SOVIET PHYSICS JETP VOLUME 29, NUMBER 4 OCTOBER, 1969

THEORY OF THE MUONIUM MECHANISM OF μ^+-MESON DEPOLARIZATION IN MEDIA WITH TENSOR RELAXATION OF THE MUONIUM ELECTRON SPIN

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Submitted November 23, 1968

Zh. Eksp. Teor. Fiz. 56, 1419–1431 (April, 1969)

A theory of depolarization in media is developed, in which the tensor nature of the relaxation rate is related to the presence of an external field. A new effect that should occur in semiconductors is predicted, namely that slow precession in a superstrong magnetic field is insensitive to the magnitude of the damping, and hence it should be possible to observe muonium under conditions in which direct methods would be ineffective. Formulas are obtained for polarization at an instant that is infinite with respect to the time of the chemical reaction. Formulas are also derived for the time dependence of the polarization in a number of limiting cases. In particular, formulas are obtained for the case when the recently-discovered muonium-precession beats can be observed in a magnetic field.

UNTIL recently, for the analysis of experiments on depolarization of μ^+ mesons, the degree to which the relaxation rate of the electron spin can be regarded as a scalar was immaterial. Indeed, the small tensor addition vanishes in longitudinal fields at a field value on the order of 50–100 G. The relaxation time connected with this addition is quite large, on the order of hundreds of nanoseconds. In experiments, even radioscopic ones, it is possible to disregard such a small addition, provided one does not deal with media in which the muonium has a long lifetime especially if in such media there is a large electron density, and the temperatures are infralow, so that the scalar component of the relaxation rate is suppressed. The large electron density leads to a sharp decrease of the frequency connected with the hyperfine splitting in muonium, and it becomes necessary to consider both the slow and fast relaxation rates. In this paper we confine ourselves to the influence of the tensor relaxation in the absence of an external radio-frequency field, and to the most realistic and simple case in which the tensor properties are determined exclusively by the external fields, i.e., the tensor properties of the relaxation times are connected with the physics of the relaxation phenomenon itself, and not with features of the crystal structure. Until the dependence of the relaxation time on the external magnetic field is determined, such a theory can be purely formalistic and can apply to any mechanism of tensor relaxation. We shall first confine ourselves to the case in which the experiments are carried out without time-varying fields.

It should be noted that a certain caution must be exercised with transferring the well known equations for the relaxation times from the theory of paramagnetic and nuclear resonances to the case of μ^+-meson depolarization: a single muonium atom can diffuse in certain media up to the very act of chemical reaction, and the kinematic effects connected with the diffusion may exert their own influence in the determination of which of the regions around the active center make the largest contribution to the tensor component of the relaxation time. Tensor effects are produced merely for the sole reason that the external magnetic field plus the internal random field form an elliptic field. It must be emphasized that in this case we cannot regard the influence of local magnetic fields separately from the equations for the density matrix, for it is necessary here to introduce different values of attenuation for the different polarization components.

A. CASE OF LONGITUDINAL FIELDS

If the preferred direction is only the field itself, then there are two different relaxation times of the electron spin, as is the case in the theory of magnetic resonance (the Bloch equation).

Using the same directions of the axes as in[1], we obtain in lieu of the system (11) in[1]

$$\begin{aligned}
d\rho_{10}/d\tilde{t} &= \rho_{32}-\rho_{23},\\
d\rho_{23}/d\tilde{t} &= \rho_{10}-\rho_{01}+2x\zeta\rho_{33}+2x\rho_{22}-\gamma_{\perp}\rho_{23},\\
d\rho_{32}/d\tilde{t} &= \rho_{01}-\rho_{10}-2x\zeta\rho_{22}-2x\rho_{33}-\gamma_{\perp}\rho_{32},\\
d\rho_{01}/d\tilde{t} &= \rho_{23}-\rho_{32}-\gamma_{\parallel}\rho_{01},\\
d\rho_{22}/d\tilde{t} &= 2x\zeta\rho_{32}-2x\rho_{23}-\gamma_{\perp}\rho_{22},\\
d\rho_{33}/d\tilde{t} &= -2x\zeta\rho_{23}+2x\rho_{32}-\gamma_{\perp}\rho_{33}.
\end{aligned} \tag{1}$$

We have retained in (1) the notation of[1]: $\zeta = m_e/m$ is the ratio of the magnetic moments of the μ^+ meson and the electron, $\gamma = 4\nu/\omega_0$ is a dimensionless electron-spin relaxation parameter, 2ν is the time of relaxation of the electron spin, $\tilde{t}$ is the dimensionless time, $x = H/H_0$ is the dimensionless magnetic field, and ρ are the components of the density matrix. We are interested primarily in the dependence of the average polarization of the magnetic field, if the dependence of $\gamma_{\parallel}$ and $\gamma_{\perp}$ on the longitudinal field is known.

However, the information obtained from experiment is richer, since it may contain also time relationships. We shall not present analytic solutions in terms of the inconvenient Cardan formulas, as was done in a paper by Yakovleva[2], and develop a perturbation theory in analogy with the paper of Nosov and Yakovleva[3].

