INFLUENCE OF NONLINEAR EFFECTS ON CURRENT INSTABILITY IN A PLASMA

A. S. KINGSEP

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The effect of nonlinear scattering of waves by particles and of weak plasma inhomogeneity on current instability in a plasma is considered. The noise energy density in the stationary state and the heating law for the plasma components are estimated. It is shown that under experimental conditions nonlinear scattering of waves is a more effective mechanism of damping than Coulomb collisions. The conditions under which weak radial inhomogeneity of the plasma may be a more effective mechanism of establishment of the stationary state than nonlinear wave scattering are established. It is shown that longitudinal inhomogeneity cannot ensure a true stationary state of current instability.

INTRODUCTION

THIS is a continuation of work by Rudakov and Korablev[1], who presented for the first time an analytic solution of the problem of current instability in the quasilinear approximation. In particular, an attempt was made to find the stationary spectrum of ion-acoustic noise corresponding to the steady state, in which the increment γ vanishes.

It was shown that such a spectrum, if it does exist, should have the form[1)]

$$N_{\mathbf{k}} = N_0 N(\vartheta)\delta(k - k_0), \tag{1}$$

where k_0 is defined by the conditions

$$\gamma(k_0; \vartheta) = 0; \qquad \frac{\partial}{\partial \mathrm{k}}\gamma(k_0; \vartheta) = 0. \tag{2}$$

From the condition that the dynamic friction force due to the noise radiation be equal to the force exerted on the particles by the electric field E, we can estimate N_0:

$$N_0 \approx \frac{1}{2\pi\omega_{pi}^2}\frac{m}{e}En/f^e\left(\frac{\omega}{k}\right). \tag{3}$$

In the case when the plasma is strongly non-isothermal, namely when the condition

$$T_e/T_i > 2\ln\left[(T_e/T_i)^{3/2}(M/m)^{1/2}\right] \equiv 2L$$

is satisfied, the phase velocity of the steady-state oscillation spectrum is

$$\left(\frac{\omega}{k}\right)^2_{k=k_0} \approx 2\frac{T_i}{M}L, \tag{4}$$

and the frequency is then close to the ion plasma frequency ω_{pi}. A formal calculation of the stationary distribution $N(\vartheta)$ yielded in[1] the following result:

$$N(\vartheta) = \cos^2\vartheta\frac{4 - 3\cos\vartheta}{(1 - \cos\vartheta)^2}. \tag{5}$$

The oscillation energy density corresponding to such a spectrum becomes infinite. To eliminate this divergence, it is necessary to forego the stationarity condition in the quasilinear approximation.

The absence of a stationary solution can be simply explained on the basis of the momentum conservation law. Indeed, in the quasilinear approximation, the oscillations receive momentum from the electrons and transferred only to the resonant ions. The decrement of damping by the ions is of the order of magnitude of

$$\gamma_i \sim f^i(\omega/k) \sim n_p T_p^{-3/2},$$

where n_p and T_p are respectively the density and temperature of the resonant ions. From the relation

$$\frac{d}{dt}n_p T_p = \int \gamma_i \omega_{\mathbf{k}} N_{\mathbf{k}} d\mathbf{k} \sim T_p^{-3/2}$$

we can see that $T_p \sim t^{2/5}$ and $\gamma_i \sim t^{-3/5}$, so that, taking into account the low density of the resonant ions, we can assume that in the steady state $\gamma_i \approx 0$. The total momentum transferred per unit time to the ions, $\int \gamma_i \mathbf{k} N_{\mathbf{k}} d\mathbf{k}$, also vanishes. Thus, the oscillations lose their ability to transfer momentum to the ions, and consequently in the quasilinear approximation it is impossible to obtain a stationary solution: the noise must increase. In order to obtain somehow a stationary solution of the current-instability problem, it is necessary to take into account some additional damping mechanism that ensures momentum transfer to the ions. One such mechanism, "collision" damping, was considered by Kovrizhnykh in[2]. Two other mechanisms, nonlinear wave interaction and damping due to the plasma inhomogeneity, are investigated in the present paper.

