectors l, m, and n depend only on the coordinates x, y, and z. All the components of the vectors give us nine three-dimensional functions, which we subject to nine conditions:

$$\begin{aligned} &\mathbf{l}\,\mathrm{rot}\,\mathbf{l} = \lambda, \quad \mathbf{m}\,\mathrm{rot}\,\mathbf{l} = 0, \quad \mathbf{n}\,\mathrm{rot}\,\mathbf{l} = 0,\\ &\mathbf{l}\,\mathrm{rot}\,\mathbf{m} = 0, \quad \mathbf{m}\,\mathrm{rot}\,\mathbf{m} = \mu, \quad \mathbf{n}\,\mathrm{rot}\,\mathbf{m} = 0,\\ &\mathbf{l}\,\mathrm{rot}\,\mathbf{n} = 0, \quad \mathbf{m}\,\mathrm{rot}\,\mathbf{n} = 0, \quad \mathbf{n}\,\mathrm{rot}\,\mathbf{n} = \nu;\\ &\lambda, \mu, \nu = \mathrm{const.}\end{aligned} \tag{3.2}$$

In addition, we can always fix the condition

$$\mathbf{l}[\mathbf{mn}] = 1. \tag{3.3}$$*

We shall show later that the system (3.2)—(3.3) can be readily solved by choosing a certain special coordinate system x, y, z. Using the formulas of Appendix C of[1], we can readily see that the components R^0_l, R^0_m, R^0_n, R^l_m, R^l_n, and R^m_n, projected on the triad l, m, n, vanish identically, and for the remaining components we have

$$R_l{}^l = \frac{1}{abc}[abc(\ln a)^{\cdot}]^{\cdot} + \frac{1}{2a^2b^2c^2}[a^4\lambda^2 - (b^2\mu - c^2\nu)^2] = 0,$$
$$R_m{}^m = \frac{1}{abc}[abc(\ln b)^{\cdot}]^{\cdot} + \frac{1}{2a^2b^2c^2}[b^4\mu^2 - (a^2\lambda - c^2\nu)^2] = 0, \tag{3.4}$$

*[mn] ≡ m x n.

$$R_n^n = \frac{1}{abc}[abc(\ln c)^{\cdot}]^{\cdot} + \frac{1}{2a^2b^2c^2}[c^4\nu^2 - (a^2\lambda - b^2\mu)^2] = 0;$$
$$R_0^0 = (\ln abc)^{\cdot\cdot} + (\ln a)^{\cdot 2} + (\ln b)^{\cdot 2} + (\ln c)^{\cdot 2} = 0. \quad (3.5)$$

It is impossible to solve such equations exactly, but we can investigate them by the method employed above. We note that now the problem is simpler, since the scalar products of the vectors by their curls are specified constants, and the functions a, b, and c depend only on the time.

It is easy to see that we need actually know only the solution (2.17)—(2.18). We shall assume that $\lambda = \mu = \nu > 0$. Choosing the initial point $t_0 \ll 1$ and sufficiently small λ, such as to make

$$\lambda^2 t_0^{4p_1} \ll 1, \quad (3.6)$$

we can start with the metric

$$a^2 \sim t^{2p_1}, \quad b^2 \sim t^{2p_2}, \quad c^2 \sim t^{2p_3}. \quad (3.7)$$

The procedure is again repeated as before, the only difference being that now we shall alternately include into consideration all three scalar diagonal products **l** curl **l**, **m** curl **m**, **n** curl **n**, depending on which of the functions increases during the stage under consideration. In each such stage, we use the solution (2.17)—(2.18), making it continuous with the end of the previous solution. As a result we obtain an infinite number of oscillations down to t = 0. The direction of the axis along which the expansion of the scales takes place (negative exponent) will also change an infinite number of times.

Let us examine in greater detail the behavior of the numerical values of the exponents p. If at the initial instant of time the inequality $p_1 < p_2 < p_3$ $(u > 1)$ holds, then as result of a finite number of oscillations we arrive at $u < 1$. This violates the inequality for the exponents, which are now in the order $p_1 < p_3 < p_2$. However, we can make the transformation $u \to 1/u$ and interchange the indices 2 and 3 (or the axes y and z), and thus restore the inequality $p_1 < p_2 < p_3$. The reduction process will then continue until we separate the entire integer part from 1/u, and again arrive at a parameter $u' < 1$, etc. If the initial value of the parameter u is a rational number, i.e., it is a ratio of two natural numbers m/n, then by two operations $u \to u - 1$ and $u \to 1/u$ we can reduce this number to zero. Thus, when the initial parameter is a rational number, we arrive at a solution with $p_1 = p_2 = 0$, $p_3 = 1$. However, for an irrational number the process under consideration can never stop. We can come very close to u = 0, and then again move away from this value. Thus, we arrive at the need for investigating the properties of our solution at values of the parameter u close to zero.