The determinant of the system (1) is

$$D_{\rm VI}=[(\lambda+\gamma_{\perp})^2+(2x(1-\zeta))^2]\{\lambda(\lambda+\gamma_{\parallel})[(\lambda+\gamma_{\perp})^2+(2x(1+\zeta))^2] + 2(2\lambda+\gamma_{\parallel})(\lambda+\gamma_{\perp})\}. \tag{2}$$

The minor corresponding to a certain initial polarization of the μ^+ meson, multiplied by $(-\lambda)$, is

$$M_{\mathrm{VI}} = [(\lambda+\gamma_\perp)^2 + 4x^2(1-\zeta)^2]\{\lambda(\lambda+\gamma_\parallel)[(\lambda+\gamma_\perp)^2 + 4x^2(1+\zeta)^2] + 2\lambda(\lambda+\gamma_\perp)\}. \quad (3)$$

From (2) and (3) we see that to find the eigenvalues it is necessary to solve a fourth-degree equation.

The ratio of expressions (3) and (2) is equal to the average observable polarization:

$$P_\infty = 1 - (\tau\omega_0)^2\left(\frac{1}{2} + \nu_\perp\tau\right)\left\{(1+2\nu_\perp\tau)^2 + (\omega_0\tau)^2\left[\frac{(1+2\nu_\perp\tau)(1+\nu_\parallel\tau)}{(1+2\nu_\parallel\tau)} + x^2(1+\zeta)^2\right]\right\}^{-1}. \quad (4)$$

In (4) we have made the substitution $\lambda \to 1/\tau$, where, as usual, τ is the time of entering in the chemical reaction, and changed over to dimensional symbols. For the quantity $P/2(1-P)$ we now have

$$\frac{1}{2}\frac{P_\infty}{(1-P_\infty)} = \frac{1+2\nu_\perp\tau}{(\tau\omega_0)^2} + \frac{1/2}{(1+2\nu_\parallel\tau)} + \frac{x^2(1+\zeta)^2}{(1+2\nu_\perp\tau)}. \quad (5)$$

Although expression (5) differs little in form from the corresponding expression of[4], the three-point relation is no longer satisfied, since $\nu_\parallel$ and $\nu_\perp$ depend on the magnitude of the field, i.e., on x^2. However, at sufficiently strong fields, when $\nu_\perp$ and $\nu_\parallel$ become equal and independent of the field, the three-point relation, i.e., a straight line on the axes $(P/(1-P))$ and x^2, is restored.

We are interested, however, not only in the average polarization but also in the time variation. To find the eigenvalues we must solve an equation obtained by equating to zero the expression in the curly brackets of (2):

$$\lambda^4 + \lambda^3[2\gamma_\perp + \gamma_\parallel] + \lambda^2[\gamma_\perp^2\gamma_\parallel + 4x^2(1+\zeta)^2 + 4] + \lambda[\gamma_\parallel\gamma_\perp^2 + 4\gamma_\parallel x^2(1+\zeta)^2 + 4\gamma_\perp + 2\gamma_\parallel] + 2\gamma_\parallel\gamma_\perp = 0. \quad (6)$$

We seek the roots corresponding to small attenuation; provided we do not deal with strongly contaminated semiconductors, only such roots are of physical interest, since the tensor character of the relaxation time will affect the time variation only when the relaxation time is sufficiently large, i.e., on the order of 10^{-9} sec and higher. Only in very highly contaminated semiconductors can the characteristic time of binding of the μ^+ meson with the medium be so large. (The time 10^{-8} sec already corresponds to an effective dielectric constant $\epsilon = (128)^{1/6} = 2.2$, i.e., to muonium, whose dimensions are increased by a factor of 5.)

Confining ourselves to terms quadratic in $\gamma_\parallel$ and $\gamma_\perp$, we obtain

$$\lambda_{1,2} = \frac{1}{8}[1 + x^2(1+\zeta)^2]^{-1}\{-[4\gamma_\parallel x^2(1+\zeta)^2 + 4\gamma_\perp + 2\gamma_\parallel] \pm [(4\gamma_\parallel(1+\zeta)^2x^2 + 4\gamma_\perp + 2\gamma_\parallel)^2 + 32\gamma_\parallel x^2\gamma_\perp(1+\zeta)^2]^{1/2}\}. \quad (7)$$

There is no tensor component, i.e., $\gamma_\perp = \gamma_\parallel = \gamma$, then expression (7) goes over into the well known roots[3]

$$\lambda_1 = -\gamma, \quad \lambda_2 = -\frac{1/2\gamma}{1 + x^2(1+\zeta)^2}. \quad (8)$$

For control purposes, we obtain also the roots corresponding to fast attenuation. We assume for this purpose that $\lambda \sim 1$ and retain in the equation only the terms of order of unity; then, making successive approximations up to first order in γ, we obtain

$$\lambda_{3,4} = \pm 2i\sqrt{1 + x^2(1+\zeta)^2} - \{4(\gamma_\parallel + \gamma_\perp)x^2(1+\zeta)^2 + 3(2\gamma_\perp + \gamma_\parallel)\}\{\pm 3i\sqrt{1+x^2(1+\zeta)^2} + 2[1 + x^2(1+\zeta)^2]\}^{-1}. \quad (9)$$

Let us find now the coefficient C_k of the corresponding eigenvalues:

$$C_k = -M_{\mathrm{VI}}(\lambda_k) \,/\, [D_{\mathrm{VI}}'(\lambda_k)\cdot(-\lambda_k)], \quad (10)$$

where M_{VI} is the minor given by expression (3) and D_{VI}' is the derivative of the determinant (2) with respect to the eigenvalue λ, both taken at the point $\lambda = \lambda_k$.

