CRITICAL CURRENT OF THIN FILMS FOR DIFFUSE REFLECTION FROM THE WALLS

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Submitted October 31, 1968

Zh. Eksp. Teor. Fiz. 56, 1590–1607 (May, 1969)

Boundary conditions for the Green's function are obtained for diffuse reflection of electrons by the sample boundaries. The critical current and excitation spectrum of current-carrying thin films are found and the excitation spectrum in strong magnetic fields is determined. The dependence of the penetration depth on the field strength is considered.

In the limiting case of strongly contaminated superconductors ($l \ll d$), the problem of calculating the critical current of thin films was solved by Maki[1]. When $l \ll d$, it is convenient to use a quasiclassical method, which makes it possible to reduce Gor'kov's equations[2] to a simpler system of equations for the generalized distribution function, which equals the Green's function at the coinciding points[3,4]. Such a reduction is possible in an arbitrary magnetic field and without assuming a Born character of the scattering by impurities[4]. To solve these equations, it is necessary to know the boundary conditions. It is usually assumed that the electrons are diffusely scattered from the surface of the metal. To obtain the boundary conditions for an arbitrary gap and the arbitrary magnetic field, we replace below the diffusely reflecting boundary by a boundary coated with a thin layer of scattering centers with specially chosen scattering amplitude. This choice is carried out in such a way as to obtain, when applied to the normal metal, the usual diffuse boundary condition for the distribution function. The obtained boundary condition is used to calculate the critical current and the excitation spectrum of thin films. We also consider the question of the dependence of the depth of penetration on the field.

1. DERIVATION OF BOUNDARY CONDITIONS

In the quasiclassical approximation, the system of equations for the Green's function

$$G_{\mathbf{p}}(r) = \frac{i}{\pi}\int G_{\mathbf{p}}(\mathbf{r};\,\xi)\,d\xi$$

is given by[4]

$$\left(\mathbf{v}\frac{\partial}{\partial \mathbf{r}}\right)G_{\mathbf{p}}(\mathbf{r}) + G_{\mathbf{p}}(\mathbf{r})\hat{\omega} - \hat{\omega}G_{\mathbf{p}}(\mathbf{r}) = 0,$$

$$\hat{\omega} = \omega\tau_z - ie(\mathbf{v}\mathbf{A})\tau_z - i\hat{\Delta} + in\Sigma_{\mathbf{pp}}(\mathbf{r}), \tag{1}$$

$$\Sigma_{\mathbf{pp}'}(\mathbf{r}) = \chi_{\mathbf{pp}'} - \frac{i\vartheta}{4}\int \chi_{\mathbf{pp}_1}G_{\mathbf{p}_1}(\mathbf{r})\Sigma_{\mathbf{p}_1\mathbf{p}'}(\mathbf{r})\,d\Omega_{\mathbf{p}_1};$$

$$\hat{\Delta} = \begin{pmatrix} 0 & \Delta \\ -\Delta^* & 0 \end{pmatrix}, \quad G_{\mathbf{p}}^2(\mathbf{r}) = 1, \quad \mathrm{Sp}\,G_{\mathbf{p}}(\mathbf{r}) = 0, \tag{2}$$

where $\mathbf{v}$ is the electron velocity, n the impurity concentration, $\vartheta = mp_0/2\pi^2$ is the density of states on the Fermi surface, and $\chi_{\mathbf{pp}'}$ is connected with the scattering amplitude by the relation

$$f_{\mathbf{pp}'} = \chi_{\mathbf{pp}'} - \frac{i\vartheta}{4}\int \chi_{\mathbf{pp}_1}f_{\mathbf{p}_1\mathbf{p}'}\,d\Omega_{\mathbf{p}_1}. \tag{3}$$

We replace the diffusely-reflecting boundary by a boundary coated with a thin layer of scattering centers (the free path in this layer is much smaller than the thickness of the layer), and we choose the scattering amplitude $\tilde{f}_{\mathbf{pp}'}$ of these centers in such a way as to obtain the diffuse boundary condition in the limit as $\Delta \to 0$. In the thin boundary layer, we can omit from the system (1) and (2) the terms proportional to ω, Δ, and A. At distances much larger than the free path inside the layer, the Green's function is of the form

$$G_{\mathbf{p}}(\mathbf{r}) = (\mathbf{n}_0\boldsymbol{\tau}), \tag{4}$$

where $\mathbf{n}_0$ is a unit vector independent of the angles and coordinates, $\mathbf{n}_0^2 = 1$. We choose the first two unit vectors $\mathbf{n}_1$ and $\mathbf{n}_2$ in such a way that

$$(\mathbf{n}_i\mathbf{n}_k) = \delta_{ik}, \quad i = 0, 1, 2, \\ (\mathbf{n}_1\boldsymbol{\tau})(\mathbf{n}_0\boldsymbol{\tau}) = -i(\mathbf{n}_2\boldsymbol{\tau}). \tag{5}$$

To obtain the diffuse boundary condition in the limit as $\Delta \to 0$, it is necessary to stipulate that the scattering amplitude in the boundary layer $\tilde{f}_{\mathbf{pp}'}$ be different from zero only on the hemisphere

$$p_z p_z' > 0. \tag{6}$$

The z axis is directed here along the inward normal to the surface. In all other respects, the function $\tilde{f}_{\mathbf{pp}'}$ is arbitrary. Writing the Green's function $G_{\mathbf{p}}(\mathbf{r})$ in the form

$$G_{\mathbf{p}}(\mathbf{r}) = iC_1(\mathbf{n}_1\boldsymbol{\tau}) + C_0(\mathbf{n}_0\boldsymbol{\tau}) + C_2(\mathbf{n}_2\boldsymbol{\tau}) \tag{7}$$

and substituting this expression in the system (1) and (2), we obtain a system of equations for the coefficients C_0, C_1, and C_2. As will be seen from the following, C_0 is constant accurate to terms of order $\exp(-\delta/v\tilde{\tau}) \ll 1$:

$$C_0 = 1 + O(\exp(-\delta/v\tilde{\tau})); \tag{8}$$

here δ is the thickness of the surface layer and $v\tilde{\tau}$ is the electron mean free path in this layer. With the same accuracy, the equations for C_0, C_1, and C_2 are of the form

$$\left(\mathbf{v}\frac{\partial}{\partial \mathbf{r}}\right)(C_1 + C_2) + \tilde{\tau}^{-1}(C_1 + C_2) - \tilde{n}v\int \tilde{\sigma}_{\mathbf{pp}_1}(C_1 + C_2)_{\mathbf{p}_1}\,d\Omega_{\mathbf{p}_1} = 0,$$

$$\left(\mathbf{v}\frac{\partial}{\partial \mathbf{r}}\right)(C_1 - C_2) - \tilde{\tau}^{-1}(C_1 - C_2) + \tilde{n}v\int \tilde{\sigma}_{\mathbf{pp}_1}(C_1 - C_2)_{\mathbf{p}_1}\,d\Omega_{\mathbf{p}_1} = 0,$$

$$\left(\mathbf{v}\frac{\partial}{\partial \mathbf{r}}\right)C_0 = O(C_1^2 - C_2^2); \quad \tilde{\tau}^{-1} = \tilde{n}v\int \tilde{\sigma}_{\mathbf{pp}_1}\,d\Omega_{\mathbf{p}_1}. \tag{9}$$

From the system (9), under the condition (6), we obtain the boundary condition on the Green's function $G_{\mathbf{p}}(\mathbf{r})$:

$$p_0(C_1+C_2)_{p_z>0}=\frac{1}{\pi}\int_{p_z<0}(C_1+C_2)|p_z|d\Omega_{\mathbf p}=0,$$

$$p_0(C_1-C_2)_{p_z<0}=\frac{1}{\pi}\int_{p_z>0}(C_1-C_2)p_z d\Omega_{\mathbf p}=0. \tag{10}$$

Thus, the Green's function $G_p(\mathbf r)$ on the surface is given by

$$G_{\mathbf p}(\mathbf r)=iC_1(\mathbf n_1\tau)+(\mathbf n_0\tau)+C_2(\mathbf n_2\tau), \tag{11}$$

where the vectors $\mathbf n_0$, $\mathbf n_1$, and $\mathbf n_2$ satisfy the condition (5) and the coefficients C_1 and C_2 satisfy the condition (10).

