SOVIET PHYSICS JETP VOLUME 29, NUMBER 4 OCTOBER, 1969

RADIAL SELF-FOCUSING OF AN ELECTRON BEAM DUE TO THE TWO-STREAM INSTABILITY IN A PLASMA

V. B. KRASOVITSKIĬ

Physico-technical Institute, Ukrainian Academy of Sciences

Submitted April 3, 1968

Zh. Eksp. Teor. Fiz. 56, 1252–1261 (April, 1969)

The possibility is discussed of using collective processes to achieve radial self-focusing of a beam of charged particles. It is shown that this possibility exists when a bounded beam interacts with the plasma under conditions such that the two-stream instability can be excited. The radial beam compression occurs under the effect of high-frequency pressure forces due to the surface plasma waves generated by the beam. The focusing time is of the order of the reciprocal growth rate for the two-stream instability.

ONE of the most promising new methods for the acceleration of charged particles is the self-stabilized beam[1] which makes it possible to attain high magnetic fields in regions which the accelerated particles are located. One of the important features of this method is the use of the electromagnetic radiation of electrons that oscillate in the transverse potential well of uncompensated electron-ion beams, this feature making it possible to obtain radial compression of the beam. Since the generation of transverse oscillations requires partial conversion of longitudinal motion into transverse motion, and since this effect is due to the relatively slow process of multiple scattering of electrons on ions, the beam compression time is found to be rather long (of the order of a second or more). Inasmuch as the electrons can traverse enormous distances in this time, the self-stabilized beam can only be used for cyclic accelerators.[1]

The beam compression time can be reduced significantly if focusing is achieved through more rapid processes (as compared with binary collisions) involved in coherent radiation of the beam electrons.[1)] In this case there is a real possibility of obtaining a straight, self-focused electron beam under laboratory conditions. Below we consider one of the possible methods of obtaining this focusing; specifically, we consider the self-focusing of an electron beam moving through a plasma under conditions such that the two-stream instability can be excited.[2] It is reasonable that the most convenient method for this purpose lies in the excitation of oscillations due to transverse ordered motion of the beam electrons[2)] because under these conditions there is no significant retardation of the beam in the longitudinal direction due to the feedback effect of the oscillations on the beam motion. However, the excitation of these oscillations can be realized most effectively in the presence of a magnetic field, taking account of which would cause a considerable complication of the problem; in the present work we shall consider the simpler case of oscillations associated with the longitudinal motion of the electrons. The analysis of this effect for the case of a finite magnetic field is the topic of further investigations. It should also be noted that the energy lost in the beam due to the excitation of oscillations can be compensated by means of an external accelerating field.

In the case we are considering the beam compression is caused by a high-frequency pressure[3,4] due to plasma waves excited by the beam.[3)] Since the focusing force is a "gradient" force, to realize radial focusing it is necessary that the amplitude of the focusing field increase in the radial direction. This condition can be satisfied if use is made of surface plasma waves,[1,6-8] for example, by passing a beam through a channel in a plasma.

The interaction of a low-density electron beam $n_1 \ll n_0$ with a plasma can be described by a system of equations consisting of Maxwell's equations for the fields, the linearized hydrodynamic equations of motion for the electrons in the plasma, and the equations of motion for the beam electrons. In the linear approximation the solution of this system can be written in the form

$$h(t, r, z) = \int h_k(r) \exp[i(kz - \omega t)]\, dk.$$

Then the amplitude of the electric field of the k-th harmonic of a slow axially symmetric E wave as a function of the coordinate r inside the plasma channel is described by the following equation:

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{dE_{zk}}{dr}\right) - k^2 E_{zk} = \frac{4\pi}{\omega}\left[ik^2 j_{zk} + k\frac{1}{r}\frac{d}{dr}(r j_{rk})\right] = g_k(r), \quad (1)$$

where j_{zk} and j_{rk} are the Fourier components of the beam current, which can be found from the linearized equations of motion of the beam. Since the beam density is assumed to be small, in solving Eq. (1) by successive approximations (in the parameter $n_1/n_0 \ll 1$)[9] we find the field components E_{zk}, E_{rk} and $E_{\varphi k} = \omega E_{rk}/ck$:

[1)]The idea of using collective processes to achieve radial self-focusing of beams of charged particles was proposed by Ya. B. Faĭnberg several years age. It also appears, to the best of our knowledge, that similar suggestions were made independently by O. I. Yarkov.