Since M_{VI} and D_{VI} contain the same factor $[(\lambda+\gamma_\perp)^2 + 4x^2(1-\zeta)^2]$, we actually have

$$A_k = -M_{\mathrm{IV}}(\lambda_k) \,/\, [D_{\mathrm{IV}}'(\lambda_k)\cdot(-\lambda_k)], \quad (11)$$

where

$$M_{\mathrm{IV}} = \lambda(\lambda+\gamma_\parallel)[(\lambda+\gamma_\perp)^2 + 4x^2(1+\zeta)^2] + 2\lambda(\lambda+\gamma_\perp),$$
$$D_{\mathrm{IV}} = \lambda(\lambda+\gamma_\parallel)[(\lambda+\gamma_\perp)^2 + 4x^2(1+\zeta)^2] + 2(2\lambda+\gamma_\parallel)(\lambda+\gamma_\perp). \quad (12)$$

In fact, we can always omit the part in the curly brackets of (9). Substituting then (9) and (7) in (12) and (10), we obtain

$$C_1 = -\frac{16\gamma_\perp(r-1/2)}{(\delta-\varphi)\delta} - \frac{[\gamma_\parallel + 2\gamma_\perp(r-1)]}{r\delta},$$
$$C_2 = -\frac{16\gamma_\parallel\gamma_\perp(r-1/2)}{(\delta+\varphi)\delta} + \frac{[\gamma_\parallel + 2\gamma_\perp(r-1)]}{r\delta},$$
$$C_3 = C_4 = 1/4r. \quad (13)$$

Here

$$r = 1 + x^2(1+\zeta)^2, \quad \varphi = 4x^2(1+\zeta)^2\gamma_\parallel + 4\gamma_\perp + 2\gamma_\parallel,$$
$$\delta = \sqrt{\varphi^2 - 32\gamma_\parallel\gamma_\perp r}.$$

It is easy to see that for $x = 0$, when $r = 1$, $\varphi = 4\gamma_\perp + 2\gamma_\parallel$, $\delta = 4\gamma_\perp - 2\gamma_\parallel$, $A_1 = 1/2$, and $A_2 = 0$; for $\gamma_\parallel = \gamma_\perp = \gamma$, when $\delta = 2\gamma(2r-1)$, $\varphi = 2\gamma(2r+1)$, and $A_2 = 0$.

The time dependence of the polarization is determined, as usual, by the following expression:

$$P(t) = \sum_{k=1}^{4} C_k\left[\int_0^t \frac{1}{\tau}\exp\left\{-t'\left(\frac{1}{\tau} + \frac{\lambda_k\omega_0}{2}\right)\right\}dt' + \exp\left\{-t\left(\frac{1}{\tau} + \lambda_k\frac{\omega_0}{2}\right)\right\}\right]. \quad (14)$$

Hence

$$P(t) = P_\infty + \frac{\tau\omega_0}{2}\sum_{k=1}^{4}\frac{C_k\lambda_k}{1 + \lambda_k\tau\omega_0/2}\exp\left\{-t\left(\frac{1}{\tau} + \frac{\lambda_k\omega_0}{2}\right)\right\} \quad (15)$$

To complete the picture, we present here λ_k and $C_k\lambda_k$ in compact notation:

$$\lambda_1 = \frac{\delta-\varphi}{8r}, \quad \lambda_2 = -\frac{(\varphi+\delta)}{8r}, \quad \lambda_3 = 2ir, \quad \lambda_4 = -2ir; \quad (16)$$

$$\lambda_1C_1 = \frac{-16\gamma_\parallel\gamma_\perp(r-1/2)}{8r\delta} - \frac{(\delta-\varphi)[\gamma_\parallel + 2\gamma_\perp(r-1)]}{8r^2\delta}$$
$$\lambda_2C_2 = \frac{16\gamma_\parallel\gamma_\perp(r-1/2)}{8r\delta} - \frac{[\gamma_\parallel + 2\gamma_\perp(r-1)](\delta+\varphi)}{8r^2\delta}, \quad (17)$$
$$\lambda_3C_3 = \frac{i}{2}, \quad \lambda_4C_4 = -\frac{i}{2}.$$

We call attention to the following circumstance: if we consider substances (in which stopping takes place) such that $\omega_0 \gg \max(\nu_\perp, \nu_\parallel)$, then ω_0 is in this case itself sufficiently large in the sense that times $1/\omega_0$ cannot be attained in experiment, and the time dependence can be determined only by the first two terms in the sum of (15). If there is no field, then only one term remains in the sum (in addition, $\nu_{\parallel}$ and $\nu_{\perp}$ become equal and the coefficient A_2 vanishes).