2. CALCULATION OF THE CRITICAL VALUE OF A

The critical value of the vector potential **A** is determined from the condition that Δ vanish. In a thin film with current, **A** is directed along the film and is independent of the coordinates. In the approximation linear in Δ, we obtain from (1) and (2)

$$G=iC_1\tau_x+\tau_z\operatorname{sign}\omega+C_2\tau_y\operatorname{sign}\omega,$$

$$\Delta=\frac{|\lambda|mp_0}{8\pi^2}T\sum_\omega\int[C_1+C_2\operatorname{sign}\omega]d\Omega_{\mathbf p}; \tag{12}$$

$$\left(\mathbf v\frac{\partial}{\partial\mathbf r}\right)(C_1+C_2)+2\left[|\omega|+\frac{1}{2\tau}-ie(\mathbf{vA})\operatorname{sign}\omega\right](C_1+C_2)$$
$$-vn\int\sigma_{\mathbf{pp}_1}(C_1+C_2)_{\mathbf p_1}d\Omega_{\mathbf p_1}=2\Delta\operatorname{sign}\omega,$$
$$\left(\mathbf v\frac{\partial}{\partial\mathbf r}\right)(C_1-C_2)-2\left[|\omega|+\frac{1}{2\tau}-ie(\mathbf{vA})\operatorname{sign}\omega\right](C_1-C_2)$$
$$+vn\int\sigma_{\mathbf{pp}_1}(C_1-C_2)_{\mathbf p_1}d\Omega_{\mathbf p_1}=2\Delta\operatorname{sign}\omega. \tag{13}$$

The boundary condition for the coefficients C_1 and C_2 is written in the form

$$p_0(C_1+C_2)_{(\mathbf{pn})>0}=\frac{1}{\pi}\int_{(\mathbf{pn})<0}|(\mathbf{pn})|(C_1+C_2)_{\mathbf p}d\Omega_{\mathbf p},$$
$$p_0(C_1-C_2)_{(\mathbf{pn})<0}=\frac{1}{\pi}\int_{(\mathbf{pn})>0}(\mathbf{pn})(C_1-C_2)_{\mathbf p}d\Omega_{\mathbf p}, \tag{14}$$

where **n** is the inward normal.

The boundary condition (14) goes over into (10) if the vector $\mathbf n_0$ is suitably chosen. To find the equation for Δ, we need only the first equation of (13). The solution of this equation is

$$C_1+C_2=2(\operatorname{sign}\omega)\int_{-d/2}^{d/2}\mathcal K_\omega(\mathbf p,y,y_1)\Delta(y_1)dy_1. \tag{15}$$

The coordinate system is chosen here such that the axes x and z are directed along the film, and $y=\pm d/2$ on the film boundaries.

The kernel $\mathcal K_\omega(\mathbf p, y, y')$ satisfies the equation

$$\left(\mathbf v\frac{\partial}{\partial\mathbf r}\right)\mathcal K_\omega(\mathbf p,y,y')+2\left[|\omega|+\frac{1}{2\tau}-ie(\mathbf{vA})\operatorname{sign}\omega\right]\mathcal K_\omega(\mathbf p,y,y')$$
$$-vn\int\sigma_{\mathbf{pp}_1}\mathcal K_\omega(\mathbf p_1,y,y')d\Omega_{\mathbf p_1}=\delta(y-y') \tag{16}$$

and the first boundary condition (14). We shall need henceforth only $\omega>0$, so that we shall imply $\omega>0$ throughout. In the case of isotropic scattering, Eq. (16) can be reduced to the integral equation

$$\mathcal K_\omega(\mathbf p,y,y')=\mathcal K^0_{\omega+1/2\tau}(\mathbf p,y,y')$$
$$+vn\int\sigma_{\mathbf{pp}_1}\int_{-d/2}^{d/2}\mathcal K^0_{\omega+1/2\tau}(\mathbf p,y,y_1)\mathcal K_\omega(\mathbf p_1,y_1,y')dy_1d\Omega_{\mathbf p}, \tag{17}$$

where the kernel $\mathcal K^0_\omega(\mathbf p, y, y')$ satisfies Eq. (16) at $\sigma=1/\tau=0$, as well as the first boundary condition of (14). From (12) and (15) we obtain

$$\Delta(y)=\frac{|\lambda|mp_0}{2\pi^2}T\sum_{\omega>0}\int d\Omega_{\mathbf p}\int_{-d/2}^{d/2}\mathcal K_\omega(\mathbf p,y,y_1)\Delta(y_1)dy_1. \tag{18}$$

Solving Eq. (16) at $\sigma=1/\tau=0$, we obtain the following expression for the kernel $\mathcal K^0_\omega(\mathbf p, y, y')$:

$$\mathcal K_\omega^0(\mathbf p,y,y')=e^{-\lambda y}\Big\{[e^{-\lambda d/2}\alpha(y')\theta(\sin\varphi)+e^{\lambda d/2}\beta(y')\theta(-\sin\varphi)]$$
$$+\frac{e^{\lambda y'}}{v\sin\theta|\sin\varphi|}[\theta(\sin\varphi)\theta(y-y')+\theta(-\sin\varphi)\theta(y'-y)]\Big\}, \tag{19}$$

where

$$\lambda=\frac{2(\omega-ievA\cos\theta)}{v\sin\theta\sin\varphi},$$
$$\alpha(y)=\beta(-y)=\frac{D(d/2+y)+\Gamma(d)D(d/2-y)}{v[1-\Gamma^2(d)]},$$
$$\Gamma(d)=2\int_0^1dt\cdot t\exp\left(-\frac{2|\omega|d}{vt}\right)J_0\left(\frac{2eAd\sqrt{1-t^2}}{t}\right),$$
$$D(y)=2\int_0^1dt\exp\left(-\frac{2|\omega|y}{vt}\right)J_0\left(\frac{2eAy\sqrt{1-t^2}}{t}\right). \tag{20}$$

We note that from formulas (14)—(18) we can easily obtain an integral equation for the determination of the critical magnetic field at an arbitrary amplitude of the scattering by impurities. In this case we must put $A=Hy$ in lieu of $A=\text{const}$.