[2)]This ordered motion, in particular, can be due to betatron oscillations in cyclic accelerators.

[3)]The confinement of a plasma in a cylindrical wave guide by means of an external traveling TE mode has been realized experimentally. [5]

$$E_{zk} = AI_0(kr) + \int_0^r [I_0(kr)K_0(k\xi) - I_0(k\xi)K_0(kr)]\, g_k(\xi)\,\xi d\xi,$$
$$E_{rk} = -iAI_1(kr) - i\int_0^r [I_1(kr)K_0(k\xi) + K_1(kr)I_0(k\xi)]\, g_k(\xi)\,\xi d\xi. \quad (2)$$

Correspondingly the fields outside the beam are given by

$$E_{zk} = BI_0(kr); \quad E_{rk} = -iBI_1(kr) \quad (2a)$$

(A and B in (2a) are arbitrary constants).

The dispersion equation, which describes the dependence of the frequency ω on the wave vector k, can be determined from the continuity conditions on the tangential components of the fields E_z and H_φ at the boundary between the beam and the plasma. Assuming that the beam radius a is small compared with wavelength ($ka \ll 1$), we have

$$1 - \frac{\omega_0^2}{\omega^2} + \frac{4\pi k^2}{\omega}\ln\left(\frac{C}{\pi ka}\right)\int_0^a \sigma_k(r)\,r dr = 0, \quad (3)$$

where $j_{zk} = \sigma_k E_{zk}$ and C = 0.577 is Euler's constant. It should be noted that the dispersion equation (3) depends only on the longitudinal component of the beam current. This situation holds only for a thin beam $ka \ll 1$ because under these conditions the transverse current appears in the integrand in Eq. (3) under the derivative sign and thus does not make a contribution to the integral when $\sigma_k(a) = 0$ that is to say, if the beam density vanishes at the boundary.

In describing the properties of the beam we make use of the kinetic equation for the beam distribution function $F(t, \mathbf{r}, \mathbf{v})$:

$$\frac{\partial F}{\partial t} + v_r\frac{\partial F}{\partial r} + \frac{v_\varphi}{r}\frac{\partial F}{\partial \varphi} + v_z\frac{\partial F}{\partial z} + \frac{e}{m}E_z\frac{\partial F}{\partial v_z} + \left(\frac{e}{m}E_r + \frac{v_\varphi^2}{r}\right)\frac{\partial F}{\partial v_r} - \frac{v_r v_\varphi}{r}\frac{\partial F}{\partial v_\varphi} = 0. \quad (4)$$

(the variables, v_r, v_φ, and φ are independent so that differentiation with respect to the space angle φ is subject to the explicit dependence of F on φ).

This equation can be simplified by making use of the axial symmetry of the problem. Assuming that the distribution function can be written in the form

$$F(t, r, z, v_r, v_\varphi, v_z) \equiv f(t, r, z, v_r, v_z)\,\delta(v_\varphi),$$

we average (4) over the variable v_φ. Under these conditions the third and seventh terms vanish and the last term can be combined with the second. As a result we obtain the following equation for the distribution function f:[4)]

$$\frac{\partial f}{\partial t} + \frac{1}{r}\frac{\partial}{\partial r}(rv_r f) + v_z\frac{\partial f}{\partial z} + \frac{e}{m}E_z\frac{\partial f}{\partial v_z} + \frac{e}{m}E_r\frac{\partial f}{\partial v_r} = 0. \quad (4a)$$