4. INVESTIGATION OF THE CASE WHEN $u \ll 1$ OR $u \gg 1$

According to (2.20), the exponents p_1, p_2, and p_3 are invariant against transformations of the parameter $u \to 1/u$, and consequently the cases $u \ll 1$ and $u \gg 1$ are equivalent. In order to retain the customary order of exponents $p_1 < p_2 < p_3$, we shall consider the case $u \gg 1$.

At a large value of the parameter u, as we have seen from the qualitative considerations, two functions (assume for concreteness that these are a and b) vary quasiperiodically, and the third function (c) decreases with decreasing time. It turns out however, that the character of the solution at large values of the parameter u can also be investigated quantitatively. The possibility of solving the system of equations (3.4) —(3.5) in this case is based on the fact that, owing to the decrease of the function c, there always occurs an instant of time (still at large values of u), when this function becomes smaller than the two others. We introduce the new variable

$$d\tau = dt / abc, \quad (4.1)$$

Then the equations (3.4) and (3.5) assume the form (the index τ denotes henceforth differentiation with respect to this variable)

$$(\ln a^2)_{\tau\tau} = (\mu b^2 - \nu c^2)^2 - \lambda^2 a^4,$$
$$(\ln b^2)_{\tau\tau} = (\lambda a^2 - \nu c^2)^2 - \mu^2 b^4,$$
$$(\ln c^2)_{\tau\tau} = (\lambda a^2 - \mu b^2)^2 - \nu^2 c^4, \quad (4.2)$$

$$(\ln a)_\tau (\ln b)_\tau + (\ln a)_\tau (\ln c)_\tau + (\ln b)_\tau (\ln c)_\tau = {}^1/_2 (\ln abc)_{\tau\tau}. \quad (4.3)$$

Let us examine the behavior of the solution in that region of the variable τ, with one of the functions, say c, becoming so small that it can be neglected compared with the two others a and b. In this case Eqs. (4.2) and (4.3) yield

$$(\chi + \varphi)_{\tau\tau} = 0, \quad (4.4)$$
$$(\chi - \varphi)_{\tau\tau} = 2\lambda^2 (e^{2\varphi} - e^{2\chi}), \quad (4.5)$$
$$\psi_\tau (\chi_\tau + \varphi_\tau) = -\chi_\tau \varphi_\tau + \lambda^2 (e^\chi - e^\varphi)^2. \quad (4.6)$$

We have used here the notation

$$a^2 = e^\chi, \; b^2 = e^\varphi, \; c^2 = e^\psi \quad (4.7)$$

and we have put $\lambda = \mu = \nu$, which obviously does not decrease the generality of the investigation. Furthermore, the last equation of (4.2) will not be written out, for in the case $\chi_\tau + \varphi_\tau \neq 0$[7)] it is the consequence of the system (4.4)—(4.6). We write the solution of the Eq. (4.4) in the form

$$\chi + \varphi = p(\tau - \tau_0), \quad (4.8)$$

where p = const, and we assume for concreteness that $p > 0$. Introducing the new variable

$$\xi = \frac{4\lambda}{p} e^{p(\tau - \tau_0)/2} \quad (4.9)$$

and the notation $\chi - \varphi = q$, we obtain two equations

$$q_{\xi\xi} + \frac{1}{\xi} q_\xi + \operatorname{sh} q = 0, \quad (4.10)$$

$$\psi_\xi = -\frac{1}{2\xi} + \frac{1}{8}\xi (q_\xi^2 + 2\operatorname{ch} q - 2). \quad (4.11)$$

Since $p > 0$, variation of τ from $+\infty$ to $-\infty$ corresponds to variation of ξ from $+\infty$ to zero.

