SOVIET PHYSICS JETP VOLUME 29, NUMBER 3 SEPTEMBER, 1969

# QUANTIZATION OF THE MAGNETIC FLUX OF SUPERCONDUCTING RINGS AND BOSE CONDENSATION

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Submitted August 29, 1968

Zh. Eksp. Teor. Fiz. 56, 963-974 (March, 1969)

The dependence of free energy on magnetic flux is constructed for a system of charged bosons in a thin superconducting ring. This permits one to obtain the critical value of the magnetic flux, below which undamped currents can flow at quantized values of the magnetic flux. The ratio of the lifetime to the relaxation time of the metastable macroscopic current state can also be estimated. Peculiarities of Bose condensation in a parallelepiped with strongly differing edges are considered in the Appendix. A consequence of these peculiarities is that superfluid and superconducting properties, and in particular quantization of the magnetic flux, may appear at temperatures which are smaller than that of three-dimensional Bose condensation.

## INTRODUCTION

THE quantization of the magnetic flux in superconducting rings is connected with existence of maxima on the plot of the free energy against the magnetic flux.[1] Bloch and Rorschach[2] considered the conditions for the presence of such minima in a gas of charged bosons at zero temperature. They found that in cylinders with thin walls the minima of the free energy occur at discrete values of the magnetic flux so long as the field inside the cylinder does not exceed $H^*d/R$, where d is the wall thickness, R the cylinder radius, and $H^*$ the critical field for the Meissner effect.

In [3,4] they considered the case when a periodic dependence of the free energy on the magnetic field can occur in systems of various dimensions at $T > 0$, and the associated quantization of the flux. In the present paper we consider this question in greater detail for charged bosons in a thin ring.

The constructed dependence of the free energy on the magnetic field is used to obtained the critical value of the magnetic flux at $T > 0$, above which the minima of the free energy vanish. For a hollow cylinder, the critical value of the magnetic field is connected in this case with the critical field for the Meissner effect by the same relation, as that obtained by Bloch and Rorschach[2] for T = 0.

In addition, we consider in the Appendix the features of Bose condensation in a parallelepiped with differing edge dimensions, which can be significant both for the quantization of the magnetic flux and for other superconducting and superfluid properties.

In Sec. 3 we calculate the relation between the lifetime and the relaxation time of a metastable current state, which is very important for the possibility of observing quantization of the magnetic flux in undamped currents.

## 1. HAMILTONIAN OF THE SYSTEM AND ENERGY OF THE MICROSTATE

We consider a thin ring with inside radius R and outside Radius R+d, with $d \ll R$. We introduce the coordinate x = R, which is connected with the angle of rotation about the axis of the ring. When a magnetic field is present inside the ring, the electromagnetic vector potential A in the volume of the ring differs from zero and is directed along the x axis. It can be regarded as constant in the volume of the ring, accurate to small quantities of order d/R (see [4]). Then the Hamiltonian of a system of N particles with charge e and mass m can be written in the form

$$H = \sum_{j=1}^{N} \frac{(\hat{p}_j - eA/c)^2}{2m} + H' + \frac{\Phi^2}{2L}, \qquad (1)$$

where $\hat{p}_j$ is the canonical momentum[1)] of the j-th particle, connected with the coordinate x, $\Phi = 2\pi RA$ is the magnetic flux of the ring, and L is the self-inductance of the ring. We have separated in the Hamiltonian the critical energy of the magnetic field produced by the current flowing through the ring.

We shall solve the problem in the single-particle approximation, choosing as the single-particle functions the eigenfunctions of the operator $p_j$. For each microstate of N particles, in the absence of external magnetic fields, the magnetic flux $\Phi$ should be connected with the current

$$J = \sum_{j=1}^{N} \frac{(p_j - eA/c)}{m} \frac{e}{2\pi R}$$

by the equation

$$\Phi = 2\pi RA = \frac{LJ}{c} = \frac{e}{2\pi Rm} \sum_{j=1}^{N} (p_j - \frac{e}{c}A) \frac{L}{c}. \qquad (2)$$