Writing the distribution function in the form

$$f = f_0 + f_1 = f_0 + \int f_k e^{i\Phi_k}\, dk,$$

(where f_0 is the average, slowly varying function while f_1 is the ensemble of oscillations with random phases) we obtain the following system of equations for the functions f_0 and f_k:

$$\frac{\partial f_0}{\partial t} + \frac{1}{r}\frac{\partial}{\partial r}(rv_r f_0) + \frac{e}{m}\left\langle \mathbf{E}\frac{\partial f_1}{\partial \mathbf{v}}\right\rangle = 0,$$
$$-i(\omega - kv_z)f_k + \frac{1}{r}\frac{\partial}{\partial r}(rv_r f_k) + \frac{e}{m}E_k\frac{\partial f_0}{\partial v_z} - i\frac{e}{m}\frac{kr}{2}E_k\frac{\partial f_0}{\partial v_r} = 0. \quad (5)$$

[the formula for the fields that appears in (5) is assumed to be given and determined by Eq. (2)].

Solving the second equation in (5) we can write the function f_k in the following form:

$$f_k = \frac{e}{m}E_k\sum_{m=0}^{\infty}(-i)^{m+1}\frac{v_r^m}{(\omega - kv_z)^{m+1}}\frac{\partial^m}{\partial r^m}\left[\frac{\partial f_0}{\partial v_z} - i\frac{kr}{2}\frac{\partial f_0}{\partial v_r}\right]. \quad (6)$$

At the beginning of the process, when the beam is still monoenergetic, it is sufficient to determine the changes in time and coordinate of the moments of the function f_0 rather than the function itself, that is, the density n_1, the mean velocity $n_1u_\perp$ and the temperature $T_\perp$.[10] To compute these quantities we make use of the following system of equations, which are obtained from the first equation in (5) [taking account of Eq. (6)]:

$$\frac{\partial n_1}{\partial t} + \frac{1}{r}\frac{\partial}{\partial r}(rn_1u_\perp) = 0,$$
$$\frac{\partial}{\partial t}(mn_1u_\perp) + \frac{1}{r}\frac{\partial}{\partial r}(2rn_1T_\perp) = -\frac{e^2}{2m}\int k^2E_k^2e^{2\gamma_k t}\frac{\gamma_k^2 dk}{(\Delta_k^2 + \gamma_k^2)^2}r^2\frac{\partial n_1}{\partial r},$$
$$\frac{\partial T_\perp}{\partial t} = \frac{e^2}{4m}\int k^2E_k^2e^{2\gamma_k t}\frac{\gamma_k dk}{\Delta_k^2 + \gamma_k^2}r^2. \quad (7)$$

The growth rates γ_k that appear in Eq. (7) are determined by Eq. (3), in which we substitute the current $j_{zk} = \sigma_k E_k = e\int v_z f_k d\mathbf{v}$:

$$1 - \frac{\omega^2_0}{\omega^2} - \frac{4\pi k^2e^2}{m(\omega - kv_0)^2}\ln\left(\frac{C}{\pi ka}\right)\int_0^a n_1(r)\,r dr = 0 \quad (8)$$

(v_0 is the directed velocity of the beam).

The dispersion equation in (8) describes the interaction of a monoenergetic bounded beam moving through a plasma. Solving this equation we can find the growth rate for the most unstable mode $k_0 = \omega_0/v_0$:

$$\gamma_0 = \frac{\sqrt{3}}{2}\left[\frac{4\pi e^2}{mv_0^2}\ln\left(\frac{Cv_0}{\pi\omega_0 a}\right)\int_0^a n_1(r)\,r dr\right]^{1/3}, \quad \Delta_0 = k_0v_0 - \omega_0 = \frac{\gamma_0}{\sqrt{3}}. \quad (9)$$