Let us consider the regions $\xi \gg 1$ and $\xi \ll 1$. An analysis of Eq. (4.10) shows that in the asymptotic

[7)] If $\chi_\tau + \varphi_\tau = 0$, then it follows from (4.6) that $\varphi = \chi$ = const, and we arrive at the trivial solution a = b = const, c = t, which does not satisfy the initial conditions of interest to us, $u \neq 0$ or $u \neq \infty$.

region $\xi \gg 1$ its solution is given by

$$q = \frac{4A}{\sqrt{\xi}}\left[\sin\alpha - \frac{A^2}{12\xi}(6\sin\alpha + \sin 3\alpha) + \frac{6A^4-1}{8\xi}\cos\alpha + O\left(\frac{1}{\xi^2}\right)\right], \quad (4.12)$$

$$\alpha = \xi - \xi_0 + A^2 \ln \xi, \quad (4.13)$$

where A and ξ_0 are arbitrary constants.

In the region $\xi \ll 1$ we have

$$q = 2k \ln \xi + 4 \ln s \quad (4.14)$$

(k, s = const). Calculating the correction δq to the principal approximation (4.14), we obtain it in the form of a linear combination of two powers ξ^{2-2k} and ξ^{2+2k}. It follows therefore that the requirement $\delta q \ll q$ when $\xi \ll 1$ yields

$$-1 < k < 1. \quad (4.15)$$

We now consider the function ψ. From (4.12)–(4.15) and from the equation (4.11) we obtain

$$\psi = \begin{cases} \text{const} + 2A^2\xi + \left(A^4 - \frac{1}{2}\right)\ln\xi + \frac{A^2}{2\xi}\cos 2\alpha + O\left(\frac{1}{\xi^2}\right), & \xi \gg 1. \\ \text{const} + \frac{k^2-1}{2}\ln\xi, & \xi \ll 1 \end{cases} \quad (4.16)$$

Consequently, when $\xi \gg 1$,

$$a^2 = \frac{p}{4\lambda}\xi \exp\left[\frac{2A}{\sqrt{\xi}}\sin(\xi - \xi_0 + A^2\ln\xi)\right] + O\left(\frac{1}{\sqrt{\xi}}\right),$$

$$b^2 = \frac{p}{4\lambda}\xi \exp\left[-\frac{2A}{\sqrt{\xi}}\sin(\xi - \xi_0 + A^2\ln\xi)\right] + O\left(\frac{1}{\sqrt{\xi}}\right), \quad (4.17)$$

$$c^2 = c_0^2 \xi^{A^4 - 1/2} e^{2A^2\xi}\left[1 + O\left(\frac{1}{\xi}\right)\right]$$

and when $\xi \ll 1$

$$a^2 = \frac{ps^2}{4\lambda}\xi^{1+k}, \quad b^2 = \frac{p}{4\lambda s^2}\xi^{1-k}, \quad c^2 = \bar{c}_0^2 \xi^{(k^2-1)/2}. \quad (4.18)$$

In the region where the solution (4.17) is valid, it is possible in principle to express all the functions in terms of t. The connection between the synchronous time t and the variable ξ is given by the formula

$$\lambda(t - t_0) = \frac{c_0}{2A^2}\xi^{1/2 A^4 - 1/4} e^{A^2\xi}\left[1 + O\left(\frac{1}{\xi}\right)\right]. \quad (4.19)$$

Thus, in this rather wide region (with decreasing t) the functions a and b oscillate against the background of a slow decrease. As to the function c, it decreases monotonically approximately like $c^2 \sim t^2$.

At a certain value of t corresponding to the variable $\xi \sim 1$, we fall into a region which is already described by a solution of the form (4.18), namely, the function c begins to increase (since $k^2 < 1$), and one of the functions a (or b) begins to decrease. We then return quite rapidly, as soon as the function a becomes smaller than the two others, to the regime described by formulas (4.17).

Thus, the solution of the system of Eqs. (4.2) goes asymptotically through prolonged periods described by formulas (4.17), and short periods described by (4.18). The prolonged periods (4.17) correspond to a variation of the parameter u from a very large value to $u \sim 1$, discussed in Sec. 3. As indicated, during the course of the evolution the solution will come close to the Kasner type with exponents $p_1 = p_2 = 0$, and $p_3 = 1$. Numerical calculations were also made of Eqs. (4.2) with arbitrary initial conditions satisfying (4.3).