1. ALLOWANCE FOR NONLINEAR DAMPING

The main nonlinear effect that contributes to the evolution of the spectrum of the ion-acoustic waves is in our case the scattering of sound by the thermal ions or, in other words, the scattering of quasiparticles by particles. The weak nonlinear damping will be appreciable only at small angles (the symmetry axis is the direction of the electric field **E**), while at large angles the distribution of the quasiparticles has, as before, the form (5). We seek a stationary spectrum at small angles in the form

$$N_{\mathbf{k}} = N_0[\vartheta^2/2 + \varepsilon(\vartheta)]^{-2}\delta(k - k_0 + \Delta k(\vartheta)), \tag{6}$$

where $\epsilon(\vartheta)$ and $\Delta k(\vartheta)$ vanish at finite values of ϑ, and by

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1)The main symbols are:

$$N_{\mathbf{k}} = \frac{d\varepsilon}{d\omega_k}\frac{k^2}{8\pi}\left|\varphi_{\mathbf{k}}\right|^2$$

is the quasiparticle distribution function; k, ϑ, and φ are the coordinates in momentum space, f^α are the distribution functions, and T_α the temperatures of the components α = i,e. For details see [1].

the same token solution (6) is joined to the stationary solutions (1) and (5).

We rewrite the system (2) in the form

$$\gamma_l+\gamma_n=0, \qquad \frac{\partial}{\partial k}(\gamma_l+\gamma_n)=0. \tag{2'}$$

Here γ_l is the increment obtained from the quasilinear theory[1], and when the $k(\vartheta)$ relation is taken into account it takes the form

$$\gamma_l=2\gamma_l^0\left\{\int_z^1\frac{ydy}{\sqrt{y^2-z^2}\sqrt{1-y^2}\,I(y)}\int_0^y\frac{z'(z-z')N(z')dz'}{\sqrt{y^2-z'^2}}-\frac{1}{k_0}\int_z^1\frac{ydy}{\sqrt{y^2-z^2}\sqrt{1-y^2}\,I^2(y)}\int_0^y\frac{zz'N(z')dz'}{\sqrt{y^2-z'^2}}\times\int_0^y\frac{z'^2N(z')[\Delta k(z)-\Delta k(z')]}{\sqrt{y^2-z'^2}}dz'\right\};$$

$$z=\cos\vartheta,\quad z'=\cos\vartheta',\quad \gamma_l^0=\frac{\pi\omega_{pi}^4}{2k_0^3}\frac{M}{mn}f^e\left(\frac{\omega}{k}\right),$$

$$I(y)=\int_0^y\frac{z'^2N(z')dz'}{\sqrt{y^2-z'^2}}. \tag{7}$$

The nonlinear part of the increment γ_n, which takes into account the scattering of the sound by the ions, can be easily obtained if it is recognized that short-wave ion sound, which represents simply ionic plasma oscillations, builds up in a sufficiently non-isothermal plasma. For such oscillations, the contribution of the electrons results only in a small correction to the nonlinear increment. We can therefore use directly the known result for the scattering of Langmuir oscillations by electrons, and write in analogy with them*

$$\gamma_n=\frac{\sqrt{\pi}}{Mn}\sum_{k'}\frac{\omega_k-\omega_{k'}}{c_i|\mathbf{k}-\mathbf{k}'|^3}[\mathbf{k}\mathbf{k}']^2\frac{(\mathbf{k}\mathbf{k}')^2}{k^2k'^2}N_{\mathbf{k}'}.$$

If we now go in this expression to the limit $\vartheta,\vartheta'\ll 1$, where ϑ and ϑ' are the polar angles of the vectors $\mathbf{k}$ and $\mathbf{k}'$, and use expression (6) for the plasmon distribution function, we finally get

$$\gamma_n=-\gamma_n^0\int\frac{[\Delta k(\vartheta)-\Delta k(\vartheta')]}{k_0(\vartheta'+\vartheta)[\vartheta'^2/2+\varepsilon(\vartheta')]^2}K\left(2\frac{\sqrt{\vartheta\vartheta'}}{\vartheta+\vartheta'}\right)\vartheta'd\vartheta'. \tag{8}$$

Here $K(x)$ is the complete elliptic integral,

$$\gamma_n^0=\frac{2\sqrt{\pi}N_0\omega_{pi}k_0}{Mnc_iD_e^2},\qquad c_i=\sqrt{\frac{2T_i}{M}}.$$