In Eqs. (8) and (9) only the mean beam velocity

$$\bar{n}_1 = \frac{2}{a^2}\int_0^a n_1(r)\,r dr,$$

appears and this quantity, in accordance with the first equation in (7), is independent of time. Thus, we can integrate the last equation in (7) and find the quantity $T_\perp$ in explicit form. Under these conditions the system in (7) reduces to the following equation for the function $n_1(t, r)$:

$$\frac{\partial^2 n_1}{\partial t^2} - \frac{1}{2}\left(\frac{3e}{4m}\right)^2\int\frac{k^2E_k^2}{\gamma_k^2}e^{2\gamma_k t}dk\cdot\frac{1}{r}\frac{\partial}{\partial r}\left[r\frac{\partial(r^2n_1)}{\partial r}\right] = 0. \quad (10)$$

In deriving this equation we have taken $\Delta_k \approx \gamma_k/\sqrt{3} \approx \gamma_0/\sqrt{3}$; this step is valid since the oscillation spectrum excited by a monoenergetic beam will be rather narrow $\Delta_k \sim \gamma_0/v_0$.

[4)]The variable V_φ can be eliminated from Eq. (4) because in the present case (in the absence of the field component E_φ) the r component of the particle motion is independent of the φ-component if the condition $V_\varphi(t = 0) = 0$ is satisfied at the initial time. This can be easily demonstrated if one considers the system of equations for the characteristics that correspond to Eq. (4)

$$\dot{v}_r(t) = \frac{e}{m}E_r + \frac{1}{r}v_\varphi^2(t); \quad \dot{v}_\varphi(t) = -\frac{1}{r}v_r(t)v_\varphi(t),$$

which has the solution $V_r \equiv V_r(t)$ and $V_\varphi \equiv 0$.

In (10) we carry out the substitution of variables

$$\tau = \gamma_0 t, \quad x = \ln(r/a), \quad y = r^2 n_1$$

and then consider the Fourier transform in terms of the variable x;

$$y(\tau, x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} y(\tau, q) e^{iqx} dq, \quad y(\tau, q) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} y(\tau, x) e^{-iqx} dx. \quad (11)$$

We then obtain the following second-order equation for the function $y(\tau, q)$:

$$\frac{d^2y}{d\tau^2} + 4\alpha^2(\tau) y = 0, \quad \alpha^2(\tau) = \frac{1}{2}\left(\frac{3e}{4m}\right)^2 e^{2\tau} \int \frac{k^2 E_k^2}{\gamma_k^4} dk. \quad (12)$$

The solution of this equation can be expressed in terms of Bessel functions and is of the form[5)]

$$y(\tau, q) J_0[q\alpha(0)] = y(0, q) J_0[q\alpha(\tau)]. \quad (13)$$

In order to determine the function $y(\tau, x)$ we multiply both sides of (13) by $\exp(iqx)$ and integrate with respect to q. Since both sides of the relation are transformed in the same way we need only compute the integral in the right side:

$$Q = \frac{1}{2\pi} \int_{-\infty}^{\infty} y(0, x') dx' \int_{-\infty}^{\infty} J_0(\alpha q) e^{iq(x-x')} dq. \quad (14)$$

Carrying out the integration with respect to q by means of the relation

$$\frac{1}{2\pi} \int_0^{\infty} J_0(\alpha q) e^{iq(x-x')} dq = \frac{1}{\pi} \frac{1}{\sqrt{\alpha^2 - (x-x')^2}} \equiv \delta(\alpha - |x - x'|) \quad (15)$$

and converting to the variables r and r′, we can write the quantity Q in the following form:

$$Q = \int_0^r n_1(0, r') \delta(r' - re^{-\alpha}) r'^2 dr' + \int_r^a n_1(0, r') \delta(r' - re^{\alpha}) r'^2 dr'. \quad (16)$$

The first integral in (16) is small ($\sim\exp[-2\alpha]$) compared with the second and the second can be reduced to the following form if the left side of (13) is introduced:

$$n_1(t, r) = \exp\{2[\alpha(t) - \alpha(0)]\} n_1[0, r\exp[\alpha(t) - \alpha(0)]], \quad r \leqslant a \exp - [\alpha(t) - \alpha(0)];$$
$$n_1(t, r) = 0, \quad r > a \exp - [\alpha(t) - \alpha(0)]. \quad (17)$$

According to (17) the radius of the beam $R(t) = a \exp - [\alpha(t) - \alpha(0)]$ diminishes in time while the density of particles in the region $r < R(t)$ increases uniformly over the entire volume of the beam.