In the case when a magnetic field is applied to the film, the parameter eHd^2 can be arbitrary. For a thin film with current, the corresponding parameter eAd is always small. Therefore all the kernels can be averaged over the coordinates. Assuming for simplicity an isotropic character of scattering by the impurities, we obtain from (18) and (20)

$$1=\frac{|\lambda|mp_0}{2\pi^2}T\sum_{\omega>0}\mathcal K_\omega, \tag{21}$$

where

$$\mathcal K_\omega=\left[(\mathcal K^0_{\omega+1/2\tau})^{-1}-\frac{1}{4\pi\tau}\right]^{-1},$$
$$\mathcal K_\omega^0=\frac{1}{d}\int d\Omega_{\mathbf p}\int\int_{-d/2}^{d/2}\mathcal K_\omega^0(\mathbf p,y,y_1)dy\,dy_1. \tag{22}$$

Let us consider the case of a large mean free path compared with the thickness of the film, $l\gg d$. Then the values of importance in (21) are $\omega\ll v/d$. Using the relation

$$\int_0^1\frac{dt}{t}\exp\left(-\frac{\alpha}{t}\right)J_0\left(\frac{\beta\sqrt{1-t^2}}{t}\right)=\int_\alpha^\infty dx\frac{\exp(-\sqrt{x^2+\beta^2})}{\sqrt{x^2+\beta^2}}$$
$$=K_0(\beta)-\int_0^\alpha dx\frac{\exp(-\sqrt{x^2+\beta^2})}{\sqrt{x^2+\beta^2}}, \tag{23}$$

we obtain the following expression for $\mathcal K^0_\omega$:

$$\mathcal K_\omega^0=\frac{2\pi}{\omega}\begin{cases}\left[1+eAd\dfrac{evA}{2\omega}\left(\dfrac12+\ln\dfrac{1}{\gamma eAd}\right)\right]^{-1}, & eAd\gg\dfrac{\omega d}{v},\\[2ex] \left[1+eAd\dfrac{evA}{2\omega}\left(1+\ln\dfrac{v}{2\gamma\omega d}\right)\right]^{-1}, & eAd\ll\dfrac{\omega d}{v},\end{cases} \tag{24}$$

where $\ln\gamma=C$ is the Euler constant.

From (21), (22), and (24) we obtain an equation for the critical value of A:

$$\psi\left[\frac12+eAd\frac{evA}{4\pi T}\left(\frac12+\ln\frac{1}{\gamma eAd}\right)\right]-\psi\left(\frac12\right)=\ln\left(\frac{T_c}{T}\right),$$

$$eA \gg \xi_0^{-1},\ (\tau v)^{-1}; \tag{25}$$

$$\psi\left[\frac{1}{2}+eAd\frac{evA}{4\pi T}\left(1+\ln\left(\frac{v\tau}{\gamma d}\right)\right)\right]-\psi\left(\frac{1}{2}\right)=\ln\left(\frac{T_c}{T}\right),$$

$$eA \ll (\tau v)^{-1},\quad d \ll \tau v \ll \xi_0. \tag{26}$$

Here $\psi(x)$ is the Psi function. From (25) and (26) it follows that even when $T \to 0$ we have $(eAd)^2 \sim d/\xi_0 \ll 1$.

Here T_c, formulas (21)–(24) enable us to find the dependence of Δ and of the current density j on A, and consequently the critical current j_{cr}. The appropriate calculation will be made at the end of Sec. 3.

3. CRITICAL CURRENT OF PURE FILMS

As indicated in Sec. 2, for thin films ($d \ll \xi_0$) we always have $eAd \ll 1$, so that we can assume that Δ is independent of the coordinates. We seek the Green's function $G_p(\mathbf{r})$ in the form

$$G_p(\mathbf{r}) = iC_1\tau_x + C_0(\mathbf{n}_0\boldsymbol{\tau}) + C_2(\mathbf{n}_2\boldsymbol{\tau}), \tag{27}$$

where

$$(\mathbf{n}_0\boldsymbol{\tau}) = \alpha\tau_z + \beta\tau_y,\quad (\mathbf{n}_2\boldsymbol{\tau}) = -\beta\tau_z + \alpha\tau_y,\quad \alpha^2+\beta^2=1. \tag{28}$$

Substituting the expression (27) for the Green's function $G_p(\mathbf{r})$ in (1), we obtain a system of equations for the coefficients C_0, C_1, and C_2. Solving this system, we obtain

$$\begin{pmatrix} C_1 \\ C_0 \\ C_2 \end{pmatrix} = D_1\begin{pmatrix} 0 \\ B_1 \\ B_2 \end{pmatrix} + D_2\begin{pmatrix} \sqrt{B_1^2+B_2^2}\,\mathrm{sh}(\lambda y) \\ B_2\,\mathrm{ch}(\lambda y) \\ -B_1\,\mathrm{ch}(\lambda y) \end{pmatrix}, \tag{29}$$

where

$$\lambda = \frac{2\sqrt{B_1^2+B_2^2}}{v\sin\theta\sin\varphi},\qquad \begin{aligned} B_1 &= \beta\Delta + \alpha(\omega - ievA\cos\theta), \\ B_2 &= \alpha\Delta - \beta(\omega - ievA\cos\theta). \end{aligned} \tag{30}$$

Here the axes x and z are directed along the film, $y = \pm d/2$ on the film boundaries, and the field is parallel to the z axis. Using the boundary conditions (10), we obtain the following expressions for the coefficients D_1 and D_2:

$$D_1 = \left[B_1\,\mathrm{ch}\left(\frac{\bar\lambda d}{2}\right)+\sqrt{B_1^2+B_2^2}\,\mathrm{sh}\left(\frac{\bar\lambda d}{2}\right)\right]\left[(B_1^2+B_2^2)\,\mathrm{ch}\left(\frac{\bar\lambda d}{2}\right)\right.$$
$$\left.+B_1(B_1^2+B_2^2)^{1/2}\,\mathrm{sh}\left(\frac{\bar\lambda d}{2}\right)\right]^{-1}, \tag{31}$$

$$D_2 = B_2\left[(B_1^2+B_2^2)\,\mathrm{ch}\left(\frac{\bar\lambda d}{2}\right)+B_1(B_1^2+B_2^2)^{1/2}\,\mathrm{sh}\left(\frac{\bar\lambda d}{2}\right)\right]^{-1},$$

where

$$\bar\lambda = 2(B_1^2+B_2^2)^{1/2}/v\sin\theta|\sin\varphi|. \tag{32}$$

The condition (10) yields one more equation for the determination of the coefficients α and β:

$$\int_0^1 z\,dz\int_0^{2\pi}\frac{1}{Z(z,\varphi)}[\alpha\Delta-\beta(\omega-ievA\sqrt{1-z^2}\cos\varphi)]\,d\varphi = 0,$$

$$Z(z,\varphi) = [\Delta^2+(\omega-ievA\sqrt{1-z^2}\cos\varphi)^2]^{1/2}\cdot$$
$$\times\mathrm{cth}\left[\frac{d}{vz}(\Delta^2+(\omega-ievA\sqrt{1-z^2}\cos\varphi)^2)^{1/2}\right]$$
$$+(\beta\Delta+\alpha(\omega-ievA\sqrt{1-z^2}\cos\varphi)). \tag{33}$$

Calculating the integrals in (33) in the region $\omega d/v \ll 1$ of interest to us, we obtain

$$\alpha\Delta-\beta\omega = \alpha\beta eAd\frac{evA}{2}\left[\frac{1}{2}+\ln\frac{1}{\gamma eAd}+(1-\alpha^2)f(\alpha)\right]\qquad evA \gg \omega,$$

$$\alpha\Delta-\beta\omega = \alpha\beta eAd\frac{evA}{2}\ln\left(\frac{v}{2\gamma d\sqrt{\omega^2+\Delta^2}}\right)\qquad evA \ll \omega, \tag{34}$$

where

$$f(\alpha) = \int_0^\infty\frac{dx}{x^3}\frac{\sin^4 x}{\cos^2 x+\alpha^2\sin^2 x}. \tag{35}$$

The asymptotic expressions for the function $f(\alpha)$ are

$$f(\alpha) = \begin{cases} \dfrac{7}{\pi^2(\alpha^2)^{1/2}}\zeta(3) - \displaystyle\int_0^\infty\frac{dt}{t^3}\left(\frac{1}{\mathrm{ch}^2 t}+t+\frac{1}{2}e^{-2t}-\frac{3}{2}\right), & \alpha \ll 1 \\ \ln 2 + (1-\alpha^2)(8\ln 2 - 3\ln 3), & 1-\alpha^2 \ll 1 \end{cases} \tag{36}$$

where $\zeta(x)$ is the Riemann Zeta function.