From (2) it follows that

$$A = \frac{c}{e} \langle p \rangle \frac{D}{D+1}, \quad \langle p \rangle = \frac{1}{N} \sum_{j=1}^{N} p_j, \qquad (3)$$

where $\langle p \rangle$ is the average momentum of the particle, and the dimensionless quantity D is equal to

[1)]The term "momentum" is used throughout for the canonical momentum.

$$D=\frac{e^2N}{mc^2}\frac{L}{(2\pi R)^2}. \quad (4)$$

We represent the momentum of each particle in the form

$$p_j=\langle p\rangle+\tilde{p}_j. \quad (5)$$

Taking (2), (3), and (5) into account, the energy of the microstate is

$$E=\sum_{j=1}^{N}\left\{\frac{(p_j-eA/c)^2}{2m}+\frac{1}{2}\frac{e}{c}A\frac{p_j-eA/c}{m}+\varepsilon_{Hj}\right\}$$
$$=\sum_{j=1}^{N}\left\{\frac{p_j(p_j-eA/c)}{2m}+\varepsilon_{Hj}\right\}=\frac{N\langle\tilde{p}\rangle^2}{2m(D+1)}+\sum_{j=1}^{N}\left\{\frac{p_j^2}{2m}+\varepsilon_{Hj}\right\}. \quad (6)$$

By $\epsilon_{Hj}$ we mean the signal-particle energy connected with the term H′ in the Hamiltonian (1).

Thus, by expressing A in terms of the average momentum, we have eliminated A and Φ from the expression for the energy of the microstate.

## 2. DEPENDENCE OF THE FREE ENERGY ON THE MAGNETIC FLUX

The free energy of an ensemble of microstates with specified Φ, meaning also with specified ⟨p⟩, can be written in the form of a sum of two terms

$$F=E_0+\tilde{F}, \quad (7)$$

where $E_0 = N\langle p\rangle^2/2m(D+1)$, and $\tilde{F}$ is the free energy of the ensemble of microstates with energies $\tilde{E}$, defined by the second terms in (6)

$$\tilde{E}=\sum_{j=1}^{N}\tilde{\varepsilon}_j=\sum_{j=1}^{N}\left\{\frac{\tilde{p}_j^2}{2m}+\varepsilon_{Hj}\right\}. \quad (8)$$

The investigated statistical ensemble can be characterized by the set of quantities n(k, q), which characterize the average number of particles at the level with quantum numbers k and q.

By q are meant the quantum numbers determining the energy $\epsilon_{Hj}$, and the quantum number k determines the momentum $p_j$,

$$\tilde{p}_j=(k-\alpha)h/2\pi R, \quad (9)$$

where $\alpha = t/N$, and t is an integer smaller than N; in practice, α changes continuously from zero to one. The value of α is determined by the value of the average momentum ⟨p⟩, which can assume the following values:

$$\langle p\rangle=(s+\alpha)h/2\pi R, \quad (10)$$

where s is an integer. For values of ⟨p⟩ that differ by a multiple of $h/2\pi R$, the quantity α is one and the same, meaning that the spectrum of the momenta $\tilde{p}_j$ is the same. It follows therefore that $\tilde{F}$ is a periodic function of ⟨p⟩ with a period $h/2\pi R$.

In addition to the usual conditions for the canonical ensemble

$$\sum_{k,q}n(k,q)=N \quad (11)$$

the numbers n(k, q) are subject to the additional limitation

$$\sum_{k,q}\tilde{p}(k)n(k,q)=0. \quad (12)$$

If the distribution n(k, q) for the bose particles is derived by any one of the methods known in statistics,[5] with allowance for the conditions (11) and (12), then we get

$$n(k,q)=\frac{z\exp(-\varepsilon'(k,q)/k_BT)}{1-z\exp(-\varepsilon'(k,q)/k_BT)} \quad (13)$$

By $\epsilon'(k, q)$ we mean a quantity which in general is not equal to the energy $\tilde{\epsilon}(k, q)$ of the single-particle level:

$$\varepsilon'(k,q)=\frac{h^2(k-\delta)^2}{2m(2\pi R)^2}+\varepsilon_H(q). \quad (14)$$

The constants whose selection makes it possible to satisfy conditions (11) and (12) are z and δ ($0 \le \delta \le 1$).