The self-focusing of the beam which is analyzed above admits of a simple physical explanation: as the instability develops, surface plasma waves are generated and the amplitudes of the fields of these waves increase in the radial direction from the beam axis toward the periphery. Under these conditions the particles in the beam are in a high-frequency potential well whose depth and wall curvature increase with time. As a result the particles collect at the bottom of the well, i.e., close to the point r = 0, where the force that acts on the beam vanishes.

In order to evaluate the efficiency of this focusing method we must estimate the quantity $\alpha(t)$ that appears in Eq. (17). Since the limits of applicability of the theory are bounded by the inequality $E \ll mv_0\gamma_0^2/e\omega_0$, strictly speaking this quantity is small: $\alpha(t) \ll 1$. On the other hand, the formulas given above only give a qualitative description of the process and cannot be used to make rigorous quantitative calculations. Extrapolating the results that have been obtained to the case of stronger fields $E \sim mv_0\gamma_0^2/e\omega_0$ we find $\alpha \sim 1$ so that significant focusing of the beam can only occur up to the end of the hydrodynamic stage in the development of the instability, that is to say, in a time $T \sim 1/\gamma_0$.

As the instability develops the longitudinal temperature of the beam

$$T_{\parallel} = \frac{e^2}{2m} \int E_k^2 \frac{dk}{\Delta_k^2 + \gamma_k^2} e^{2\gamma_k t}$$

increases and this violates the monoenergetic condition $\gamma_0^2 \gg k_0^2 T_{\parallel}/m$ even at small field amplitudes $E \sim mv_0\gamma_0^2/e\omega_0$. Further growth in the oscillation amplitudes is accompanied by strong smearing of the beam in longitudinal velocity and an expansion of the oscillation spectrum in wave vector. The interaction of the beam with the plasma in this stage of the development of the instability can be analyzed by solving (5) and (6) in the quasilinear approximation.[11-14]

The quasilinear growth rate γ_k can be determined by substituting the beam current in Eq. (3).

$$\gamma_k = \frac{\pi}{2} \frac{\omega_0^3}{n_0} \ln\left(\frac{C}{\pi k a}\right) \int_0^a r\, dr \int k\delta(\omega_0 - kv_z) \frac{\partial f_0}{\partial v_z} dv_z. \quad (18)$$

The equation in (5) together with the equation for the amplitude of the electric field

$$\partial E_k^2 / \partial t = 2\gamma_k E_k^2, \quad (19)$$

represent a closed system of equations for the problem. Noting that the function f_0 appears in the expression for the growth rate (18) averaged over cross-section, we introduce the function

$$F(t, v_z) = k^2 \ln\left(\frac{C}{\pi k a}\right) \int_0^{\infty} dv_r \int_0^a f_0 r\, dr. \quad (20)$$

Now, integrating the first equation in (5) with respect to r and substituting (18) and (19) we obtain the following system of quasilinear equations:

$$\frac{\partial F}{\partial t} = \pi \frac{e^2}{m^2} \frac{\partial}{\partial v_z} \int E_k^2 \delta(\omega_0 - kv_z) \frac{\partial F}{\partial v_z} dk,$$
$$\frac{\partial E_k^2}{\partial t} = \pi \frac{\omega_0^3}{kn_0} \int E_k^2 \delta(\omega_0 - kv_z) \frac{\partial F}{\partial v_z} dv_z, \quad (21)$$

which coincides with the system obtained for an infinite beam.[10-11]

It follows from (21) that the spectrum of oscillations excited by the beam in the plasma is determined completely by the longitudinal motion of the particles and is independent of the transverse motion. In view of this feature, by solving Eq. (21) we can determine E_k^2 and then treat the transverse motion in the specified field.