These calculations confirm fully the picture obtained by the analytic method. The figure shows plots of typical changes of the functions a, b, and c. Equations (4.2) were solved under the condition $\lambda = \mu = \nu = 1$ (this can always be done by simply redefining a, b, and c in the case when $\lambda, \mu, \nu > 0$), and under the following initial data at the point $\tau = 0$: $a^2 = 1$, $b^2 = 2$, $c^2 = 1.5$, $(a^2)_\tau = 0.5$, $(b^2)_\tau = 0$, (6), $(c^2)_\tau = -10.65$, satisfying Eq. (4.3). The function c^2 decreases quite rapidly, and is of the order of 10^{-19} already when $\tau \sim 8$. The function a^2 also decreases monotonically starting with $\tau \sim 70$, and subsequently becomes smaller than c^2. Then (as is clear from a qualitative analysis of the equations, and also from other diagrams which are not presented here), a prolonged period similar to that shown here again sets in, but with much smaller amplitudes and oscillation frequencies of the functions b^2 and c^2.

But will this yield a solution with a fictitious singularity in the general case, when in the region p_1, $p_2 \approx 0$, $p_3 \approx 1$, there are included arbitrary perturbations satisfying the complete system of Einstein's equations? Unfortunately, it is impossible to investigate this process rigorously. But it is possible to advance a simple consideration, based on the assumption that when u $u \to 0$ $(u \to \infty)$ we are located near a flat space described by the metric (compare (4.17) and (4.19)),

$$-ds^2 = -dt^2 + dx^2 + dy^2 + t^2 dz^2. \quad (4.20)$$

Since (4.20) describes a flat space with a fictitious coordinate singularity it is natural to expect that such a solution is stable against small perturbations. During the process of the evolution of the solution of Eqs. (4.2) and (4.3) with a physical singularity, we come close to a stable metric (4.20) and we stay for a long time near it. This gives grounds for assuming that allowance for the dependence of the metric (3.1) on all the coordinates (which is not taken into account in the particular case considered here) leads to elimination of the physical singularity.

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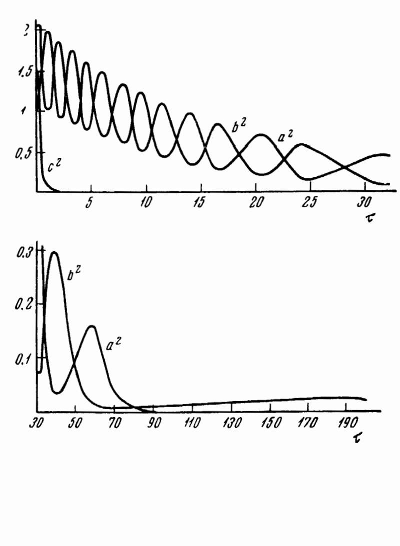

APPENDIX

Let us indicate here certain particular solutions of the system (4.2) and (4.3), and also a solution of Eqs. (3.2).

The system (4.2), (4.3) has an exact particular solution in the case when

$$\lambda a^2 = \mu b^2 \tag{A.1}$$

(or the same for any arbitrary other pair of functions). The solution is of the form

$$c^2 = \frac{2|p|}{|\nu|\,\mathrm{ch}(2p\tau - c_1)}, \tag{A.2}$$

$$\lambda a^2 = \begin{cases} \frac{1}{2}|p|\frac{\mathrm{ch}(2p\tau - c_1)}{\mathrm{ch}^2(p\tau - c_2)}, & \nu > 0 \\ \frac{1}{2}|p|\frac{\mathrm{ch}(2p\tau - c_1)}{\mathrm{sh}^2(p\tau - c_2)}, & \nu < 0 \end{cases}. \tag{A.3}$$

Here p, c_1, c_2 are arbitrary constants and it is assumed that $\lambda, \mu > 0$. Let us consider, for example, the case $\nu > 0$. When $\tau \to -\infty$ (we assume that $\lambda = \mu = \nu = 1$, $c_1 = c_2 = 0$, and $p = 1/2$) we have

$$c^2 = \frac{1}{\mathrm{ch}\,\tau} = 2e^\tau + O(e^{2\tau}), \quad a^2 = b^2 = \frac{\mathrm{ch}\,\tau}{4\,\mathrm{ch}^2(\tau/2)} = \frac{1}{2} + O(e^\tau). \tag{A.4}$$

In synchronous time, this is the metric (0, 0, 1) ($t \to 0$, $\tau \to -\infty$). The same holds also for $\nu < 0$. We note that by means of rotation in this plane, which is connected with the functions a and b, it is always possible to cause one of the scalar diagonal products of the form l curl l to vanish. Such a rotation is missing only when all the functions a, b, and c are different. But by taking the solution (A.2), (A.3) as the zero-th approximation, one can attempt to perturb it in such a way as to obtain a general solution with a small difference $\lambda a^2 - \mu b^2$.