Thus, by performing experiments in different longitudinal fields and by studying the time dependence, it is possible to reconstruct $\nu_{\parallel}$ and $\nu_{\perp}$ provided the time of entry in the chemical reaction τ is sufficiently long.

To complete the picture let us consider also a case corresponding to such rapid electron-spin relaxation times that the spin-spin coupling is almost broken. In spite of the fact that such a case is apparently exotic, it may turn out to be important for semiconducting ferrites, at very low temperatures, when on the one hand chemical reactions are slowed down, and on the other hand the probability of exchange scattering of the free electron by the muonium electron is decreased, i.e., the scalar component of the reciprocal relaxation time of the electron spin is decreased. At the same time, at low temperature[1)] the tensor component of the reciprocal electron-spin relaxation time not only fails to decrease, but even increases. At large electron densities, as a result of the decrease of the Debye length in semiconductors, a strong decrease of ω_0 may occur, and the case $\omega_0 \ll \nu_{\perp}$ may turn out to be realistic.

Retaining in (6) terms of order γ^2, γ, and 1 and terms corresponding to an external field for the case of slow attenuation, $\lambda \sim 1/\gamma$, we obtain

$$\lambda_1 \cong \frac{-2}{\gamma_\perp + 4x^2(1+\zeta)^2/\gamma_\perp}. \tag{18}$$

It thus turns out that the case considered by Zel'dovich, in the presence of a tensor relaxation time, does not differ from the corresponding case with a scalar relaxation time. Using (11) and (12) we obtain immediately that $A_1 = 1$. This completes the analysis of the behavior in a longitudinal field.

B. CASE OF TRANSVERSE FIELD

Just as in the case of a longitudinal field, we confine ourselves to large and small relaxation rates for the time variation of the polarization and to arbitrary relaxation times for the average residual polarization.

In lieu of the system (15) of[1], where it is assumed that the field is directed along the 2 axis and the initial μ^+-meson momentum is directed along the (−1) axis, we derive a system of equations with tensor γ: to this end it suffices to make the substitution $\gamma \to \gamma_{\parallel}$ if γ is a factor of ρ_{12} and ρ_{32}, and $\gamma \to \gamma_\perp$ in the remaining cases. Without writing out this system, we see immediately that it is possible to introduce here, too, the complex variables

$$\rho_\mu = \rho_{10} + i\rho_{30}, \quad \rho_e = \rho_{01} + i\rho_{03}, \quad \rho_\mu{}^t = \rho_{21} + i\rho_{23}, \quad \rho_e{}^t = \rho_{12} + i\rho_{23} \tag{19}$$

and to make the substitution $\gamma \to \gamma_\parallel$ in Eq. (17) of the same reference if γ is the factor of ρ_e^t, and $\gamma \to \gamma_\perp$ in all other cases. We then obtain

$$d\rho_\mu / d\tilde{t} = -i\rho_e{}^t + i\rho_\mu{}^t + 2i\zeta x\rho_\mu,$$

$$d\rho_e / d\tilde{t} = i\rho_e{}^t - i\rho_\mu{}^t - (\gamma_\perp + 2ix)\rho_e,$$

$$d\rho_e{}^t / d\tilde{t} = i\rho_e - i\rho_\mu - (\gamma_\parallel - 2i\zeta x)\rho_e{}^t,$$

$$d\rho_\mu / d\tilde{t} = i\rho_\mu - i\rho_e - (\gamma_\perp + 2ix)\rho_\mu{}^t. \tag{20}$$

The determinant of the system is

$$D_{\text{IV}} = A^2B^2 - (A+B)^2 - i\gamma_\parallel[AB^2 - (A+B)], \quad A = i(\lambda + \gamma_\parallel - 2i\zeta x), \quad B = i(\lambda + \gamma_\perp - 2ix). \tag{21}$$

The minor corresponding to a certain initial polarization of the μ^+ meson, multiplied by $(-\lambda)$, is, in the same notation,

$$M_{\text{IV}} = i\lambda[AB^2 - (A+B)]. \tag{22}$$

In accordance with[1], the average polarization in a transverse magnetic field is defined as a complex quantity, the modulus of which is equal to the polarization vector, and the phase is equal to the additional angle through which the μ^+ meson must be rotated in order for the precession to occur at a frequency $\tilde{\omega}_\mu$ after entering in the chemical reaction, as if the precession were to start at the very instant when the meson enters into the substance. In other words, the average polarization, or more accurately the polarization at an infinite instant of time (since we assume that the observations cannot be carried out as yet within the time of the chemical reaction), is equal to

$$\mathbf{P}_\infty = \lim_{\tilde{T}\to\infty}\left[\exp(i\tilde{\omega}_\mu\tilde{T})\int_0^{\tilde{T}}\rho_\mu(\tilde{t})\exp\left[-\tilde{t}\left(\frac{1}{\tilde{\tau}} + i\tilde{\omega}_\mu\right)\right]\frac{d\tilde{t}}{\tilde{\tau}}\right]. \tag{23}$$

We have emphasized here that the complex quantities $\mathbf{P}$ and ρ_μ are by their nature equivalent to vectors.