The ordering parameter Δ and the current density j are expressed in terms of the Green's function by means of the formulas

$$\Delta = -\frac{|\lambda| mp_0}{8\pi^2}T\sum_\omega\int[\beta C_0+\alpha C_2]\,d\Omega_{\mathbf{p}},$$

$$j = -\frac{iep_0}{4\pi^2}T\sum_\omega\int\mathbf{p}[\alpha C_0-\beta C_2]\,d\Omega_{\mathbf{p}}. \tag{37}$$

Substituting here the expression for the coefficients C_0 and C_2 from formulas (29)–(32), we obtain in the region $evA \gg \pi T$

$$\Delta = \frac{|\lambda| mp_0}{2\pi}T\sum_\omega\beta(\omega),$$

$$j = -eAd\frac{ep_0^2}{2\pi}T\sum_\omega(1-\alpha^2)\left[\ln\frac{1}{\gamma eAd}+(1-\alpha^2)\chi(\alpha)\right], \tag{38}$$

where α and β are determined by formulas (28) and (34), and

$$\chi(\alpha) = \int_0^\infty\frac{dt}{t^2}\frac{\sin^3 t\cos t}{\cos^2 t+\alpha^2\sin^2 t}. \tag{39}$$

For the function $\chi(\alpha)$ we have

$$\chi(\alpha) = \begin{cases} \dfrac{1}{2}\displaystyle\int_0^\infty\frac{dx}{x^2}\left(\frac{\mathrm{sh}\,2x}{\mathrm{ch}^2 x}-\sin 2x\right)-\frac{14(\alpha^2)^{1/2}}{\pi^2}\zeta(3), & \alpha \ll 1 \\ {}^1/_2\ln 2 + {}^1/_3(1-\alpha^2)(8\ln 2-3\ln 3), & 1-\alpha^2 \ll 1 \end{cases} \tag{40}$$

In the temperature region $d/\xi_0 \ll (T_c - T)/T_c \ll 1$ we obtain from (34) and (37) the following expressions for Δ and for the current:

$$\frac{T_c-T}{T_c} = \frac{7\Delta^2}{8\pi^2 T_c^2}\zeta(3)+\frac{\pi}{8T_c}(eAd)(evA)\left(\frac{1}{2}+\ln\frac{1}{\gamma eAd}\right),$$

$$j = -eAd\frac{ep_0^2}{8\pi T}\Delta^2\ln\frac{1}{\gamma eAd}. \tag{41}$$

With logarithmic accuracy we obtain from (41) the critical current

$$j_{cr} = \frac{4ep_0^2(T_c-T)^{3/2}}{63\zeta(3)}\left[\frac{3\pi d}{v}\ln\left(\frac{v}{d(T_c-T)}\right)\right]^{1/2}. \tag{42}$$

The critical current is reached when

$$(eAd)_{cr.c} \approx 4\left[\frac{d(T_c-T)}{3\pi v}\Big/\ln\left(\frac{v}{d(T_c-T)}\right)\right]^{1/2}. \tag{43}$$

This quantity is smaller by a factor $\sqrt{3}$ than the critical value eAd, at which Δ vanishes.

In a very narrow region near T_c, $(T_c - T)/T_c \ll d/\xi_0$, we get in lieu of formulas (41) and (42)

$$j = -eAd\frac{ep_0^2}{8\pi T}\Delta^2\left[1+\ln\left(\frac{v}{2\pi\gamma dT_c}\right)+\frac{1}{3}\ln 2+\frac{\zeta'(2)}{\zeta(2)}\right],$$

$$\Delta^2 = \frac{8\pi^2T_c^2}{7\zeta(3)}\left[\frac{T_c-T}{T_c}-eAd\frac{evA}{8T_c}\pi\left(1+\ln\frac{v}{2\pi\gamma dT_c}+\frac{1}{3}\ln 2+\frac{\zeta'(2)}{\zeta(2)}\right)\right] \tag{44}$$

$$j_{cr} = \frac{4ep_0^2}{21\zeta(3)}(T_c-T)^{3/2}\left(\frac{2\pi d}{3v}\right)^{1/2}\left[1+\ln\frac{v}{2\pi\gamma dT_c}+\frac{1}{3}\ln 2+\frac{\zeta'(2)}{\zeta(2)}\right]^{1/2}. \tag{44}$$

The dependence of the critical current on the temperature away from T_c can be determined from formulas (34), (35), and (38). However, it is impossible to obtain a closed expression for the critical current in analytic form, and a numerical calculation is necessary. At T = 0 we obtain from (34) and (38), with logarithmic accuracy

$$\ln\frac{\Delta_0}{\Delta}=\frac{\pi}{4}x,\quad j=-\frac{mp_0\Delta^2}{2\pi A}x\left(1-\frac{4}{3\pi}x\right),\quad x\leqslant 1; \tag{45}$$

$$\ln\frac{\Delta_0}{\Delta}=\ln(x+\sqrt{x^2-1})+\frac{x}{2}\arcsin\left(\frac{1}{x}\right)-\frac{1}{2x}\sqrt{x^2-1},$$
$$j=-\frac{mp_0\Delta^2}{\pi^2 A}x\left[\arcsin\left(\frac{1}{x}\right)-\frac{2x}{3}+\frac{1}{3}\left(2+\frac{1}{x^2}\right)\sqrt{x^2-1}\right],\quad x\geqslant 1, \tag{46}$$

where

$$x=eAd\frac{evA}{2\Delta}\ln\frac{1}{\gamma eAd}, \tag{47}$$

and Δ_0 is the value of the gap in the absence of the current.

Analogous formulas are obtained in an analysis of superconductors of small dimensions in a strong magnetic field[1,5]. The values x > 1 correspond to gapless superconductivity. In our case, the points x ≥ 1 lie in the unstable region, to the right of the point of the current maximum on the (j, A) diagram.

From (45) we obtain an equation for x_0, at which the maximum current is reached

$$1-\frac{4x_0}{3\pi}=2\left[1-\frac{8x_0}{3\pi}-\frac{\pi x_0}{4-\pi x_0}\right]. \tag{48}$$

Solving Eq. (48), we get $x_0 = 0.3$, and substituting this value of x_0 in the expression for the curve (45), we obtain

$$j_{\rm cr}=0.027ep_0^2\Delta_0^{3/2}\left(\frac{d}{v}\ln\frac{v}{\Delta_0 d}\right)^{1/2}. \tag{49}$$

We note that the character of the reflection of the electrons from the surface affects strongly the magnitude of the critical current. In the case of specular reflection, the density of the critical current does not depend in general on the thickness of the film.

The results obtained in this section pertain to the case of pure films. Near T_c, however, we can obtain an expression for j_{cr} also in the case $d \ll l = \tau v$, using only the form of the kernel $\mathcal{K}_\omega$ (formulas (21)–(24)). When Δ is finite, we obtain in lieu of (21)

$$1=\frac{|\lambda|mp_0}{2\pi}T\sum_\omega\left\{\frac{1}{\sqrt{\omega^2+\Delta^2}}-eAd\frac{evA}{2\omega^2}\left(1+\ln\frac{v}{2\gamma d(|\omega|+1/2\tau)}\right)\right\},$$
$$\tau v\gg d,\quad T_c\gg T_c-T. \tag{50}$$

From (50) in the vicinity of $\tau v \ll \xi_0$ we obtain an expression for Δ:

$$\frac{T_c-T}{T_c}=\frac{7\Delta^2}{8\pi^2T_c^2}\zeta(3)+eAd\frac{evA}{8T}\pi\left(1+\ln\frac{v\tau}{\gamma d}\right). \tag{51}$$

The current is equal to the derivative of the thermodynamic potential Ω_s with respect to the potential A:

$$j=-\frac{\delta\Omega_s}{\partial A}=-\frac{\partial\Omega_s}{\partial\Delta^2}\frac{\partial\Delta^2}{\partial A}. \tag{52}$$