D_e and D_i are the Debye radii. We shall assume the functions $\Delta k(\vartheta)$ and $\epsilon(\vartheta)$ to be analytic when $\vartheta=0$. Then at small angles we have

$$\Delta k(\vartheta)\approx k_0(\delta-g\vartheta^2/2),\qquad \varepsilon(\vartheta)\approx\varepsilon_0+\varepsilon_1\vartheta^2/2. \tag{9}$$

We note that the quantity δ does not enter in general in the expression for the increment, and allowance for ϵ_1 leads only to a multiplication of γ_n by a certain coefficient of the order of unity. In order to estimate ϵ_0 and g, we put in the expressions for the increment $\vartheta<\sqrt{\epsilon_0}$, when it is possible to expand the elliptic integral in a series. In this region, the increment, accurate to ϑ^2, assumes the following form:

$$\gamma\approx\gamma_l^0(1-g)\int_0^\xi\frac{d\eta}{\sqrt{\eta(\xi-\eta)}}\int_\eta^1\frac{(\xi'-\xi)d\xi'}{(\xi'+\varepsilon_0)^2\sqrt{\xi'-\eta}}\cdot\times\left[\int_\eta^1\frac{d\xi'}{(\xi'+\varepsilon_0)^2\sqrt{\xi'-\eta}}\right]^{-1}-\gamma_n^0\frac{\pi g}{4}\int_0^\infty\frac{\vartheta'^2d\vartheta'}{[\varepsilon_0+\vartheta'^2/2]^2} \tag{10}$$

where $\xi=\vartheta^2/2$ and $\xi'=\vartheta'^2/2$. The upper limit of the integral, which enters in γ_n, can be regarded as infinite, corresponding to a transition from small angles to finite angles.

Substituting (10) in the system (2) and estimating the integrals contained therein, we obtain

$$g={}^2/_3,$$

$$\varepsilon_0=\zeta\left(\frac{\gamma_n^0}{\gamma_l^0}\right)^{2/3}\sim\left[\frac{mN_0k_0^4}{2\sqrt{\pi}M\omega_{pi}^3c_iD_e^2f^e(\omega/k)}\right]^{2/3},\qquad \zeta\sim 1; \tag{11}$$

$$k(\vartheta)\approx k_0(1+\vartheta^2/3),$$

$$N_{max}(\vartheta)=N(0)\approx N_0(\gamma_l^0/\gamma_n^0)^{1/3}\sim N_0\left[\frac{m}{M}\left(\frac{T_i}{T_e}\right)^2\frac{neD_i}{E}\right]^{1/3}. \tag{11'}$$

The estimates (11) were obtained under the assumption that $N(\vartheta)$ assumes a maximum value at $\vartheta=0$. If, on the other hand, $N_{max}=N(\vartheta_0)$ when $\vartheta_0\neq 0$, then (11) and (11′) still remain of the correct order of magnitude. The solution of the system (2), (7), and (8) can in this case have no singularities at small angles. Indeed, such a function $N(\vartheta)$ would make the nonlinear part of the increment equal to $-\infty$, whereas its linear part γ_l would remain finite (γ_l would contain ratios of diverging integrals of the same kind).

Let us establish the law governing the growth of the temperatures of the ions and electrons in the case when collisions can be neglected, and the heating of the nonresonant ions is due only to the nonlinear wave scattering. Denoting by W the energy density, we can write (see also[3]):

$$\frac{dW_i/dt}{d(W_i+W_e)/dt}=\frac{\int\gamma_n(\mathbf{k})(\omega_k-\omega_{k'})N_{\mathbf{k}}d\mathbf{k}}{neEv_0}\sim\frac{1}{v_0}\frac{\int\gamma_n(\mathbf{k})(\omega_k-\omega_{k'})N_{\mathbf{k}}d\mathbf{k}}{\int\gamma_l(\mathbf{k})(k-k')N_{\mathbf{k}}d\mathbf{k}}\sim\frac{1}{v_0}\left|\frac{\partial\omega}{\partial k}\right|_{k=k_0}, \tag{12}$$

where v_0 is the current velocity. Putting

$$\frac{T_{0e}}{T_{0i}}\gg 2\ln\left[\left(\frac{T_e}{T_i}\right)^{3/2}\sqrt{\frac{M}{m}}\right]\equiv 2L,$$

we get

$$\frac{1}{v_0}\frac{\partial\omega}{\partial k}\sim\left(\frac{v_0}{c_s}\right)^2\sim 2L\frac{T_i}{T_e},$$

whence

$$\frac{\dot{T}_e}{\dot{T}_i}\approx\frac{T_e}{T_i}(2L)^{-1}. \tag{13}$$

Neglecting the small change of the quantity with temperature, we obtain the final result in the form

$$\frac{T_e}{T_{0e}}\approx\left(\frac{T_i}{T_{0i}}\right)^{1/2L}\varphi(L), \tag{14}$$

where $\varphi(L)$ is a slowly varying function. It is seen from (14) that as the components become heated, the ratio T_e/T_i decreases.