We define the polarization as an analytic function of the parameter:

$$\mathbf{P}(\lambda) = -\lambda\left\{\gamma_\parallel + \frac{i[A^2B^2 - (A+B)^2]}{[AB^2 - (A+B)]}\right\}^{-1}; \tag{24}$$

Then

$$\mathbf{P}_\infty = \mathbf{P}(1/\tau + i\tilde{\omega}_\mu). \tag{25}$$

We recall that here $\tilde{\tau}$ and $\tilde{\omega}_\mu$ are quantities made dimensionless relative to $\omega_0/2$: $\tilde{\tau} = \tau\omega_0/2$ and $\tilde{\omega}_\mu = 2\omega_\mu/\omega_0$. On the basis of (21)–(25) we have

$$\mathbf{P}_\infty = -\frac{1}{\tilde{\tau}}\left\{\gamma_\parallel + i\frac{[A_0{}^2B_0{}^2 - (A_0+B_0)^2]}{[A_0B_0{}^2 - (A_0+B_0)]}\right\}^{-1};$$

$$A_0 = i\left(\frac{1}{\tilde{\tau}} + \gamma_\parallel\right) + 2x\left(\zeta - \frac{\tilde{\omega}_\mu}{2x}\right),$$

$$B_0 = i\left(\frac{1}{\tilde{\tau}} + \gamma_\perp\right) - 2x\left(1 + \frac{\tilde{\omega}_\mu}{2x}\right). \tag{26}$$

In the case of primary interest to us, when the μ^+-meson precession frequency after entering into the chemical reaction is equal to the precession frequency of the μ^+ meson, i.e., $\zeta = \tilde{\omega}_\mu/2x$, we obtain from (26), going over simultaneously to the usual dimensional notation

$$\mathbf{P}_\infty = \left[1 + i\frac{\tau\omega_0}{2}\frac{B_1(A_1+B_1)}{A_1B_1{}^2 - (A_1+B_1)}\right]^{-1};$$

$$B_1 = i\left(\frac{2}{\omega_0\tau} + \frac{4\nu_\perp}{\omega_0}\right) - 2x(1+\zeta),$$

$$A_1 = i\left(\frac{2}{\omega_0\tau} + \frac{4\nu_\parallel}{\omega_0}\right). \tag{27}$$

In analogy with[4], we consider in lieu of $\mathbf{P}$ the quantity $\mathbf{P}/(1 - \mathbf{P})$, for which we get from (27)

[1)]The fact that effects of random magnetic fields and of the dipole-dipole interaction proper should become more intense at low temperature was pointed out to the author by V. I. Selivanov, to whom the author is deeply grateful for this remark, particularly in connection with the fact that, to a certain degree, it stimulated the present work.

$$\frac{\mathbf{P}}{1-\mathbf{P}} = -\frac{2}{i\tau B_1\omega_0} + \frac{2A_1B_1}{i\tau\omega_0(A_1+B_1)}. \qquad (28)$$

For the quantity $q = \mathbf{P}/2(1 - \mathbf{P})$ we obtain the following expressions:

$$\operatorname{Re} q = \frac{d_\parallel}{\tau\omega_0(d_\parallel^2+4y^2)} + \frac{d_\perp}{\omega_0\tau} - \frac{2d_\perp^2(d_\parallel+d_\perp)}{\omega_0\tau[(d_\parallel+d_\perp)^2+4y^2]},$$
$$\operatorname{Im} q = \frac{-2y}{\tau\omega_0(d_\parallel^2+4y^2)} + \frac{2yd_\perp^2}{\tau\omega_0[(d_\parallel+d_\perp)^2+4y^2]}. \qquad (29)$$

We have introduced here the notation

$$d_\parallel = \frac{2}{\omega_0\tau}(1+2\nu_\parallel\tau), \quad d_\perp = \frac{2}{\omega_0\tau}(1+2\nu_\perp\tau); \qquad (30)$$

$$y = x(1+\zeta) = \frac{H}{H_0}\left(1+\frac{m_e}{m_\mu}\right). \qquad (31)$$

Formulas (29)–(31) in the scalar case go over into the corresponding formulas of[4]. All the relations obtained in[4] remain valid for q and P.

It may turn out from a comparison of formulas (5) and (29) that they contradict each other at a zero field. In fact this is not so: the limits as the field approaches zero must be taken also with the same relaxation rates, since only the field causes the difference between the relaxation rates. At the same time, if we consider formally the system of Eqs. (1) and (19), then we see immediately the difference between them: in one case a relaxation rate along the initial polarization is preferred, and in the other it is preferred in one of the transverse directions. This is also the reason why at very large relaxation rates a difference is obtained between the limiting formulas: the entire difference is determined by the field, and the field influences the relaxation in different manners.