We shall henceforth consider rings of macroscopic dimensions, satisfying the condition

$$M=\frac{m}{k_BT}\left(\frac{h}{2\pi Rm}\right)^2=\frac{1}{2\pi}\frac{\lambda^2}{R^2}\gg 1, \quad (15)$$

where $\lambda = (2\pi\hbar^2/k_BTm)^{1/2}$ is the thermal wavelength.

Condition (15) makes it possible to replace summation over k by integration outside the region of the Bose condensation, for a definite total number of particles.

However, the integration with respect to k, i.e., with respect to the momentum $\tilde{p}$, causes the left side of (11) to become independent of δ; it follows from (12) that δ = α. The free energy $\tilde{F}$ does not depend in this approximation on α, meaning also on the magnetic flux.[6] The dependence of $\tilde{F}$ on α and the associated quantization of the magnetic flux can take place in the region of bose condensation, when one term in the sum over k can make a contribution comparable with the value of the total sum. The largest contribution to the sum over k can give either the levels with k = 0 or with k = 1.

We therefore represent (11) and (12) in the form

$$N_{0c}+N_{1c}+N_b=N, \quad (16)$$

$$-\alpha N_{0c}+(1-\alpha)N_{1c}+(\delta-\alpha)N_b=0, \quad (17)$$

Where $N_{0c}$ is the number of particles at the levels with k = 0, $N_{1c}$ is the number of particles at the levels with k = 1, and $N_b$ is the number of particles at the remaining levels. The expressions for $N_{0c}$, $N_{1c}$, and $N_b$ are:

$$N_{0c}=\sum_q z\left\{\exp\left(M\frac{\delta^2}{2}+\frac{\varepsilon_H(q)}{k_BT}\right)-z\right\}^{-1}, \quad (18)$$

$$N_{1c}=\sum_q z\left\{\exp\left(M\frac{(1-\delta)^2}{2}+\frac{\varepsilon_H(q)}{k_BT}\right)-z\right\}^{-1}, \quad (19)$$

$$N_b=\sum_q\int dkz\left\{\exp\left(M\frac{(k-\delta)^2}{2}+\frac{\varepsilon_H(q)}{k_BT}\right)-z\right\}^{-1}. \quad (20)$$

In (20), the integral with respect to k replaces the sum over all k except k = 0 and k = 1.

Further calculations will be made for the region of parameters where, besides the condition (15), the following inequality is satisfied:

$$M=\frac{1}{2\pi}\frac{\lambda^2}{R^2}\gg\frac{1}{N_c}, \quad (21)$$

and the quantity $N_c = N_{0c} + N_{1c}$ will be the called the number of particles in the condensate.

In order for $N_{0c}$ or $N_{1c}$ to be able to assume macroscopic values comparable with N, it is necessary that the aenominators in (18) or (19) vanish, i.e., either that

$$z \approx \exp\left(M\frac{\delta^2}{2}+\frac{\varepsilon_H(q_0)}{k_BT}\right),$$

or

$$z \approx \exp\left(M\frac{(1-\delta)^2}{2}+\frac{\varepsilon_H(q_0)}{k_BT}\right),$$

where $\epsilon_H(q_0)$ is the smallest value of $\epsilon_H(q)$.

The indicated circumstance, together with the condition (21), can lead to the following singularities of the dependence of $N_{0c}$, $N_{1c}$, and $N_b$ on z and δ:

1) When δ differs from ½, we get the inequalities $N_{0c} \gg N_{1c}$ and $N_{1c} \gg N_{0c}$ when $\delta < 1/2$ and $\delta > 1/2$, respectively.

2) Changes of δ and the ensuing small changes of z near unity do not affect the value of $N_b$, where there is no divergence as $z \to 1$. In this connection, neither $N_b$ nor $N_c = N - N_b$ depends on α, i.e.,

$$N_c = wN, \qquad N_b = (1-w)N, \tag{22}$$

where w is independent of α.