The stationary solution of the system in (21) is found in[10-12]. It is shown in these references that in a time of the order of the reciprocal growth rate a plateau develops on the beam distribution function and a steady-state spectrum of superthermal oscillations is excited:

$$E_k^2 = \frac{m^2\omega_0}{e^2 n_0} v_z^3 \int_{v_1}^{v_2} [F_{\infty}(v_z) - F_0(v_z)] dv_z, \quad (22)$$

[5)]We do not take account of the second solution of Eq. (12) $N_0(\alpha q)$ which has a singularity as $\alpha \to 0$; it is physically obvious that the density must remain finite as the electric field vanishes.

where F_0 is the initial distribution function while F_∞ is the height of the plateau, which is a known quantity and is determined from the conservation of the total number of particles:

$$F_\infty = \frac{\bar{n}_1}{v_2 - v_1} \tag{23}$$

(v_2 and v_1 are respectively the upper and lower limits of the plateau).

It is clear physically that the establishment of the oscillation spectrum and the termination of the growth of the fields means that the radial focusing of the beam is terminated and a stationary radial distribution of particles is established. The equation that describes the dependence of the beam distribution function on transverse velocity v_r and coordinate r can be obtained by substituting $f_0 = f_\perp(r, v_r)F_\infty$ in (5) and averaging over the longitudinal velocity

$$\frac{1}{r}\frac{\partial}{\partial r}(rv_r f_\perp) - Dr^2\frac{\partial^2 f_\perp}{\partial v_r^2} = 0, \tag{24}$$

$$D = \pi\frac{e^2}{4m^2(v_2 - v_1)}\int_{-\infty}^{\infty} dv_z \int k^2 E_k^2 \delta(\omega_0 - kv_z)\,dk.$$

Substituting E_k^2 from Eq. (22) in Eq. (24) and carrying out the integration over the variables k and v_z we can find the coefficient D:

$$D = \frac{\pi}{8}\frac{\bar{n}_1}{n_0}\omega_0^3. \tag{25}$$

(we have made use of the estimates of[10]: $v_2 - v_0 \cong v_0(\bar{n}_1/n_0)^{1/3}$, $v_2 \gg v_\perp$).

In Eq. (24) we have carried out the substitution of variables

$$x = {}^1\!/_3 Dr^3; \quad W = v_r^{-1/2} r f_\perp; \quad y = {}^2\!/_3 v_r^{3/2}$$

and have used the Fourier method for solutions: $W(x, y) = W_1(x)W_2(y)$

$$\frac{1}{W_1}\frac{dW_1}{dx} = \frac{1}{W_2}\left[\frac{1}{y}\frac{d}{dy}\left(y\frac{dW_2}{dy}\right) - \frac{1}{9y^2}W_2\right] = -\lambda^2. \tag{26}$$

(The constant on the right side of Eq. (26) is taken to be negative since it is clear physically that when $x \to \infty$ the distribution function must vanish.)

The solution of Eq. (26) that vanishes when $y \to 0$ is of the form

$$W(x, y) = \int_0^\infty A(\lambda)e^{-\lambda^2 x} J_{1/3}(\lambda y)\,d\lambda. \tag{27}$$

Here, the coefficient $A(\lambda)$ is as yet unknown; to determine this coefficient we make use of the Hankel integral formula, which allows us to write an arbitrary function in the form of an integral that contains the Bessel functions:

$$W(0, y) = \int_0^\infty J_{1/3}(\lambda y)\lambda\,d\lambda \int_0^\infty W(0, u)J_{1/3}(\lambda u)\,u\,du. \tag{28}$$