Denoting a^2, b^2, and c^2 in accordance with (4.7), we write for each function the expansions

$$\chi = \chi_0 + \chi_1 + \chi_2 + \ldots, \quad e^\chi = e^{\chi_0} + e^{\chi_0}\chi_1 + e^{\chi_0}(\chi_2 + {}^1/_2\chi_1^2) + \ldots, \tag{A.5}$$

where χ_1 and χ_2 are quantities of first and second orders of smallness, and χ_0 is taken from the exact solution (A.2), (A.3). Writing the equations of first and second orders of smallness and solving them in the asymptotic region $\tau \to -\infty$, we obtain

$$\begin{gathered} \chi = \chi_0 + \alpha_3 + \alpha_4\cos\tau + \alpha_5\sin\tau + {}^1/_2(\alpha_4^2 + \alpha_5^2)\tau + \\ + (\alpha_3\alpha_5 - {}^1/_2\alpha_3\alpha_4)\cos\tau - (\alpha_3\alpha_4\tau + {}^1/_2\alpha_3\alpha_5)\sin\tau + O(\alpha^3), \\ \varphi = \varphi_0 + \alpha_3 - \alpha_4\cos\tau - \alpha_5\sin\tau + {}^1/_2(\alpha_4^2 + \alpha_5^2)\tau - \\ - (\alpha_3\alpha_5 - {}^1/_2\alpha_3\alpha_4)\cos\tau + (\alpha_3\alpha_4\tau + {}^1/_2\alpha_3\alpha_5)\sin\tau + O(\alpha^3), \\ \psi = \psi_0 + \alpha_1 + \alpha_2\tau + {}^1/_4(\alpha_4^2 + \alpha_5^2)\tau^2 - \\ - {}^1/_8(\alpha_4^2 - \alpha_5^2)\cos 2\tau - {}^1/_4\alpha_4\alpha_5\sin 2\tau + O(\alpha^3). \end{gathered} \tag{A.6}$$

Here α are arbitrarily small constants, and terms of the order e^τ and higher have been omitted. Altogether, the solution yields five arbitrary constants, i.e., as to many as should be possessed by the general solution of the system (4.2), (4.3). It follows from (A.6) that the expansions are good only up to certain limited values of τ, since they contain divergent expressions[8)]

$$\chi, \varphi \sim {}^1/_2(\alpha_4^2 + \alpha_5^2)\tau, \qquad \psi \sim {}^1/_4(\alpha_4^2 + \alpha_5^2)\tau^2. \tag{A.7}$$

This means that the functions a and b will not approach a constant value asymptotically, but will start to decrease. On the other hand, the functions c will begin to increase, i.e., we again arrive at oscillations. Such a result is obtained in any attempt to construct a solution in which it is possible to neglect any function compared with the two others (see Sec. 4).

Let us consider now Eqs. (3.2). By means of three-dimensional transformations $x^\alpha = x^\alpha(\bar{x}^\beta)$ it is always possible to fix a system of coordinates by means of the conditions $l_3 = m_3 = 0$, $n_3 = 1$, after which there remain also transformations of the type

$$x^a = x^a(\bar{x}^b), \qquad x^3 = \bar{x}^3 + f(\bar{x}^b) \quad (a, b = 1,2). \tag{A.8}$$

Then we get $\mathbf{l}\cdot\mathbf{m}\times\mathbf{n} = l_1 m_2 - l_2 m_1$, and the difference of the equations m curl l = 0 and l curl m = 0 yields

$$\mathbf{m}\,\mathrm{rot}\,\mathbf{l} - \mathbf{l}\,\mathrm{rot}\,\mathbf{m} = (l_1 m_2 - l_2 m_1)_{,3} = 0.$$