By making these assumptions, we can readily determine from (16) and (17) the dependence of $N_{0c}$, $N_{1c}$, and $N_b$ on α. When $0 < \alpha(1-w)/2$, we have

$$\delta = \frac{\alpha}{1-w}, \qquad N_{0c} = wN, \qquad N_{1c} = 0. \tag{23}$$

When $(1-w)/2 < \alpha < (1+w)/2$ we have

$$\delta = \frac{1}{2}, \qquad N_{0c} = \left(\frac{1+w}{2}-\alpha\right)N, \qquad N_{1c} = \left(\alpha-\frac{1-w}{2}\right)N. \tag{24}$$

When $(1+w)/2 < \alpha < 1$ we have

$$\delta = \frac{\alpha-w}{1-w}, \qquad N_{0c} = 0, \qquad N_{1c} = wN. \tag{25}$$

For the free bosons, the individual terms in the sum over k can be comparable with the value of the entire sum only in the region of the bose condensation.

For macroscopic rings, the approximate equations for the sums over q in (18) and (19) may include a term corresponding to the level with the lowest energy, and also one-dimensional and two-dimensional integrals.

The one-dimensional and two-dimensional integrals may also enter in the expression (20) for $N_d$, together with a three-dimensional integral. This is shown in the Appendix. It is easy to verify that the term comparable with the entire sum in the sum over k can lead to the existence, over the periphery of the circle, of an "off diagonal long range order" (ODLRO), which, according to Yang,[7] should always accompany the appearance of superfluid or superconducting properties.

Equations (23)–(25) make it possible to obtain the dependence of $\widetilde{F}$ on α. The condensate makes no contribution to the entropy, which is determined by an integral independent of α:

$$\widetilde{F} = \widetilde{E} - TS = \sum_{k,q}\tilde{\varepsilon}(k,q)n(k,q) - TS =$$

$$= \widetilde{F}_0 + \frac{\hbar^2}{m(2\pi R)^2}\left[\frac{\alpha^2}{2}N_{0c}+\frac{(1-\alpha)^2}{2}N_{1c}+\frac{(\delta-\alpha)^2}{2}N_b\right], \tag{26}$$

where $\widetilde{F}_0$ is the value of the free energy at α = 0.

Substituting (23)–(25) in (26), we obtain

$$\widetilde{F} = \widetilde{F}_0 + \begin{cases} \dfrac{\hbar^2 N}{m(2\pi R)^2}\dfrac{w}{1-w}\dfrac{\alpha^2}{2}, & \alpha < \dfrac{1-w}{2}, \\ \dfrac{\hbar^2 N}{m(2\pi R)^2}\left[\dfrac{\alpha-\alpha^2}{2}-\dfrac{1-w}{8}\right], & \left|\dfrac{1}{2}-\alpha\right| < \dfrac{w}{2}, \\ \dfrac{\hbar^2 N}{m(2\pi R)^2}\dfrac{w}{1-w}\dfrac{(1-\alpha)^2}{2}, & \alpha > \dfrac{1+w}{2}. \end{cases} \tag{27}$$

It follows from (23)–(25) that when $|1/2 - \alpha| > w/2$ the condensate is concentrated at levels with one value of k, and when α changes from $(1-w)/2$ to $(1+w)/2$, the condensate goes over to levels with other values of k. At the points $\alpha = (1-w)/2$ and $\alpha = (1+w)/2$, the analyticity of the $\widetilde{F}$ as a function of the magnetic flux is violated. In the study of the superfluidity of a Bose gas, such nonanalytic points appear on the dependences of the thermodynamic properties on the rotation velocity.[8]

Figure 1 shows a plot of $(F - \widetilde{F}_0)/NMk_BT$ (see (9)) against the quantity

$$\langle p\rangle\frac{R}{\hbar} = \Phi\frac{c}{eh}\frac{D}{D+1}$$

for the values w = ½ (solid line) and w = 1 (dashed line). The curves were plotted for D = 74. The minima of the free energy F are found at the points $m_0, m_1, m_2, \ldots$ at magnetic-flux values

$$\Phi = \frac{hc}{e}\frac{Dw}{Dw+1}s, \tag{28}$$

where s is the index of the point $m_s$. At the points $a_s$ and $b_s$ we have $\alpha = (1-w)/2$ and $\alpha = (1+w)/2$, respectively.