Near T_c we have[2]

$$\Omega_s-\Omega_n=\int_0^{|\lambda|}\frac{d(1/|\lambda|)}{d\Delta}\Delta^2d\Delta=-\frac{7\Delta^4}{32}\zeta(3)\frac{mp_0}{\pi^4T_c^2}. \tag{53}$$

Differentiating (53) with allowance for (51), we obtain

$$j=-eAd\frac{ep_0^2}{8\pi T_c}\Delta^2\left(1+\ln\frac{v\tau}{\gamma d}\right). \tag{54}$$

From (51) and (54) we obtain an expression for the critical current

$$j_{\rm cr}=\frac{4ep_0^2}{21\zeta(3)}(T_c-T)^{3/2}\left(\frac{2\pi d}{3v}\right)^{1/2}\left(1+\ln\frac{v\tau}{\gamma d}\right)^{1/2},$$
$$d\ll\tau v\ll\xi_0. \tag{55}$$

4. EXCITATION SPECTRUM

At T = 0, the density of states is expressed in terms of the Green's function $G_p(r)$ in accordance with the formula

$$\rho=\frac{1}{2}\rho_0\,{\rm Im}\left[\frac{i}{4\pi}{\rm Sp}\,\tau_z\int G_{-i\omega}(\mathbf{p})d\Omega_{\mathbf{p}}\right], \tag{56}$$

where ρ_0 is the density of states in the normal metal, $G_{-i\omega}$ is the analytic continuation of the function G_ω from the imaginary axis to the real axis. It can be easily shown that when evA < Δ, there is a finite gap in the spectrum. In the region evA > Δ there appears a finite density of states at ε = 0. The region evA ∼ Δ is of no interest, since this is the region of very small currents. We shall therefore consider directly the case evA ≫ Δ. From (27), (29), and (56) it follows that when ε = 0 we can write for the density of states

$$\rho=\rho_0\,{\rm Im}\,i\alpha(\omega=0). \tag{57}$$

In the stable region of the currents, the density of states at ε = 0 turns out to be logarithmically small

$$\rho=\rho_0\frac{7\zeta(3)}{\pi^2}eAd\frac{evA}{2\Delta}\Big/\left[1-eAd\frac{evA}{2\Delta}\ln\left(\frac{\Gamma}{eAd}\right)\right], \tag{58}$$

where

$$\ln\Gamma=\frac{1}{2}-\ln\gamma-\int_0^\infty\frac{dt}{t^3}\left[\frac{1}{{\rm ch}^2t}+t+\frac{1}{2}e^{-2t}-\frac{3}{2}\right].$$

The logarithmic smallness and the density of states at ε = 0 remains up to the point x = 1, which lies in the unstable region near the critical value of the vector potential A_{cr}. The density of states at ε = 0 becomes of the order of ρ_0 only when x > 1.

5. DEPENDENCE OF DEPTH OF PENETRATION ON THE MAGNETIC FIELD

The correction to the depth of penetration was first considered by Rusinov and Shapoval. In[6], the reflection of the electrons on the boundary was assumed to be specular, and in[7] it was assumed to be diffuse. The use of the diffuse boundary condition, proposed in Sec. 1, makes it possible to calculate the correction to the depth of penetration in a much simpler manner. In the London and in the strong-Pippard cases, the results agree with those of Shapoval[7], but they differ in the intermediate region δ ∼ ξ_0. Just as in[7], we consider only the case of a pure sample. We choose the coordinate system in such a way that the vector potential A(z) is directed along the x axis, and the sample occupies the half-space z > 0. As before, we seek the Green's function $G_p(r)$ in the form

$$G_{\mathbf{p}}(\mathbf{r})=iC_1\tau_x+C_0(\mathbf{n}_0\boldsymbol{\tau})+C_2(\mathbf{n}_2\boldsymbol{\tau}), \tag{59}$$

where

$$(\mathbf{n}_0\boldsymbol{\tau})=\cos\chi\left(\frac{\Delta_0}{E}\tau_y+\frac{\omega}{E}\tau_z\right)+\sin\chi\left(\frac{\omega}{E}\tau_y-\frac{\Delta_0}{E}\tau_z\right),$$

$$(\mathbf{n}_2\boldsymbol{\tau}) = -\sin\chi\left(\frac{\Delta_0}{E}\tau_y + \frac{\omega}{E}\tau_z\right) + \cos\chi\left(\frac{\omega}{E}\tau_y - \frac{\Delta_0}{E}\tau_z\right),$$

$$E = (\omega^2 + \Delta_0^2)^{1/2}. \quad (60)$$

Here Δ_0 is the magnitude of the gap in the absence of the field, and the angle χ is determined by the boundary condition (10). In the chosen gauge of the vector potential, the correction to Δ and the angle χ appear only in the second order in the field.

Expanding the coefficients C_i in powers of the vector potential A(z), and substituting this expansion in (1) in each order in the field, we obtain the system of equations

$$\left(\mathbf{v}\frac{\partial}{\partial \mathbf{r}}\right)B_1 + 2EF_1 = \frac{2ie}{m}(\mathbf{pA})\frac{\Delta_0}{E}, \quad \left(\mathbf{v}\frac{\partial}{\partial \mathbf{r}}\right)F_1 + 2EB_1 = 0,$$

$$\left(\mathbf{v}\frac{\partial}{\partial \mathbf{r}}\right)B_2 + 2EF_2 = -2E\chi + \frac{2\omega\Delta_2}{E} \pm \frac{2ie}{m}(\mathbf{pA})\frac{\omega}{E}F_1,$$

$$\left(\mathbf{v}\frac{\partial}{\partial \mathbf{r}}\right)F_2 + 2EB_2 = \frac{2ie}{m}(\mathbf{pA})\frac{\omega}{E}B_1,$$

$$\left(\mathbf{v}\frac{\partial}{\partial \mathbf{r}}\right)D_2 = \frac{2ie}{m}(\mathbf{pA})\frac{\Lambda}{E}B_1,$$

$$\left(\mathbf{v}\frac{\partial}{\partial \mathbf{r}}\right)B_3 + 2EF_3 = \frac{2ie}{m}(\mathbf{pA})\cdot$$

$$\times\left[\frac{\omega}{E}F_2 + \frac{\Delta_0}{E}D_2 + \frac{\omega\chi}{E}\right] - \frac{2\Delta_0\Delta_2}{E}F_1,$$

$$\left(\mathbf{v}\frac{\partial}{\partial \mathbf{r}}\right)F_3 + 2EB_3 = \frac{2ie}{m}(\mathbf{pA})\frac{\omega}{E}B_2 - \frac{2\Delta_0\Delta_2}{E}B_1,$$

$$\left(\mathbf{v}\frac{\partial}{\partial \mathbf{r}}\right)D_3 = \frac{2ie}{m}(\mathbf{pA})\frac{\Delta_0}{E}B_2 + \frac{2\omega\Delta_2}{E}B_1 - 2E\chi B_1, \quad (61)$$

where

$$\Delta = \Delta_0 + \Delta_2, \quad C_1 = B_1 + B_2 + B_3,$$

$$C_2 = F_1 + F_2 + F_3, \quad C_0 = 1 + D_2 + D_3. \quad (62)$$

The indices of the coefficients indicate the order of the expansion in the field.