On the other hand, if $T_e/T_i\lesssim 2L$, the heating law has a different form. Since in this case long-wave sound builds up ($kD_e\lesssim 1$), we have

$$\frac{1}{v_0}\frac{\partial\omega}{\partial k}\approx\frac{c_s}{v_0},$$

$$v_0=Lc_s\left[1+\left(\frac{T_e}{T_i}\right)^{3/2}\sqrt{\frac{M}{m}}\exp\left(-\frac{T_e}{2T_i}\right)\right].$$

*$[\mathbf{k}\mathbf{k}']\equiv\mathbf{k}\times\mathbf{k}$.

Upon heating, the ratio T_e/T_i increases, reaching 2L in the limit. The heating occurs in accordance with the law

$$T_i \approx eEc_s \approx \frac{e^2E^2}{8L^2M}t^2, \quad T_e \approx 2LT_i. \tag{15}$$

Thus, for any initial non-isothermal behavior, there is established in the limit a certain "universal" temperature ratio

$$T_e/T_i \sim 2L.$$

It is interesting to compare the results with those of Kovrizhnykh[2]. He considered the problem of the instability of current in a plasma with allowance for collisions. In particular, he obtained a stationary noise spectrum, which can be represented when $\vartheta \ll 1$ in the form

$$N(\mathbf{k}) \approx \frac{N_0}{[\vartheta^2/2 + \gamma_{\text{col}}/\gamma_l{}^0]^2}\delta(k-k_0),$$

$$\gamma_{\text{col}} \approx \sqrt{\frac{mT_e}{MT_i}}\nu_{ei}(kD_e)^2; \quad \nu_{ei} = \frac{4\pi ne^4\Lambda}{m^2c_e{}^3}, \tag{16}$$

where Λ is the Coulomb logarithm. The spectrum of the form (16) can be obtained from the following simple considerations. In the absence of ion damping, the solution of the stationary quasilinear problem is the spectrum (5) with infinite energy density. When damping is taken into account, the momentum and the energy are transferred to the ions, and the oscillation energy density is finite, a fact that can be taken into account by introducing a certain cutoff parameter in the spectrum (5):

$$N(\vartheta) \approx 4[\vartheta^2 + \vartheta_0{}^2]^{-2}. \tag{5$'$}$$

We equate the momentum acquired by the electrons from the field to the momentum transferred to the ions via the noise:

$$ne\mathbf{E} = \int \gamma_i N(\mathbf{k})\mathbf{k}\,d\mathbf{k} = \int \gamma_i \frac{W}{\omega_{pi}}\mathbf{k}\,d\mathbf{k},$$

whence, using the form of the spectrum (5′), we can arrive at the expression

$$neE \sim \frac{\gamma_i}{\vartheta_0{}^2}k^3N_0.$$

Substituting expression (3) for N_0 and using the form of γ_{l_0} from (7), we obtain

$$\vartheta_0{}^2 \sim \gamma_i/\gamma_\pi{}^0.$$

Equating $\gamma_i \equiv \gamma_{\text{col}}$, we obtain a spectrum of the form (16).

In order for the stationary distribution (16) to be established earlier than the stationary distribution (6) and (11), and therefore in order for the nonlinear interaction of the waves to be negligible, it is necessary to satisfy the inequality

$$(\gamma_{\text{col}}/\gamma_l{}^0)^{3/2} > \gamma_\pi{}^0/\gamma_l{}^0,$$

from which we can obtain with the aid of (3), (7), (8), and (16)

$$E < E_{\text{Dr}}(k_0D_e)^{7/2}\left(\frac{mT_e}{MT_i}\right)^{1/4} N_D^{-1/2}, \tag{17}$$

$E_{\text{Dr}} \sim T_e\Lambda/8\pi neD_e^4$ is the Dreicer field and N_D is the number of particles in the Debye sphere.