We present the values of the energy barriers surrounding the minimum point $m_s$. We denote by $E_-$ the energy barrier on the side of the smaller fluxes (the energy difference between the points $t_s$ and $m_s$ on Fig. 1), and by $E_+$ the energy barriers on the side of the larger fluxes (the energy difference between the points $t_{s+1}$ and $m_s$ on Fig. 1).

$$E_\pm = \frac{h^2}{m(2\pi R)^2}\frac{Nw}{16}\frac{Dw+1}{Dw}\left(1\pm\frac{2s}{Dw+1}\right)^2. \tag{29}$$

The minima of the free energy occur so long as the barrier $E_1$ does not vanish, i.e., so long as

$$s \leqslant (1+Dw)/2. \tag{30}$$

This criterion depends on the temperature only via $N_c$, and was previously obtained by Bloch and Rorschach[2] for T = 0 and for a hollow cylinder. The mag-

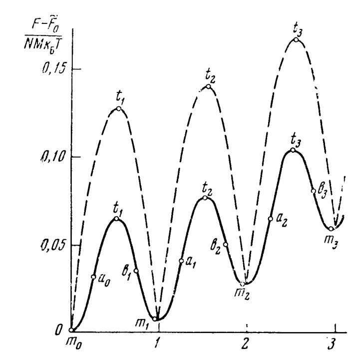


FIG. 1.

netic field for a cylinder, corresponding to the limiting value of s, is given in the introduction.

We shall need the energy differences between the point $t_s$ and $b_s(E_{t-})$, $T_{s+1}$ and $a_s(E_{t+})$, $b_s$ and $m_s(E_{m-})$, and also $a_s$ and $m_s(E_{m+})$;

$$E_{t\pm}=\frac{h^2}{m(2\pi R)^2}\frac{N}{16}\frac{(Dw+1)^2}{D(D+1)}\left(1\pm\frac{2s}{Dw+1}\right)^2, \tag{31}$$

$$E_{m\pm}=\frac{h^2}{m(2\pi R)^2}\frac{N(1-w)}{16}\frac{Dw+1}{D+1}\left(1\pm\frac{2s}{Dw+1}\right)^2. \tag{32}$$

It should be noted that the presence in the sum over k of one term that is comparable in magnitude with the entire sum, and the associated presence of ODLRO, is not the only cause of a dependence of $\widetilde{F}$ on $\alpha$. According to Bloch,[4] such a dependence takes place without ODLRO in one-dimensional and two-dimensional systems when the condition (21) is satisfied. However, the proof of the impossibility of ODLRO in one-dimensional and two-dimensional systems[9] pertains to systems with a finite one-dimensional and two-dimensional density. This is not so in our situation, as follows directly for a one-dimensional system from the condition (21), since the density N/R increases more rapidly than R with increasing R. Using the Appendix, we can verify that ODLRO takes place. In the next section it will be shown that the condition (21) is necessary for the existence of a steady metastable state for quantized values of the magnetic field.

## 3. LIFETIME AND RELAXATION TIME OF METASTABLE CURRENT STATE

The undamped currents in the superconducting ring are non-equilibrium metastable states with very long lifetimes. To observe undamped currents experimentally at values (28) of the flux $\Phi$, it is necessary, on the one hand, to have a sufficiently rapid momentum exchange between the system and the medium at values of $\Phi$ different from (28), and on the other hand this rapid momentum exchange must not lead to a considerable change of the state of the system with time after the flux $\Phi$ has reached the values (28). It is therefore of interest to find the ratio of the relaxation time and the lifetime of the metastable macrostate with average values $\Phi$ equal to (28). We shall consider the situation when the interaction of the bosons with one another is more intense than the interaction with the surrounding medium. We can then speak of a partial equilibrium within the limits of the ensembles of the microstates with identical values of the total energy E and the total momentum $P = N\langle p\rangle$. Such an ensemble will be called microcanonical, although it is narrower than the ordinary microcanonical ensemble. The nonequilibrium state can be characterized in this case by a distribution function f(E, P), which determines the probability that an arbitrarily chosen microstate belongs to an ensemble with specified E and P.