The boundary conditions for the system (61) are obtained by simply expanding (10) in the field. The equations (61) can be easily solved, and by using the expressions for Δ and for the current in terms of the Green's function $G_p(\mathbf{r})$

$$\Delta = -\frac{i|\lambda|mp_0}{8\pi^2}T\sum_\omega \mathrm{Sp}\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}\int G_\mathbf{p}(\mathbf{r})d\Omega_\mathbf{p},$$

$$j = -\frac{iep_0}{8\pi^2}T\sum_\omega \mathrm{Sp}\,\tau_z\int \mathbf{p}G_\mathbf{p}(\mathbf{r})d\Omega_\mathbf{p}, \quad (63)$$

we obtain the equation for the correction to Δ of second-order in the field and the equation for the vector potential A(z). To obtain the latter, it is necessary to use Maxwell's equation

$$-\frac{d^2\mathbf{A}}{dz^2} = 4\pi\mathbf{j}.$$

In the notation of Shapoval[7], the expression for the correction to Δ is

$$\int_0^\infty \{L(z-z_1) + L'(zz_1)\}\Delta_2(z_1)dz_1 = \int_0^\infty\int \{L_2(zz_1z_2)$$

$$+ L_2'(zz_1z_2)\}A(z_1)A(z_2)dz_1\,dz_2.$$

For the kernels we obtain the following expressions:

$$L(g) = \pi T\sum_\omega \frac{1}{E}\int_0^1\left[\Delta_0^2 + \frac{1}{4}(gvx)^2\right]\left[E^2 + \frac{1}{4}(gvx)^2\right]^{-1}dx,$$

$$L_-'(gg_1) = -2\pi Tv\sum_\omega\frac{\omega^2}{E^2}\int_0^1 dx\cdot x[2E + ivxg]^{-1}\int_0^1 dx_1\cdot x_1[2E + ivx_1g_1]^{-1},$$

$$L_{2-}'(gg_1g_2) = -\pi T\Delta_0 e^2v^3\sum_\omega\frac{\omega^2}{E^3}\int_0^1\frac{x\,dx}{2E + ivxg}$$

$$\times\int_0^1 dx_1\cdot x_1(1-x_1^2)\left[\frac{1}{(2E + ivx_1g_2)(2E + ivx_1(g_1+g_2))} + (12)\right],$$

$$L_{2-}(gg_1g_2) = (ev)^2\Delta_0\frac{-i}{g+g_1+g_2-i\nu}\,\frac{\pi T}{2}\sum_\omega\frac{1}{E^3}\int_0^1 dx(1-x^2)$$

$$\times\left\{\Delta_0^2\left[\frac{1}{(2E+ivxg_2)(2E+ivx(g+g_2))} + (12)\right]\right.$$

$$\left. -\omega^2\left[\frac{1}{(2E+ivxg)(2E+ivx(g+g_1))} + (12) + (02) + (012)\right]\right\},$$

where (12), (02), etc., are cyclic permutations of the indices. The kernels L, L′, and L_{2-} coincide with the corresponding expressions of[7]. However, the kernel L'_{2-} does not coincide. This discrepancy is apparently connected with a misprint in[7]. Following Shapoval[7], we write the equation for the vector potential A(z) in the form

$$-\frac{\partial^2 A}{\partial z^2} + \int_0^\infty \mathscr{K}(z-z_1)A(z_1)dz_1 = \int_0^\infty\int[\mathscr{K}_2(zz_1z_2)$$

$$+ \mathscr{K}_2'(zz_1z_2)]A(z_1)\Delta_2(z_2)dz_1\,dz_2 + \int_0^\infty\int\int[\mathscr{K}_3(zz_1z_2z_3)$$

$$+ \mathscr{K}_3'(zz_1z_2z_3)]A(z_1)A(z_2)A(z_3)dz_1\,dz_2\,dz_3.$$

For the kernels $\mathscr{K}$ we obtain the following expressions:

$$\mathscr{K}(g) = \frac{6\pi^2Ne^2\Delta_0^2}{m}T\sum_\omega\frac{1}{E}\int_0^1 dx(1-x^2)\left[E^2 + \frac{1}{4}(vxg)^2\right]^{-1},$$

$$\mathscr{K}_{2-}(gg_1g_2) = \frac{8mp_0}{\pi}L_{2-}(g_2g_1g),$$

$$\mathscr{K}_{2-}'(gg_1g_2) = \frac{8mp_0}{\pi}L_{2-}'(g_2g_1g),$$

$$\mathscr{K}_{3-}'(gg_1g_2g_3) = 8(ev)^4p_0^2\Delta_0^2T\sum_\omega\frac{\omega^2}{E^4}$$

$$\times\int_0^1 dx_1\frac{x_1(1-x_1^2)}{(2E+ivx_1g_3)(2E+ivx_1(g_2+g_3))}$$

$$\times\int_0^1 dx\cdot x(1-x^2)\left[\frac{1}{(2E+ivxg_1)(2E+ivx(g+g_1))} + (01)\right],$$

$$\mathscr{K}_{3-}(gg_1g_2g_3) = -\frac{3i(ev)^4mp_0\Delta_0^2}{g+g_1+g_2+g_3-i\nu}T\sum_\omega\frac{1}{E^4}\int_0^1 dx(1-x^2)^2$$

$$\times\left\{2\omega^2\left[\frac{1}{(2E+ivxg)(2E+ivx(g+g_1))(2E+ivx(g+g_1+g_2))}\right.\right.$$

$$\left. + (0123) + (01) + (012)\right]$$

$$-\Delta_0^2\left[\frac{1}{(2E+ivxg)(2E+ivxg_3)(2E+ivx(g+g_1+g_3))}\right.$$

$$\left.\left. + \frac{1}{(2E+ivxg_3)(4E+ivx(g_1+g_3))}\left(\frac{1}{2E+ivx(g_1+g_2+g_3)} + (20)\right)\right]\right\}$$

The kernels $\mathscr{K}_3$ and $\mathscr{K}'_3$ differ from the corresponding expressions of[7].

6. DENSITY OF STATES OF THE EXCITATIONS IN A MAGNETIC FIELD

The properties of thin superconducting films in a magnetic field in diffuse reflection of the electrons from the boundaries of the sample were investigated by Thompson in[8]. In particular, it was shown that the excitation spectrum has a gap, i.e., the density of states vanishes at energies below a certain value. A similar problem in specular reflection was considered in[9]. In the case of specular reflection, the excitations are characterized by a quasimomentum, and the gap in the excitation spectrum vanishes at certain directions of the quasimomentum. Traces of such anisotropy should remain also in diffuse reflection of the electrons from the sample boundaries. It will be shown below that the statement that the excitation spectrum has a gap in the case of diffuse reflection is valid only in the zeroth order in the small parameter d/ξ_0—the ratio of the film thickness to the coherence length of the superconductor. Allowance for the first order does not lead to an appreciable change of the properties of the system at $T \sim T_c$, but is significant at $T < T_c/\ln(\xi_0/d)$. This is connected with the fact that in sufficiently strong fields ($eH \gg (d\xi_0)^{-1}$) the density of states differs from zero everywhere. We confine ourselves only to consideration of a thin pure film in a magnetic field directed along its surface.