On the other hand, the very condition for the existence of ion-acoustic instability states that

$$E > \sqrt{\frac{m}{M}}\frac{\omega}{kc_s}E_{\text{Dr}}; \quad c_s = \sqrt{\frac{T_e}{M}}. \tag{18}$$

The inequalities (17) and (18) can be satisfied simultaneously if

$$\frac{T_e}{T_i} > 2LN_D^{1/5}; \quad L = \ln\left[\left(\frac{T_e}{T_i}\right)^{3/2}\sqrt{\frac{M}{m}}\right]. \tag{19}$$

Assuming for concreteness $n \sim 10^{13}$ cm^{-3} and $T_e \sim 1$ keV, we get $N_D \sim 10^7$–10^8, which T_e/T_i of the order of several hundred.

The inequality (19), which follows from (17) and (18), determines the condition for the applicability of the solution obtained in[2].

In conclusion let us say a few words concerning the limits of applicability of the results of the first part of the investigation. We have assumed throughout that the distribution functions f^e and f^i remain Maxwellian during the course of the development of the instability and of the heating. The result (5) obtained in[1], and consequently the solution (6), (11), and (11′), change very little if this assumption is not used. If the plasma is sufficiently non-isothermal, the structure of the distribution functions will influence only the value of k_0 of the steady-state oscillation spectrum. As to the results (14), (15), and (19), the structure of the distribution functions is quite important for them. For non-Maxwellian distributions f^e and f^i, the condition for the applicability of the solution obtained in[2] is determined, as before, by the inequalities (17) and (18), but not by the condition (19). The relation (12) for the heating rates of the components remains in force, but formulas (14) and (15) turn out to be inexact, although they determine correctly the general character of the heating process.

2. CURRENT INSTABILITY IN A WEAKLY INHOMOGENEOUS PLASMA

One more effect that can ensure in principle the transfer of momentum to nonresonant ions is the shift of the phase velocities of the ion-acoustic waves as a result of weak inhomogeneity of the plasma. We shall henceforth confine ourselves to a plasma of inhomogeneous density. We shall assume that the current instability developed locally, just as in a homogeneous plasma, in accordance with the results of[1] and[2]. This is possible under the condition

$$\left\langle \left|\frac{\nabla n}{n}\right| \frac{\partial\omega}{\partial k}\right\rangle \ll \gamma_\pi{}^0,$$

where γ_l^0 is determined from (7) and represents the reciprocal of the characteristic growth time of the instability. Denoting by a the characteristic scale of the inhomogeneity and using (7), we obtain

$$(ka)^{-1} \ll \sqrt{mT_i/MT_e}. \tag{20}$$

When condition (20) is satisfied, the growth time of the instability will be much shorter than the characteristic damping time, which is determined by the expression $\lambda/|\partial\omega/\partial\mathbf{k}|$, where λ is the plasmon mean free path. The behavior of the quasiparticles in such a model can be regarded in the geometrical-optics approximation, and consequently their distribution function should satisfy the Liouville equation:

$$\frac{\partial N}{\partial t}+\frac{\partial \omega}{\partial \mathbf{k}} \nabla N-\nabla \omega \frac{\partial N}{\partial \mathbf{k}}=2 \gamma N. \tag{21}$$

Assuming the process to be stationary and integrating over the characteristics with allowance for the dispersion properties of the ion sound, we obtain

$$k^{-2}+D_e^2(x)=\text{const}, \tag{22}$$

$$N(\mathbf{k}, x_2)=N(\mathbf{k}_{12}, x_1) \exp \left[2 \int_{x_1} \frac{\gamma(\mathbf{k}, x)}{(\partial \omega / \partial \mathbf{k})_x} d x\right]. \tag{23}$$

The integral in (23) is taken along the density gradient; by $\mathbf{k}_{12}$ we should mean a point in momentum space, connected with $\mathbf{k}$ by the relation (22), and the components of $\mathbf{k}_{12}$ and $\mathbf{k}$ perpendicular to the density gradient coincide.