Substituting $x = 0$ in Eq. (27) and comparing Eqs. (27) we can now determine $A(\lambda)$ and find $W(x, y)$:

$$W(x, y) = \int_0^\infty W(0, u)\,u\,du \int_0^\infty e^{-\lambda^2 x} J_{1/3}(\lambda y)J_{1/3}(\lambda u)\,\lambda\,d\lambda. \tag{29}$$

Integrating with respect to the variable λ we have

$$W(x, y) = \frac{1}{2x}\int_0^\infty W(0, u)\exp\left(-\frac{y^2 + u^2}{4x}\right) I_{1/3}\left(\frac{yu}{2x}\right)u\,du. \tag{30}$$

In order to find the radial distribution of density in the focused beam we now multiply both sides of Eq. (30) by $\sqrt{v_r}$ and integrate with respect to v_r from zero to infinity:

$$n_1(x) = \frac{1}{2}\sqrt{\frac{\pi}{x}}\int_0^\infty W(0, u)\exp\left(-\frac{u^2}{8x}\right) I_{1/6}\left(\frac{u^2}{8x}\right)u\,du. \tag{31}$$

The function $W(0, u)$ that appears in Eq. (31) is the velocity distribution function for the beam particles at the beam axis. If the beam is monochromatic in velocity $W(0, u) \sim \delta(u)$, then in accordance with Eq. (31) the density is nonzero only at the point $r = 0$, that is to say, the beam becomes infinitesimally thin. Under actual conditions, in which the initial velocity spread in the beam is finite, the transverse dimensions of the beam are nonzero. Assuming that

$$f_\perp(r = 0) = \frac{2}{\sqrt{\pi v_T^2}}n_1(0)\exp\left(-\frac{v_r^2}{v_T^2}\right),$$

we write Eq. (31) in the following form:

$$n_1(r) = 2n_1(0)\sqrt{\frac{v_T^3}{3r^3 D}}\int_0^\infty \xi^{5/2}\exp\left(-\xi^2 - \frac{v_T^3}{6r^3 D}\xi^3\right) I_{1/6}\left(\frac{v_T^3}{6r^3 D}\xi^3\right)d\xi \tag{32}$$

($n_1(0)$ is the particle density at the axis of the focused beam.)

In general the integral in (32) cannot be computed so that we consider the asymptotic expression of (32), making use of the parameter $\Delta = v_T^3/3_r^3 D$. Near the axis, with $\Delta \gg 1$, writing

$$I_{1/6}\left(\frac{\Delta\xi^3}{2}\right) \approx \frac{1}{\sqrt{\pi\Delta\xi^3}}\exp\left\{\frac{1}{2}\Delta\xi^3\right\}$$

we have

$$n_1(r) = n_1(0), \quad \Delta \gg 1. \tag{33}$$

Far from the axis, with $\Delta \ll 1$ we can take $I_{1/6}(\Delta\xi^3/2) \sim (\Delta\xi^3)^{1/6}$. Then the quantity $n_1(r)$ is found to be

$$n_1(r) \sim \Delta^{2/3} n_1(0) \ll n_1(0). \tag{34}$$

According to Eqs. (33) and (34) the beam density is independent of the coordinate r within the region $r \ll v_T/D^{1/3}$ and falls off rapidly when $r \gg v_T/D^{1/3}$. The quantity

$$R_0 = \frac{v_T}{D^{1/3}} \sim \frac{v_T}{\omega_0}\left(\frac{n_0}{n_1}\right)^{1/3}$$

thus determines the beam radius in steady state. The particle density at the axis can be estimated making use of the conservation of particles. Integrating both sides of (32) with respect to r from zero to a and assuming that the right side vanishes when $r > R_0$, we have

$$n_1(0) \approx \frac{a^2}{R_0^2}\bar{n}_1. \tag{35}$$

It follows from Eq. (35) that a significant focusing of the beam occurs when $R_0^2 \ll a^2$, that is to say, if the energy associated with the thermal motion of particles is not too large:

$$v_T \ll (\bar{n}_1/n_0)^{1/3}\omega_0 a. \tag{36}$$

As the beam density increases the Coulomb repulsion forces also increase: $F_C \approx 2\pi n_1 r$. However, if the density of the beam is small compared with the density of the plasma, the force F_C is found to be small compared with the high-frequency focusing force [the right side of the second equation in (7)] and polarization effects can be neglected. Furthermore, the Coulomb forces are reduced in a relativistic beam.[1]