Thus, $\mathbf{l}\cdot\mathbf{m}\times\mathbf{n}$ does not depend on the variable z, and by choosing the Jacobian of the remaining two-dimensional transformations we can make l·m n equal to unity. Solving then Eqs. (3.2) under the conditions

$$l_3 = m_3 = 0, \qquad n_3 = 1, \qquad \mathbf{l}[\mathbf{mn}] = 1, \tag{A.9}$$

we obtain a general solution that contains two arbitrary functions of the variables x, y, and a certain number of arbitrary one-dimensional functions. However, after satisfying (A.9), the transformations (A.8) contain two other two-dimensional functions, and again a certain number of one-dimensional functions that depend on those coordinate conditions, which fix the remaining two-dimensional functions. It can be shown that by a special choice of the indicated arbitrary functions it is possible to reduce the solution to the form

$$\begin{gathered} l_1 = \sqrt{\lambda}\sqrt{\beta - \nu y^2}\sin\sqrt{\lambda\mu}\,z, \quad l_2 = \frac{\cos\sqrt{\lambda\mu}\,z}{\sqrt{\mu}\sqrt{\beta - \nu y^2}}, \quad l_3 = 0; \\ m_1 = -\sqrt{\mu}\sqrt{\beta - \nu y^2}\cos\sqrt{\lambda\mu}\,z, \quad m_2 = \frac{\sin\sqrt{\lambda\mu}\,z}{\sqrt{\lambda}\sqrt{\beta - \nu y^2}}, \quad m_3 = 0; \\ n_1 = -\nu y, \quad n_2 = 0, \quad n_3 = 1, \end{gathered} \tag{A.10}$$

where β is an arbitrary constant. In accordance with (A.10), the metric assumes the form

$$\begin{aligned} -ds^2 = -dt^2 &+ [a^2\lambda(\beta - \nu y^2)\sin^2\sqrt{\lambda\mu}\,z + b^2\mu(\beta - \nu y^2)\cos^2\sqrt{\lambda\mu}\,z \\ &+ c^2\nu^2 y^2]\,dx^2 + \left[\frac{a^2\cos^2\sqrt{\lambda\mu}\,z}{\mu(\beta - \nu y^2)} + \frac{b^2\sin^2\sqrt{\lambda\mu}\,z}{\lambda(\beta - \nu y^2)}\right]dy^2 + c^2\,dz^2 \\ &+ \left(a^2\sqrt{\frac{\lambda}{\mu}} - b^2\sqrt{\frac{\mu}{\lambda}}\right)\sin 2\sqrt{\lambda\mu}\,z\,dx\,dy - 2c^2\nu y\,dx\,dz. \end{aligned} \tag{A.11}$$

There can arise different cases, depending on the signs of the constants λ, μ, ν, β. In the case $\nu > 0$, the metric (A.11) reduces by a transformation $y = \sqrt{\beta/\nu}\cos\bar{y}$ to the form of a metric of the ninth type (in accordance with the Bianchi classification) admitting of a group of motions G_3 on V_3. Then the particular solution (A.2), (A.3) turns out to be Taub's solution[4]

We present finally formulas for the invariants of the metric (3.1). The latter are the eigenvalues of the λ matrix of the Riemann tensor in bivector space. We note that in the calculation of the Petrov matrices M, and N in the triad al, bm, cn, the latter are immediately obtained in canonical form of type I:

$$\lambda_1 = \ddot{a}/a, \quad \lambda_2 = \ddot{b}/b, \quad \lambda_1 + \lambda_2 + \lambda_3 = 0; \tag{A.12}$$

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8)The remaining divergences are only illusory, since they are connected with the variation of the argument of the exponential of the zero-th approximation of the function c, and the variation of the frequencies of the first approximation of the functions a and b.

$$\lambda_4 = \frac{1}{2abc}\left[\mu b\dot{b} + \nu c\dot{c} - 2\lambda a\dot{a} + \frac{\dot{b}}{b}(\lambda a^2 - \nu c^2) + \frac{\dot{c}}{c}(\lambda a^2 - \mu b^2)\right] \quad \text{(A.13)}$$

$$\lambda_5 = \frac{1}{2abc}\left[\lambda a\dot{a} + \nu c\dot{c} - 2\mu b\dot{b} + \frac{\dot{a}}{a}(\mu b^2 - \nu c^2) + \frac{\dot{c}}{c}(\mu b^2 - \lambda a^2)\right]$$

$$\lambda_4 + \lambda_5 + \lambda_6 = 0.$$

A qualitative analysis shows that the indicated invariants increase on approaching the singularity $t = 0$.

[1] E. M. Lifshits and I. M. Khalatnikov, Usp. Fiz. Nauk **80**, 391 (1963) [Sov. Phys.-Usp. **6**, 495 (1963/64)].
[2] A. Z. Petrov, Prostranstva Éĭnshteina (Einstein Spaces), Fizmatgiz, 1961.
[3] R. Penrose, Phys. Rev. Lett. **14**, 57 (1965).
[4] A. Taub, Ann. Math **53**, 472 (1951).

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