We now proceed to study the time dependences of the polarization in a transverse magnetic field. To find the eigenvalues we must solve an equation obtained by equating to zero the determinant (21):

$$A^2B^2 - (A+B)^2 - i\gamma_\parallel[AB^2 - (A+B)] = 0. \qquad (32)$$

We have retained here the notations (21). We note that Eq. (32) is transformed into a very compact equation if we make a change of variable and find the quantity μ, which is determined from the equation

$$i\mu = i\lambda + 2\zeta x. \qquad (33)$$

Introducing

$$z = \gamma_\perp + 2iy, \qquad (34)$$

we get

$$A = i\mu + \gamma_\parallel, \quad B = i\mu + iz. \qquad (35)$$

Substituting (35) in (32), we get

$$\mu(\mu+\gamma_\parallel)(\mu+\gamma_\perp+2iy)^2 + (2\mu+\gamma_\parallel+\gamma_\perp+2iy)(2\mu+\gamma_\perp+2iy) = 0 \qquad (36)$$

or

$$\left(\frac{1}{\mu+\gamma_\parallel} + \frac{1}{\mu+\gamma_\perp+2iy}\right)\left(\frac{1}{\mu} + \frac{1}{\mu+\gamma_\perp+2iy}\right) = -1. \qquad (37)$$

Expression (37) goes over in the scalar case into the corresponding equation of[4].

Although (36) is simply an equation of fourth degree and can therefore be solved exactly, we shall not use the very complicated formulas and will solve it by an approximate procedure, in order to be able to perform readily a physical analysis. It is seen directly from (37) that the complex parameter z(34) is very useful for a mathematical analysis of (36) and for its solutions. Then

$$\mu(\mu+\gamma_\parallel)(\mu+z)^2 + (2\mu+\gamma_\parallel+z)(2\mu+z) = 0. \qquad (38)$$

We are interested in cases in which both parameters $\gamma_\parallel$ and z are large in absolute magnitude, or one of them is small, or else when both are small. In principle, for ferrites at very low temperatures, an intermediate case might also be interesting, but then it is more convenient to solve (36) with a computer, for an analysis of the results of the time variation would be difficult to carry out in any case without a computer; for this reason, we shall not consider this case, all the more since there is no acute experimental need for it at present.

We start with a very simple case considered by Zel'dovich, when both parameters are large; this case is characterized by the fact that owing to the very large relaxation rate of the electron spin, the coupling between the spins of the μ^+ meson and of the electron breaks, and the μ^+ meson precesses almost like a free meson. We are interested in a root having a very small real part. We note that if the imaginary part is very large, i.e., comparable with $\gamma_\parallel$ and z (this, in principle, can be realized for ferrites), then our analysis will not be applicable to such a case.

In the case of a small root, Eq. (36) goes over into

$$\mu^2z(z+\gamma_\parallel) + \mu[\gamma_\parallel z^2 + 2\gamma_\parallel + 4z] + z(\gamma_\parallel+z) = 0, \qquad (39)$$

from which we obtain immediately

$$\mu_1 \cong -\frac{1}{z} - \frac{1}{\gamma_\parallel}. \qquad (40)$$

The remaining roots will be of the order of $\gamma_\parallel$ and z, and their coefficients will be close to zero, in analogy with the results obtained in[1,4]. The coefficient C_1 is equal to unity with great accuracy. Indeed, according to (10), (21), and (22), in the case of a transverse field the coefficient corresponding to a certain eigenvalue λ is

$$C_k(\lambda_k) = \frac{AB^2-(A+B)}{2AB^2+2A^2B-2(A+B)-i\gamma_\parallel[2AB+B^2-2]}, \qquad (41)$$

from which we see immediately that even when $\gamma_\perp$ and $\gamma_\parallel$ are different, the correction term δ for the coefficient will still be of the order of

$$\operatorname{Re}\delta \sim \frac{1}{\gamma}, \quad \operatorname{Im}\delta \sim \max\left[\frac{x}{\gamma}, \frac{x^3}{\gamma^3}\right]. \qquad (42)$$

We present in explicit form expressions for the time dependence of the polarization for Zel'dovich's case. According to (22) of[1], we have

$$\mathbf{P}(t) = Q(1-\mathbf{P}_\infty) + \mathbf{P}_\infty e^{ix\zeta\omega_0 t}, \qquad (43)$$

where $\mathbf{P}_\infty$ is determined by formula (27), which goes over in the case of very large relaxation rates into

$$\mathbf{P}_\infty \approx \left\{1+\left(\frac{\tau\omega_0}{2}\right)^2\frac{[2+2(\nu_\parallel+\nu_\perp)\tau+ix(1+\zeta)\omega_0\tau]}{[1+2\nu_\perp\tau+ix(1+\zeta)\omega_0\tau](1+2\nu_\parallel\tau)}\right\}^{-1}, \qquad (44)$$

and Q is defined as follows:

$$Q = \exp\left\{-t\left[\frac{1}{\tau} + \frac{\omega_0^2}{8\nu_\parallel} + \frac{\omega_0^2\nu_\perp}{2[4\nu_\perp^2+\omega_0^2x^2(1+\zeta)^2]}\right]\right\}$$
$$\times \exp\left\{it\left[\omega_0\zeta x + \frac{\omega_0^3x(1+\zeta)}{8[4\nu_\perp^2+x^2(1+\zeta)^2]}\right]\right\}. \qquad (45)$$

Greatest interest attaches, however, to the case when $\gamma_{\|}$ is small, corresponding to beats in the μ^+-meson spin precession[5,1,6]. We begin the analysis with the limiting case when $\gamma_{\|} = 0$. It is then seen directly from (37) that the equation can be solved exactly and that its four roots are

$$\mu_{1,2,3,4} = \frac{-(z \pm 2i) \pm \sqrt{z^2 - 4}}{2}. \qquad (46)$$

From (46) we see that the tensor relaxation rate leads to an untrivial physical fact: the beats can exist in one form or another even at large values of the attenuation, provided the $\gamma_{\|}$ are small. By "large" we mean in this case $\gamma \lesseqgtr x^2$.