We choose the coordinate system in such a way that the y axis is directed transversely to the film and on the film boundaries $y = \pm d/2$, while the x axis is along the magnetic field **H**. We can then choose the vector potential **A** with a direction along the z axis and with

$$A_z = Hy. \tag{64}$$

We seek the solution of a system (1) in the form

$$G_{\mathbf{p}}(\mathbf{r}) = f_1\tau_z + \frac{1}{\sqrt{2}} f_2(\tau_y + i\tau_x) + \frac{1}{\sqrt{2}} f_3(\tau_y - i\tau_x). \tag{65}$$

Introducing in lieu of y a new dimensionless variable

$$z = \frac{\sqrt{2}(1+i)}{\sin\theta\sin\varphi}\left[\frac{\omega}{v} - ieHy\cos\theta\right]\left(\left|\frac{eH\cos\theta}{\sin\theta\sin\varphi}\right|\right)^{-1/2} \times \exp\left[-\frac{i\pi}{4}\left(2 - \operatorname{sign}\cos\theta - \operatorname{sign}\sin\varphi\right)\right] \tag{66}$$

and putting

$$\nu(y) = \frac{(1+i)\Delta(y)}{2evH\cos\theta}\left(\left|\frac{eH\cos\theta}{\sin\theta\sin\varphi}\right|\right)^{1/2} \times \exp\left[\frac{i\pi}{4}\left(2 - \operatorname{sign}\cos\theta - \operatorname{sign}\sin\varphi\right)\right], \tag{67}$$

we obtain from formulas (1) and (65) at $\Sigma = 0$

$$\frac{\partial}{\partial z}\begin{pmatrix} f_1 \\ f_2 \\ f_3 \end{pmatrix} = \frac{z}{2}\begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} f_1 \\ f_2 \\ f_3 \end{pmatrix} + \nu\begin{pmatrix} 0 & 1 & -1 \\ 1 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}\begin{pmatrix} f_1 \\ f_2 \\ f_3 \end{pmatrix}. \tag{68}$$

We confine ourselves to a strong field

$$eH \gg (d\xi_0)^{-1}. \tag{69}$$

Then the solution of the system (68) can be sought in the form of a series in powers of ν. Accurate to terms of second order in ν, the general solution of the system (68) is

$$f_1(z) = B_1 + B_2\psi(z) - B_3\chi(z) + B_1(D(z) + F(z)),$$
$$f_2(z) = \exp\left(-\frac{z^2}{4}\right)[B_2 + B_1\chi(z) + (B_2F(z) - B_3F_1(z))],$$
$$f_3(z) = \exp\left(\frac{z^2}{4}\right)[B_3 - B_1\psi(z) + (B_3D(z) - B_2D_1(z))], \tag{70}$$

where

$$\chi(z) = \int_{x_0}^{z} \nu(z)\exp\left(\frac{z^2}{4}\right)dz, \quad \psi(z) = \int_{x_0}^{z}\nu(z)\exp\left(-\frac{z^2}{4}\right)dz,$$
$$x_0 = z(y=0),$$
$$D(z) = \int_{x_0}^{z}\nu(z)\chi(z)\exp\left(-\frac{z^2}{4}\right)dz, \quad F(z) = \int_{x_0}^{z}\nu(z)\psi(z)\exp\left(\frac{z^2}{4}\right)dz,$$
$$D_1(z) = \int_{x_0}^{z}\nu(z)\psi(z)\exp\left(-\frac{z^2}{4}\right)dz, \quad F_1(z) = \int_{x_0}\nu(z)\chi(z)\exp\left(\frac{z^2}{4}\right)dz. \tag{71}$$

It can be shown that the coefficients B_i in (70) satisfy the condition

$$B_i(\theta, \varphi) = B_i(\pi - \theta, \pi + \varphi). \tag{72}$$

We shall therefore consider only the angle region $0 \le \varphi \le \pi$. In the chosen gauge of the vector potential (formula (64)), the matrices $(\mathbf{n}_i\boldsymbol{\tau})$ which enter in the boundary condition (10) and (11), can be chosen in the form

$$(\mathbf{n}_i\boldsymbol{\tau}) = \tau_x, \quad (\mathbf{n}_0\boldsymbol{\tau}) = \alpha\tau_z + \beta\tau_y, \quad (\mathbf{n}_2\boldsymbol{\tau}) = -\beta\tau_z + \alpha\tau_y,$$
$$\alpha^2 + \beta^2 = 1, \quad \alpha, \beta = \text{const}. \tag{73}$$

From (65) and (73) it follows that in the angle region $0 \le \varphi \le \pi$ the boundary condition (10), (11) can be written in the form

$$\beta f_1(z_1) - \frac{1+\alpha}{\sqrt{2}} f_2(z_1) + \frac{1-\alpha}{\sqrt{2}} f_3(z_1) = 0,$$
$$\alpha f_1(z_1) + \frac{\beta}{\sqrt{2}} f_2(z_1) + \frac{\beta}{\sqrt{2}} f_3(z_1) = 1, \tag{74}$$
$$\beta f_1(z_2) + \frac{1-\alpha}{\sqrt{2}} f_2(z_2) - \frac{1+\alpha}{\sqrt{2}} f_3(z_2) = 0,$$

$$\int^{\pi} d\varphi \int^{\pi} d\theta \sin\varphi\sin^2\theta[f_2(z_1) - f_3(z_1)] = 0, \tag{75}$$

where z_1 and z_2 are the values of z on the film boundaries:

$$z_1 = z(-d/2), \quad z_2 = z(d/2).$$

The coefficients B_i are determined by the system (74), and the integral relation (75) together with condition (74) defines the coefficients α and β.

It can be shown that the terms proportional to ν^2 and $\nu\omega d/v$ do not lead to a noticeable change of the spectrum of the system in the solution of the system (74). Omitting these terms, we obtain for the coefficients B_i the following expression:

$$\begin{pmatrix} B_1 \\ B_2 \\ B_3 \end{pmatrix} = [(1+\alpha)e^{t/2} + (1-\alpha)e^{-t/2}]^{-1} \cdot \begin{pmatrix} (1+\alpha)e^{t/2} - (1-\alpha)e^{-t/2} \\ \beta\sqrt{2}\exp\left(\frac{z_1^2}{4} + \frac{t}{2}\right) \\ \beta\sqrt{2}\exp\left(-\frac{z_1^2}{4} - \frac{t}{2}\right) \end{pmatrix} + \begin{pmatrix} \Gamma_1 \\ \Gamma_2 \\ \Gamma_3 \end{pmatrix}, \tag{76}$$

where

$$t = \frac{z_2^2 - z_1^2}{4} = \frac{2\omega d}{vx}, \quad \Gamma_1 = -\frac{\alpha\beta}{vx}[\theta_1 + \theta_2],$$
$$\Gamma_2 = \frac{1}{\sqrt{2}\,vx}\exp\left(\frac{z_1^2}{4}\right)[(1+\alpha^2)\theta_1 - \beta^2\theta_2],$$
$$\Gamma_3 = \frac{1}{vx\sqrt{2}}\exp\left(-\frac{z_1^2}{4}\right)[-\beta^2\theta_1 + (1+\alpha^2)\theta_2],$$
$$\theta_1 = \exp\left(-\frac{z_1^2}{4}\right)\int_0^{d/2}\Delta(y)\exp\left[-ieHy^2\frac{\sqrt{1-x^2}}{x}\cos\varphi'\right]dy,$$
$$\theta_2 = \exp\left(\frac{z_1^2}{4}\right)\int_0^{d/2}\Delta(y)\exp\left[ieHy^2\frac{\sqrt{1-x^2}}{x}\cos\varphi'\right]dy,$$
$$x = \sin\theta\sin\varphi, \quad \sqrt{1-x^2}\cos\varphi' = \cos\theta. \tag{77}$$

Substituting in (75) the explicit expressions for the functions f_2 and f_3, we obtain

$$\alpha\int_0^{2\pi}d\varphi'\int_0^1 dx\int_0^{d/2}dy\,\Delta(y)\exp\left[ieH\frac{\sqrt{1-x^2}}{x}\cos\varphi'\left(\frac{d^2}{4}-y^2\right)\right]$$
$$=\pi v\beta\int_1^\infty\frac{dt}{t^3}\frac{\exp(\omega td/v)-\exp(-\omega td/v)}{(1+\alpha)\exp(\omega td/v)+(1-\alpha)\exp(-\omega td/v)} \quad (78)$$

To find the spectrum it is necessary to find the coefficients α and β from the system (73) and (78), accurate to terms d/ξ_0. We find first the ordering parameter $\Delta(y)$ in the zeroth approximation in the parameter d/ξ_0. Using Eqs. (63) for $\Delta(y)$ and expression (65) for the Green's function $G_p(r)$, we obtain

$$\Delta(y)=\frac{|\lambda|mp_0}{2\sqrt{2}\pi^2}T\sum_\omega\int_0^{2\pi}d\varphi'\int_0^1 dxf_2(z).$$