We note that expression (22) has a simple physical meaning—this is the condition that the frequency of the plasmon be constant in a quasiclassical approximation. From it, in particular, it follows that the plasmon traveling in the direction of the density gradient loses momentum (becomes "red") and its principal damping mechanism is weak damping by the electrons. The momentum of a plasmon traveling in the opposite direction increases, and such a plasmon is rapidly damped by theraml ions, traversing from the point of its creation a density difference

$$\frac{\delta n}{n} \equiv \frac{|n(x_2)-n(x_1)|}{n(x_1)} \sim (k D_e)^{-2}. \tag{24}$$

Assuming that the value of the wave vector of the oscillations changes little compared with the results of the linear theory, we obtain

$$\frac{\delta n}{n} \sim \frac{1}{L} \frac{T_i}{T_e}; \quad L \sim 1 \div 10. \tag{24'}$$

We shall approximate the case of weak radial inhomogeneity in a plasma column by a simpler model: a constant density gradient perpendicular to the electric field (Fig. 1). Since in the linear theory the quasiparticles accumulate near the direction of the average electron velocity, we confine ourselves to a consideration of quasiparticles at small angles. Such a plasmon is acted upon in such a system by a force perpendicular, accurate to ϑ^2, to its momentum

$$\dot{\mathbf{k}}=-\nabla \omega \approx-\frac{\omega_{pi}}{2} \frac{\nabla n}{n} \perp \mathbf{k}. \tag{25}$$

(We assume, just as in the linear theory, that short-wave ion sound builds up—this is guaranteed by the condition (20).)

Thus, for plasmons at small angles, the influence of the inhomogeneity in first order perturbation theory leads only to a rotation of the momentum towards the smaller densities, without a change of the modulus k.

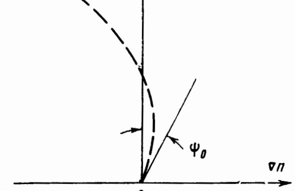

FIG. 1

In view of the absence of a symmetry axis in such a system, it is convenient to use in lieu of ϑ and φ a system of orthogonal angle coordinates: $\psi = \sin\vartheta \cos\varphi$ and $\chi = \sin\vartheta \sin\varphi$, where φ is reckoned from the ∇n direction. Then at small angles the equation of the plasmon trajectory will take the form

$$\chi=\text{const}; \quad \dot{\psi}=-\frac{\omega_{pi}}{2(k a)}; \quad a^{-1} \equiv\left|\frac{\nabla n}{n}\right|,$$

i.e., after a time interval

$$\Delta \tau(\psi_0) \approx 2 \frac{k a}{\omega_{pi}} \psi_0 \tag{26}$$

a plasmon emitted at an acute angle to the density gradient will fall in a plane perpendicular to the direction of ∇n. Thus, starting with the instant of time $\nabla\tau$, the trajectories of all the plasmons emitted at a given angle χ will be the same, as if they started from a plane perpendicular to ∇n, but with a delay (or with an advance—for negative ψ_0) determined by expression (26) (see Fig. 1, where OO′ is the plasmon trajectory).

Since at small angles the momentum of the quasiparticle remains constant in modulus, the plasmon is situated during this entire time on the border of intense ion damping. However, when the plasmon goes into the region of finite angles, its drift into the region of lower density, with velocity $\psi\partial\omega/\partial k$, which leads to a growth of the momentum, becomes appreciable. As a result, the quasiparticle is rapidly damped by the thermal ions. Thus, the effective damping time of the plasmon is determined by the time of rotation through a certain finite angle or, which is the same, by the time of the relative change of the momentum by an amount on the order of unity

$$\tau_{eff} \sim ka / \omega_{pi}. \tag{27}$$

The addition (27) to the damping time, which takes into account the angle at which the quasiparticle is emitted, is readily seen to be small, and consequently in estimating the energy density of the noise we can assume that the damping decrement

$$\gamma_i \sim -\omega_{pi} / ka$$

is the same for all the quasiparticles at small angles.