In the above we have considered self-focusing for the case of a nonrelativistic beam. If the beam velocity is close to the velocity of light $1 - v_0^2/c^2 \ll 1$ it is necessary to consider the relativistic increase in the the mass of the beam particles, which leads to a reduction in the growth rate: $\gamma^* \sim \gamma_0(1 - v_0^2/c^2)^{1/2}$ and an increase in the focusing time $T^* \sim 1/\gamma^*$:[15] At the same time the focusing efficiency is increased since the energy transferred by the beam to the field is increased. It should be noted that the method of focusing considered here applies only when $(\bar{n}_1/n_0)^{1/3} \ll (1 - v_0^2/c^2)^{1/2}$. In the opposite limit the beam excited a volume wave which leads to defocusing of the beam since the amplitude of the longitudinal field diminishes with radius.

The author wishes to thank Ya. B. Faĭnberg for suggesting this topic and for his continued interest and valuable discussions; the author is also indebted to V. I. Kurilko and V. D. Shapiro for valuable comments.

[1] G. I. Budker, Atomnaya energiya (Atomic Energy) 1, 9 (1965).

[2] A. I. Akhiezer and Ya. B. Faĭnberg, Dokl. Akad. Nauk SSSR **64**, 555 (1949); Zh. Eksp. Teor. Fiz. **21**, 1262 (1951).

[3] A. B. Gaponov and M. A. Miller, Zh. Eksp. Teor. Fiz. **34**, 242 (1958) [Sov. Phys. JETP **7**, 168 (1958)].

[4] H. Boot, S. Self and R. Shersby-Hervie, J. Electronics and Control **4**, 434 (1958).

[5] K. F. Sergeĭchev and I. R. Gekker, ZhETF Pis. Red. **5**, 183 (1967) [JETP Lett. **5**, 146 (1967)].

[6] M. A. Gorbatenko, Fizika plazmy i problemy upravlyaemogo termoyadernogo sintez (Plasma Physics and the Problem of Controlled Thermonuclear Fusion) K. D. Sinel'nikov ed., Nauchnaya mysl' 1963, P. 110.

[7] V. I. Kurilko, Zh. Tekh Fiz. **31**, 889 (1961) [Sov. Phys.-Tekh. Phys. **6**, 655 (1962)].

[8] P. A. Sturrock Phys. Rev. **117**, 1426 (1960).

[9] A. B. Mikhaĭlovskiĭ and K. Jungvirt, Zh. Tekh. Fiz. **36**, 777 (1966) [Sov. Phys.-Tech. Phys. **11**, 581 (1966)].

[10] V. D. Shapiro, Zh. Eksp. Teor. Fiz. **44**, 613 (1963) [Sov. Phys. JETP **17**, 416 (1963)].

[11] A. A. Vedenov, E. P. Velikhov and R. Z. Sagdeev, Nuclear Fusion **1**, 82 (1961).

[12] V. D. Shapiro, Izv. Vuzov, Radiofizika **4**, 867 (1961).

[13] L. I. Rudakov and A. A. Ivanov Zh. Eksp. Teor. Fiz. **51**, 1522 (1966) [Sov. Phys. JETP **24**, 1027 (1967)].

[14] B. B. Kadomtsev, Plasma Turbulence, Academic Press, New York, 1965.

[15] Ya. B. Faĭnberg and V. D. Shapiro, Vzaimodeistvie puchkov zaryazhennykh chastits s plazmoi (Interaction of Charged-particle Beams with Plasma) Nauchnaya mysl'. 1965, p. 95.

Translated by H. Lashinsky
145