In the zeroth approximation in d/ξ_0, this yields

$$\Delta(y)=\frac{|\lambda|mp_0}{2\pi}T\sum_\omega\beta(\omega)\exp\left[-eH\left(\frac{d^2}{4}-y^2\right)\right], \quad (79)$$

i.e., $\Delta(y)$ can be written in the form

$$\Delta(y)=\Delta_1\exp\left[-eH(d^2/4-y^2)\right]. \quad (80)$$

To determine the constants α, β, and Δ_1, we solve Eq. (78) in the zeroth approximation in d/ξ_0. In this approximation, the integral in the right side of (78) is equal to $\omega d/v$. Substituting the expression (80) for $\Delta(y)$ in (78) and using (73), we obtain

$$\alpha(\omega)=\frac{\omega}{[\omega^2+(\Delta_1\Phi)^2]^{1/2}},\quad \beta(\omega)=\frac{\Delta_1\Phi}{[\omega^2+(\Delta_1\Phi)^2]^{1/2}}, \quad (81)$$

where

$$\Phi=\Phi\left(1,\frac{3}{2},-\frac{eHd^2}{2}\right) \quad (82)$$

is the confluent hypergeometric function. Substituting the value of $\beta(\omega)$ from (81) and (79) we obtain an equation for Δ_1:

$$\Delta_1=\frac{|\lambda|mp_0}{2\pi}T\sum_\omega\frac{\Delta_1\Phi}{[\omega^2+(\Delta_1\Phi)^2]^{1/2}}; \quad (83)$$

in (83) it is necessary to replace Φ by unity when $\omega > evHd$. Equation (83) for Δ_1 was obtained earlier by Thompson[8].

The density of states is expressed in terms of the Green's function $G_{-i\omega}$ by means of formula (56), where $G_{-i\omega}$ is the analytic continuation of the function G_ω from the values $\omega = \pi T(2n+1)$ to the imaginary axis. Accurate to terms d/ξ_0, we get from (65) and (70)

$$\rho(\omega)=\rho_0\,\mathrm{Im}\frac{i}{4\pi d}\int f_1(-i\omega)\,dyd\Omega_p=\rho_0\,\mathrm{Im}\frac{i}{2\pi}\int_0^{2\pi}d\varphi'\int_0^1 dxB_1(-i\omega). \quad (84)$$

Substituting in (84) the expression for B_1 from formula (76), we get

$$\rho(\omega)=\rho_0\,\mathrm{Im}\left\{i\alpha+\frac{i\pi}{2}\frac{\omega d}{v}\beta^2\right.$$
$$\left.+\beta^2\int_1^\infty\frac{dt}{t^2}\frac{\exp(\omega td/v)-\exp(-\omega td/v)}{(1+\alpha)\exp(\omega td/v)+(1-\alpha)\exp(-\omega td/v)}\right\} \quad (85)$$

where the values of α and β are determined by formulas (73) and (78), with the substitution $\omega\to -i\omega$. In the zeroth approximation in d/ξ_0, only the first term remains in (85). Using expression (81) for α, we obtain[8]

$$\rho(\omega)=\rho_0\begin{cases}\dfrac{\omega}{[\omega^2-(\Delta_1\Phi)^2]^{1/2}}, & \omega>\Delta_1\Phi\\ 0, & \omega<\Delta_1\Phi,\end{cases} \quad (86)$$

where Φ is given by (82).

In the first order in d/ξ_0, the density of states differs from zero also when $\omega < \Delta_1\Phi$. From (85) it follows that to find ρ with accuracy to first order in d/ξ_0 it is necessary to know α with the same accuracy. Since $\Delta(y)$ is real in all orders in d/ξ_0, it is sufficient to know $\Delta(y)$ in the zeroth approximation in d/ξ_0 (formula (80)). Carrying out the analytic continuation of the expressions in (78) with respect to ω and replacing $\omega\to -i\omega$, we obtain, with account taken of terms of order d/ξ_0,

$$\alpha\Delta_1\Phi=-\frac{v\beta}{d}\left\{\int_1^\infty\frac{dt}{t^3}\frac{\exp(\omega td/v)-\exp(-\omega td/v)}{(1+\alpha)\exp(\omega td/v)+(1-\alpha)\exp(-\omega td/v)}\right.$$
$$\left.+(i-1)\frac{\omega d}{v}-\frac{i\pi}{2}\alpha\left(\frac{\omega d}{v}\right)^2\right\}, \quad (87)$$

where $\alpha = \alpha(-i\omega)$, $\beta = \beta(-i\omega)$, and we used expression (80) for $\Delta(y)$. The formulas (73), (85), and (87) determine the density of the states at arbitrary ratios of ω and $\Delta_1\Phi$. In the simplest case, when $\omega \ll \Delta_1\Phi$, the integral in (87) can be readily evaluated, and we obtain for α the expression

$$\alpha(-i\omega)=-\frac{i\omega}{\Delta_1\Phi}+\frac{7\zeta(3)}{\pi^2}\frac{\omega d}{v}\frac{\omega}{\Delta_1\Phi}. \quad (88)$$

Substituting this value of α in (85) and calculating the integral contained in it at $\omega \ll \Delta_1\Phi$ (which corresponds to $|\alpha| \ll 1$), we obtain

$$\rho(\omega)=\rho_0\frac{\omega d}{v}\left[\frac{\pi}{2}+\frac{21}{\pi^2}\zeta(3)\frac{\omega}{\Delta_1\Phi}\right]. \quad (89)$$

The presence or excitations with energy $\epsilon < \Delta_1\Phi$ can be observed, for example, in experiments on tunneling. Allowance for such excitations leads to the appearance at $T = 0$ of a single-particle current at $eV < \Delta_1\Phi$. For the case when one metal is normal, the expression for the current is of the form[10]

$$I=\frac{1}{eR}\int_0^{eV}d\omega\frac{\rho(\omega)}{\rho_0}. \quad (90)$$

Here V is the voltage on the contact, B is the resistance of the contact in the normal state. Substituting in (90) the value of $\rho(\omega)$ from (89), we get

$$I=\frac{V}{R}\frac{eVd}{v}\left[\frac{\pi}{4}+\frac{7}{\pi^2}\zeta(3)\frac{eV}{\Delta_1\Phi}\right],\quad eV\ll\Delta_1\Phi.$$

7. CONCLUSION

We obtained boundary conditions for the integral of the Green's function with respect to ξ for the case of diffuse reflection from the walls. These boundary conditions coincide, in the approximation linear in Δ, with the diffuse boundary conditions for the distribution function of the normal metal. It is customary to use the method of classical trajectories[7,11] for diffuse reflection of electrons from the wall, but this method is not valid in the presence of impurities in the superconductor[4]. In addition, in this method correlation functions of four and more momentum electrons appear, in the expansion in terms of the field. To find these functions it is necessary to make additional assumptions concerning the calculation method. This causes the correction of thirdorder in the field to the penetration depth to deviate at $\delta \sim \xi_0$ from the corresponding result of the classical-trajectories method[7].

An expression was obtained for the critical current near T_c. For T = 0, the critical current was calculated with logarithmic accuracy. In the remaining temperature region, algebraic equations were obtained for the determination of the critical current; these can be solved only numerically. We also obtained the excitation spectrum of thin films in the presence of a current as well as in strong magnetic fields. The presence of gapless excitations can be observed, for example, in tunnel experiments. Allowance for such excitations leads to the appearance of a single-particle current that does not vanish when $T \to 0$.

In conclusion, I am grateful to A. I. Larkin for a discussion of the problems considered here.

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Translated by J. G. Adashko
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