Further simplification of the problem can be effected in two opposite cases: 1) the energy relaxation is much more slow than the momentum relaxation; we can then consider the time variation of f(E, P) at a specified E; 2) the energy relaxation is much more rapid than the momentum relaxation; we can then speak of equilibrium within the framework of the broader ensemble considered in the preceding section and characterized by specified T and P.

In both cases the nonequilibrium state can be characterized by a distribution function f(P) that depends only on the momentum P. The equilibrium values of f(P) are proportional to $\exp(S(\Phi)/k_B)$ in the former case and $\exp(-F(\Phi)/k_BT)$ in the latter case, where $F(\Phi)$ is the free energy calculated in the preceding section, and $S = (E-F)/T$ is the entropy for the microcanonical ensemble defined above. For the microcanonical ensemble, T is a parameter whose variation satisfies the condition E = const at various values of $\Phi$. If $F(\Phi) - \widetilde{F}_0 \ll E_0$, where $E_0$ is the energy of the system when $\Phi = 0$, then T varies little with $\Phi$, and the quantities S and F/T differ by a constant that does not depend on $\Phi$.

Inasmuch as $E_0 \sim k_BTN(1-w)$ and $F-\widetilde{F}_0 \sim Nwh^2/mR^2$, the condition $F - F_0 \ll E_0$ means that $M \ll (1-w)/w$, and is violated only at sufficiently low temperature, when w is very close to unity. Let us assume that momentum exchange occurs in collisions of individual bosons with objects of the external medium. Then, for each collision, the total momentum P changes by an amount of the order of $\langle p\rangle$, which is small compared with $P = N\langle p\rangle$, and therefore the change of f(P) with time can be described by the diffusion equation in momentum space P, as is done, for example, to describe the electron distribution function in the Coulomb interaction[10]

$$\frac{\partial f}{\partial t}=-\frac{\partial}{\partial P}\left[\left(\frac{\partial F}{\partial P}\frac{f}{k_BT}+\frac{\partial f}{\partial P}\right)W\right], \tag{33}$$

where W is the diffusion coefficient, whose order of magnitude is $\langle p\rangle^2/\tau$, $\tau$ being the average time between the boson collisions that change the momentum of the system.

We define the distribution function of the metastable current state corresponding to the index s in (28) as a function proportional to the equilibrium distribution function near the point $m_s$ (see Fig. 1)

$$f_0(P)=\frac{1}{G}\exp\left(-\frac{F-F(s)}{k_BT}\right) \tag{34}$$

and equal to zero everywhere outside the interval between the points $t_s$ and $t_{s+1}$ of the free-energy maxima adjacent to the point $m_s$. In (34), F(s) denotes the free energy at the point $m_s$, and G is a normalization constant. The quantity G characterizes the interval of momenta near $m_s$, in which $f_0(P)$ differs noticeably from zero. In order for the arbitrary distribution function to assume in this momentum interval a form close to (34), it is necessary to have times on the order of the relaxation time $\tau_r$, determined from the relation

$$1/\tau_r = W/G^2. \tag{35}$$

Once the distribution function assumes a form close to (34), it changes slowly in time in accordance with $f_0(P)\exp(-t/\tau_M)$, where $\tau_M$ is the lifetime of the metastable state. To determine $\tau_M$, we put in (33) $\partial f/\partial t = 0$ and solve the obtained differential equation in the momentum interval between the points $m_{s-1}$ and $m_s$ with the boundary conditions f(P) = 0 at the point $m_{s-1}$ and $f(P) = f_0(P)$ at the point $m_s$.

Then the constant $\tau_M$ can be obtained from

$$\frac{1}{\tau_M}=-W\left(\frac{\partial f}{\partial P}+\frac{\partial F}{\partial P}\frac{f}{k_BT}\right), \tag{36}$$

where the right side contains the diffusion flux. In the general case, the expression for $\tau_M$ is somewhat complicated. We present expressions for G and $\tau_r/\tau_M$ for the most interesting particular cases. All pertain to the situation in which $E_- \gg k_B T$ and $Dw \gg 1$.