Let us write down now expressions for the roots in the case of large and small z. In the case of small z (which in particular corresponds to beats in a weak magnetic field, $x \ll 1$), we obtain

$$\mu_1 = 2i - \frac{z}{2} - \frac{z^2}{4}, \quad \mu_2 = -2i - \frac{z}{2} + \frac{z^2}{4},$$
$$\mu_3 = -\frac{z}{2} + \frac{z^2}{4}, \quad \mu_4 = -\frac{z}{2} - \frac{z^2}{4}. \qquad (47)$$

The roots 1 and 2 correspond to very rapid precession, which is practically unobservable. The roots 3 and 4 correspond to beats in the precession with frequency close to the precession frequency of the electron spin.

In the case of large $z \gg 1$, we have an asymptotic expansion

$$\mu_1 = i - \frac{2}{z}, \quad \mu_2 = -i - z + \frac{2}{z}, \quad \mu_3 = i - z + \frac{2}{z}, \quad \mu_4 = -i - \frac{2}{z}. \qquad (48)$$

In the case of large z, it is useful to write out the eigenvalues λ:

$$\lambda_1 = 2\zeta x i + i - \frac{2}{z}, \quad \lambda_2 = -i - z + \frac{2}{z} + 2\zeta x i,$$
$$\lambda_3 = 2\zeta x i + i - z + \frac{2}{z}, \quad \lambda_4 = 2\zeta x i - i - \frac{2}{z}. \qquad (49)$$

It is seen from (49) that in principle the set of roots 1 and 4 can ensure, at sufficiently large z (it is unimportant here whether they are complex or pure imaginary), modulations that can be observed at large values of the magnetic field (large $x \sim 1/2\zeta$). The case of pure imaginary z was considered in fact in[6]. In other words, it turns out that the transverse component of the relaxation rate does not hinder the existence of such oscillations, no matter how large the rate may be. To complete the description of the limiting case $\gamma_{\|} = 0$, let us calculate the coefficients for both limiting cases. At small values of z, the coefficients are

$$C_1 \cong C_2 \cong C_3 \cong C_4 \cong 1/4. \qquad (50)$$

At large z we obtain

$$C_1 \cong C_4 \cong 1/2, \quad C_2 \cong C_3 \cong 0. \qquad (51)$$

When we have considered above the case of large relaxation rates (expressions (39)–(42)), we have tacitly assumed that the fields are not very strong, i.e., the condition $|z| \gg \gamma_{\|}, \gamma_{\|} \gg 1$ is not realized.

Let us consider now this very case. From the equation in the form (37) we see directly that in this case there is one root $\mu_1 \sim -\gamma_{\|}$, another root $\mu_4 \approx [1/\gamma_{\|}]$, and two roots of the order of z. Solving Eq. (38) by successive approximations, we get $\mu_1 = -\gamma_{\|} - 4/z$ or

$$\lambda_1 = 2\zeta x i - \gamma_{\|} + \frac{8yi - 4\gamma_{\perp}}{\gamma_{\perp}^2 + 4y^2}. \qquad (52)$$

The frequency shift might be appreciable, but as seen directly from (41) the coefficient of the root μ_1 is of the order of $1/\gamma_{\|}$, i.e., it is practically equal to zero. For the second root we obtain $\mu_4 \cong -1/\gamma_{\|} - 1/z$, which coincides exactly with (40). The coefficient of this term, corresponds to the second root, is close to unity. The coefficient at the terms corresponding to the remaining roots will be close to zero like $1/z^2$, and we shall therefore disregard them.

Thus, this case does not contain anything of physical interest. However, the situation changes as soon as we decrease $\gamma_{\|}$, to the extent that it becomes comparable with unity. Eq. (37) then assumes the very simple form

$$\left(\frac{1}{\mu + \gamma_{\|}} + \frac{1}{z}\right)\left(\frac{1}{\mu}\right) = -1, \qquad (53)$$

if we consider roots much smaller than z. We obtain immediately that

$$\mu_1 = -\left(\frac{\gamma_{\|}}{2} + \frac{1}{2z}\right) + i\sqrt{1 - \frac{1}{4}\left(\gamma_{\|} - \frac{1}{z}\right)^2},$$
$$\mu_4 = -\left(\frac{\gamma_{\|}}{2} + \frac{1}{2z}\right) - i\sqrt{1 - \frac{1}{4}\left(\gamma_{\|} - \frac{1}{z}\right)^2}. \qquad (54)$$