The electronic part γ_e of the increment should be calculated from the equations of the quasilinear theory, in analogy with the procedure used in[1] and[2] in a homogeneous plasma. To simplify the problem, we assume that there is present in the plasma a magnetic field which is sufficiently weak to be able to assume the electrons in the oscillations to be non-magnetized, but still strong enough to be able to neglect the axial symmetry of the electronic distribution function:

$$\omega_{pe} \gg \omega_{He} \gg \nu_{eff},$$

where $\nu_{eff} \approx eE/mc_s$ is the effective frequency of electron scattering by the noise. In this case we can use expression (7) for the increment, but it contains not the exact distribution function $N(\vartheta, \varphi)$, but the function $\overline{N}(\vartheta)$ averaged over the azimuth. We note that the same function $\overline{N}(\vartheta)$ enters in the expression for the noise energy density

$$W \approx 2\pi\omega_{pi} k^2 \int_0^1 \overline{N}(\vartheta)\, d\cos\vartheta.$$

We represent $\overline{N}(\vartheta)$ in the form (6) and approximate the function $\epsilon(\vartheta)$ simply by the cutoff parameter ϵ_0. Then the integrals entering in (7) can be readily calculated at small angles. As a result we get the expression

$$\varepsilon_0 = -\frac{\gamma_i}{\pi\gamma_л^0} \sim \frac{m}{M}\left(\frac{kc_e}{\omega_{pi}}\right)^3 (ka)^{-1}. \tag{28}$$

The estimate (28) can be obtained also by another method, from the momentum conservation law. The derivation is perfectly analogous to the derivation of (16).

We note that the characteristic damping time is

$$\tau_{eff} \sim \frac{ka}{\omega_{pi}} \ll a \Big/ \left|\frac{\partial\omega}{\partial \mathrm{k}}\right|,$$

since the plasmon momentum changes by a factor T_e/T_i over the length of the inhomogeneity. Therefore the condition for the applicability of the result (28) is more stringent than the inequality (20), namely:

$$(ka)^{-1} \ll \sqrt{\frac{m}{M}}\,(T_i/T_e)^{3/2}. \tag{29}$$

Comparing (28) with the estimates (11) and (11′) obtained with allowance for the nonlinear effects, we can see that the influence of the inhomogeneity turns out to be more appreciable, provided that

$$E \ll \sqrt{\frac{mT_i}{MT_e}}\,\frac{neD_i}{ka}. \tag{30}$$

At the same time, the condition for the existence of the instability is

$$E \gg \sqrt{\frac{m}{M}}\,\frac{\omega}{kc_s}\,\frac{T_e\Lambda}{8\pi neD_e^4}.$$

These inequalities are compatible if

$$(ka)^{-1} \gg \frac{\Lambda}{nD_e^3}\sqrt{\frac{T_e}{T_i}}. \tag{31}$$

In turn, the inequalities (29) and (31) are compatible if

$$nD_e^3 \gg \sqrt{\frac{M}{m}}\left(\frac{T_e}{T_i}\right)^2 \Lambda,$$

which is practically always satisfied in the experiments.

Let us turn to another model. Assume that the density in the plasma has a certain profile along the electric field (Fig. 2). When the field if turned on, the instability limit will be exceeded primarily at the points x_0 and x_0', where the density is minimal, and consequently the electron velocity is maximal. Subsequently the established current velocity is such that in certain regions ($x_1 < x < x_2$, $x_1' < x < x_2'$) plasmons are created, and are then attenuated in the regions $x_2 < x < x_3$, etc. Assuming the energy density on the boundaries of the noise region to be the same, and using solution (23) of the Liouville equation, we write

$$\int_{x_1}^{x_3} \frac{\gamma(x)}{(\partial\omega/\partial \mathrm{k})_x}\,dx = 0, \tag{32}$$

under the condition that

$$\frac{n(x_1) - n(x_0)}{n(x_0)} \ll (kD_e)^2,$$

otherwise the plasmons created near the point x_1 would be damped on the thermal ions, going into the region where the density decreases. The transfer of momentum to the ions will be the result of adiabatic interaction of the waves with the medium. Indeed, the plasmon acquires momentum, going into the region of lower density, and gives it up to the medium on the path from x_0 to x_3. The average momentum density transferred per unit time

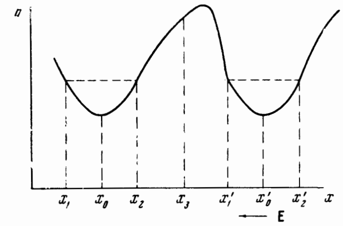

FIG. 2

$$\dot{\bar{p}} = -\frac{1}{|x_3 - x_1|}\int_{x_1} dx \int d\mathrm{k} N_{\mathrm{k}}(x)\,\frac{dk}{dx}\left(\frac{\partial\omega}{\partial \mathrm{k}}\right)_x \tag{33}$$

$$= \frac{1}{x_1 - x_3}\int_{x_1}^{x_3} dx \int dk\,\frac{dk}{dx}\left(\frac{\partial\omega}{\partial \mathrm{k}}\right)_x N(\mathrm{k}, x_1)\exp\left\{\int_{x_1} \frac{2\gamma(y)}{(\partial\omega/\partial \mathrm{k})_y}\,dy\right\}$$

is a positive quantity.