1. $E_- \simeq E_{t-} \gg k_B T$, $E_{m-} \ll k_B T$. From (31) and (32) it follows in this case that $w \sim 1$, i.e., almost all the particles are in the condensate (dashed curve in Fig. 1). Then

$$G = \sqrt{mk_B TN}\sqrt{\frac{k_B T}{E_-}}, \quad \frac{\tau_r}{\tau_M} = \sqrt{\frac{k_B T}{E_-}}\exp\left(-\frac{E_-}{k_B T}\right). \tag{37}$$

2. $E_{t-} \gg k_B T$, $E_{m-} \gg k_B T$. Then

$$G = \sqrt{mk_B TN}\sqrt{\frac{w}{1-w}}, \quad \frac{\tau_r}{\tau_M} = \sqrt{\frac{1-w}{w}}\exp\left(-\frac{E_-}{k_B T}\right). \tag{38}$$

3. $E_{t-} \ll kT$, $E_- \approx E_{m-} \gg k_B T$. In this case w is close to zero,

$$G = \sqrt{mk_B TN}\sqrt{w}, \quad \frac{\tau_r}{\tau_M} = \sqrt{\frac{E_-}{k_B T}}\exp\left(-\frac{E_-}{k_B T}\right). \tag{39}$$

In (37)–(39) we have left out coefficients of the order of unity. In all cases $\tau_r/\tau_M$ decreases exponentially with increasing $E_-/k_B T$, and in order to quantize the magnetic flux it is necessary to have $E_- \gg k_B T$. We see from (29) that to this end it is necessary to satisfy the condition (21).

We present $E_-/k_B T$ for the conditions of the experiment of Deaver and Fairbank,[11] assuming, as usual, the number of condensed bosons to be equal the number of condensed spheres. This value is $E_-/k_B T \simeq 10^9$, which should lead to astronomical lifetimes of the current state for any reasonable mechanism of momentum exchange.

## APPENDIX

We consider here bose condensation in a parallelepiped with different edge dimensions.

We consider free bosons in a parallelepiped with edges $L_x > L_y > L_z$ with periodic boundary conditions for the wave functions. The sum expressing the total number of particles is broken up into four parts, each of which is calculated[2)] within the limits of large $L_x/\lambda$, $L_y/\lambda$, and $L_z/\lambda$ ($\lambda$—thermal wavelength):

$$N = N_0 + N_1 + N_2 + N_3, \quad N_0 = \frac{z}{1-z},$$

$$N_1 = \sum_{\substack{i=-\infty \\ (i\neq 0)}}^{\infty} z\left\{\exp\left(\pi\frac{\lambda^2}{L_x^2}i^2\right) - z\right\}^{-1} = \frac{L_x}{\lambda}\operatorname{cth}\left[\sqrt{\pi(1-z)}\,\frac{L_x}{\lambda}\right]$$

$$\times\sqrt{\frac{\pi}{1-z}} - \frac{1}{1-z} = \begin{cases} \dfrac{\pi}{3}\dfrac{L_x^2}{\lambda^2}, & 1-z \ll \dfrac{\lambda}{L_x}, \\[2ex] \dfrac{L_x}{\lambda}\sqrt{\dfrac{\pi}{1-z}}, & 1-z \gg \dfrac{\lambda}{L_x} \end{cases}$$

$$N_2 = \sum_{\substack{k=-\infty \\ (k\neq 0)}}^{\infty}\sum_{i=-\infty}^{\infty} z\left\{\exp\left(\pi\frac{\lambda^2}{L_x^2}i^2 + \pi\frac{\lambda^2}{L_y^2}k^2\right) - z\right\}^{-1}$$

$$= -2\frac{L_x L_y}{\lambda^2}\ln\left(\frac{\lambda}{L_y}\sqrt{z} + \sqrt{\frac{\lambda^2}{L_y^2}z + 1 - z}\right)$$