It is clear from the foregoing that the coefficients for both roots should, generally speaking, be of the order of unity, and since the coefficients of the remaining roots at large z are practically equal to zero, the sum of the coefficients of the terms corresponding to roots 1 and 4 should be quite closely to unity. We conclude therefore that effects connected with the existence of the roots (54) are in principle observable: a shift takes place of the resonant frequency, at which precession takes place with very small frequency in a very strong field. In the absence of attenuation, such a precession was considered in[6], but it was indicated in that paper that exceedingly strong fields will be necessary. The case of semiconductors, in which ω_0 may greatly exceed the vacuum value, was not considered at all, since it was quite clear that large relaxation rates will be obtained. However, it is seen from (54) that, first, slow precession will be observed in a much wider class of cases than the recently discovered[6] beats in a weak field, and second, in the region of large $\gamma_{\|}$ and fields such that precession with the frequency of the μ^+ meson can be observed, i.e., in the region of the dip on the curves in the (γ, P) plane of[1], it is possible to carry out time investigations, provided the absolute values of $\nu_{\|}$ and $\nu_{\perp}$ are themselves sufficiently small. In other words, a possibility is uncovered of investigating ferrites and semiconductors in a wider range than before, particularly at low and infralow temperatures, at which the probability of exchange scattering by free electrons is decreased. We note that follows from (7) that a time dependence will be observed also in a longitudinal field (formula (8) for λ_2 remains valid if a transition $\gamma \to \gamma_{\perp}$ is made in it.)

We now determine the coefficients of the terms with roots (54). A relatively accurate cumbersome formula is obtained in elementary fashion by substituting (54) and (35) in (41). In two limiting cases we have $C_1 = C_4 = 1/2$ if $\gamma_{\|} \ll 2$, but if $\gamma_{\|} \gg 2$ then (since a root close to zero appears), this root, corresponding to (40), enters with weight 1. The transition formula can be written approximately in the following form:

$$C_{1,4} = \pm\left\{-\left[\frac{\gamma_{\parallel}}{2} + \frac{1}{2z} \pm R\right] z^2 - z\right\} \Big/ 2z^2 R,$$

$$R = \sqrt{\frac{1}{4}\left(\gamma_{\parallel} - \frac{1}{z}\right)^2 - 1}. \qquad (55)$$

We proceed to consider weak attenuation when $z \ll 1$. This case corresponds to corrections to expressions (46) and (47), and it follows from a comparison of[5] and[1] that it requires a very cautious analysis. This case is all the more of interest to us, since it corresponds precisely to the recently discovered beats in a weak field[6], and can be significant in a correct analysis of the aggregate of experimental data, since it is quite probable that the existing theory[1,5] may be corrected by taking into account the tensor character of the relaxation rate.

We start from an equation in the form (38) and begin with a case $\gamma \ll 1$, $z \ll 1$; we wish here to find roots $\mu \ll 1$, i.e., $\mu \sim \gamma \sim z$. We then have the equation

$$\mu^2 + \mu\left(z + \frac{\gamma_{\parallel}}{2}\right) + \frac{z}{4}(z + \gamma_{\parallel}) = 0 \qquad (56)$$

and the roots

$$\mu_1 = -\frac{z}{2}, \quad \mu_2 = -\frac{z}{2} - \frac{\gamma_{\parallel}}{2}. \qquad (57)$$

We note it is impossible to go in (57) to the limit as $\gamma_{\parallel} \to 0$, since this may cause the loss of solution properties of interest to us. We now consider the case $z^2 \sim \gamma_{\parallel}$ and $z \gg \gamma_{\parallel}$. We neglect the difference of the roots (57) and assume

$$\mu_1^0 \approx -z/2, \quad \mu = \delta + \mu_1^0. \qquad (58)$$

Substituting (58) in (38), we seek the correction $\delta \sim z^2 \sim \gamma_{\parallel}$. We have the equation

$$4\delta^2 + 2\delta\gamma_{\parallel} + z^4/16 = 0 \qquad (59)$$

and accordingly the roots

$$\mu_{1,2} = -\frac{z}{2} - \frac{\gamma_{\parallel}}{4} \pm \frac{1}{4}\sqrt{\gamma_{\parallel}^2 - \frac{z^4}{4}}. \qquad (60)$$

In the region of the point where the character of the solution changes, $\gamma_{\parallel}^2 \approx z^2/4$, it is necessary to take into account also the term $\sim z^5$; the equation then becomes

$$4\delta^2 + 2\delta\gamma_{\parallel} + \frac{z^4}{16} - \frac{z^3}{8}\gamma_{\parallel} = 0, \qquad (61)$$

and we obtain for the roots the expression

$$\mu_{1,2} = -\frac{z}{2} - \frac{\gamma_{\parallel}}{4} \pm \frac{1}{4}\sqrt{\gamma_{\parallel}^2 - \frac{z^4}{4} - \frac{z^3\gamma_{\parallel}}{2}}. \qquad (62)$$

Expression (62) in conjunction (33)–(35) determines the form of the solution. Let us find the coefficients of the slowly-varying terms:

$$C_{1,2} = \left[\gamma_{\parallel} - \sqrt{\gamma_{\parallel}^2 - \frac{z^4}{4} - \frac{z^3\gamma_{\parallel}}{2}}\right] \Big/ 4\sqrt{\gamma^2 - \frac{z^4}{4} - \frac{z^3\gamma_{\parallel}}{2}}. \qquad (63)$$

Since the remaining two roots would appear under rather exotic conditions, we shall not discuss them at all.

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Translated by J. G. Adashko
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