In the case of a periodic inhomogeneity, when the point x_0 is not the only minimum, the region $x_3 < x < x_1'$ may "collapse" at small density differentials, so that the quasiparticles can pass from one well to another. In this case the integral (32) should be taken over the period of the inhomogeneity. As before, the transfer of momentum from the waves to the medium is ensured by the adiabatic interaction. In fact, let us consider points that are symmetrical in density on two sides of the well (for example, x_1 and x_2). It is seen from the solution (23) that the number of plasmons at the point x_2, where the momentum is transferred to the medium, is always larger than the number of plasmons that acquire momentum from the medium at the point x_1, which is symmetrical with respect to density.

Finally, if the density differential in the system exceeds $(kD_e)^2$, and the stationary current velocity is sufficiently large to make the density differential between the boundary of the region $\gamma > 0$ (point x_1) and the bottom of the well x_0 reach a value on the order of $kD_e)^2$, plasmons appear, created near the point x_1 and attenuating on the thermal ions in the vicinity of the point x_0. This attenuation mechanism is analogous to the attenuation already considered by us in a system with a transverse density gradient.

There are thus simultaneously two mechanisms for the transfer of momentum from the electrons to the ions. Nonetheless, such a longitudinal inhomogeneity still cannot contribute to the establishment of a stationary state. To explain this fact, we shall use the following results of an investigation of the stability of current in a homogeneous plasma. It is easy to verify that the quasilinear increment (7) obtained in[1] will always be a positive quantity for plasmons traveling in the direction of the electron velocity ($\vartheta = 0$). Even in the case when the current velocity is smaller than the critical velocity, so that the integral energy density decreases, quasiparticles traveling in the direction $\vartheta = 0$ accumulate. This means that in the region of small angles there is always a momentum and energy flux—the spectrum tends to go over from a three-dimensional one to a one-dimensional one. The energy density remaining in a finite one-dimensional spectrum as a result of such a process can be estimated if it is taken into account that even a small nose level suffices to make the distribution function of the electrons spherically symmetrical. In this case the damping by the electrons does not lead to a loss of momentum, so that the total momentum of the waves in the initial and final states is approximately the same:

$$\int \frac{W_i(k)}{\omega} \mathbf{k}\, d\mathbf{k} = \frac{W_f}{\omega} \mathbf{k}_f,$$

where W is the energy density. Projecting this equality on the direction $\vartheta = 0$, we obtain the following ratio for the integral energy densities:

$$\frac{W_f}{W_i} = \frac{\int_0^1 W_i(\vartheta) \cos\vartheta \, d\cos\vartheta}{\int_0^1 W_i(\vartheta)\, d\cos\vartheta}. \tag{34}$$

The quantity in the right side is close to unity, since the energy density of the waves is concentrated near the direction $\vartheta = 0$.

We consider now the evolution of the spectrum of plasmons produced near the bottom of the well x_0. Going over to the region of larger densities, they can only lose momentum, and consequently, both their buildup and their attenuation are determined only by the electrons. But then in the region $x > x_2$ the quasiparticles are gathered in a one-dimensional spectrum, losing only an insignificant part of the energy. Even independently of the density profile, the spectrum will remain one-dimensional, and a new stationary state is established for it—a plateau on the electron distribution function. Inasmuch as $\gamma \equiv 0$ in such a state, the oscillations cannot attenuate outside the region $x_1 < x < x_2$, and $\gamma > 0$ in this region. As a result, the energy density of the noise begins to accumulate. Thus, the longitudinal inhomogeneity cannot ensure a truly stationary state for the instability of the current in the plasma.

We have considered in this paper only a plasma with non-uniform density. The shift of the phase velocities of the oscillations can result also from inhomogeneity of the electron temperature, and the shift of the boundary of the resonance region can be due to inhomogeneity of the ion temperature. However, weak temperature drops of the components can lead only to a weak shift of the phase velocity or of the instability boundary, and therefore can likewise not ensure an effective damping of the noise.

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