$$= \begin{cases} 2\dfrac{L_x L_y}{\lambda^2}\ln\dfrac{L_y}{\lambda}, & 1-z \ll \dfrac{\lambda}{L_y} \\[2ex] \dfrac{L_x L_y}{\lambda^2}\ln\dfrac{1}{1-z}, & 1-z \gg \dfrac{\lambda}{L_y} \end{cases} \tag{A.1}$$

$$N_3 = \sum_{\substack{n=-\infty \\ (n\neq 0)}}^{\infty}\sum_{k=-\infty}^{\infty}\sum_{i=-\infty}^{\infty} z\left\{\exp\left(\pi\frac{\lambda^2}{L_x^2}i^2 + \pi\frac{\lambda^2}{L_y^2}k^2 + \pi\frac{\lambda^2}{L_z^2}n^2\right) - z\right\}^{-1} = \frac{2}{\sqrt{\pi}}\frac{L_x L_y L_z}{\lambda^3}\int_0^\infty \frac{z\sqrt{t}\,dt}{e^t - z}$$

In the calculation of $N_1$ and $N_2$ we took into account the fact that these terms are significant only when $z \sim 1$. We shall call $N_0$, $N_1$, $N_2$, and $N_3$ the numbers of the particles of the zeroth, one-dimensional, two-dimensional, and three-dimensional phases. The aggregates of phases $N_{(1)} = N_0 + N_1$ and $N_{(2)} = N_{(1)} + N_2$ will be called the one-dimensional and two-dimensional gas. We define the temperature of the i-dimensional bose condensation from the condition that the number of particles in the i-dimensional phase equals the number of particles of the i-dimensional gas in the limit as $z \to 1$. Thus, we obtain the temperatures $T_1$, $T_2$, and $T_3$ and the corresponding thermal wavelengths $\lambda_1$, $\lambda_2$, and $\lambda_3$. The temperatures $T_3$ and $T_2$ are non-analytic points on the plots of $N_2$ and $N_3$, and also of $N_{(2)}$ and $N_{(1)}$ against the temperature:

$$N_3 = \begin{cases} N, & T > T_3 \\ N\left(\dfrac{T}{T_3}\right)^{3/2}, & T < T_3 \end{cases}; \quad N_2 = \begin{cases} N_{(2)} = 0, & T > T_3 \\ N_{(2)} = N\left(1 - \left(\dfrac{T}{T_3}\right)^{3/2}\right), & T_3 > T > T_2. \\ N\left(1 - \left(\dfrac{T_2}{T_3}\right)^{3/2}\right)\dfrac{T}{T_2} & T_2 > T \end{cases} \tag{A.2}$$

In a one-dimensional gas near the temperature $T_1$, the zeroth phase begins to be formed, but no violation of the analyticity of $N_1$ takes place. For the thermodynamic functions of the entire system as a whole, only the point $T_3$ is non-analytic, and therefore only this point can be called a phase transition point in accordance with the customary terminology. The conditions indicated above for the determination of $T_1$, $T_2$, and $T_3$ lead to the following relations between the dimensions $L_x$, $L_y$, $L_z$ and the critical temperatures $T_1$, $T_2$, and $T_3$:

$$L_x = \frac{6}{\pi}\frac{T_2}{T_1}\frac{1 - (T_1/T_3)^{3/2} - (1 - (T_2/T_3)^{3/2})T_1/T_2}{1 - (T_2/T_3)^{3/2}}L_y\ln\frac{L_y}{\lambda_2},$$

$$L_z = \frac{1}{1.306}\frac{T_2}{T_3}\frac{1}{1 - (T_2/T_3)^{3/2}}\lambda_3\ln\frac{L_y}{\lambda_2}. \tag{A.3}$$

Let us see how the critical temperatures behave in the case of an unlimited increase of the dimensions $L_x$, $L_y$, and $L_z$ and the number of particles N at a constant three-dimensional density $N/L_x L_y L_z$. This is the so-called "thermodyanamic limit." The quantity $T_3$ remains unchanged in this case. On the other hand, the behavior of the temperatures $T_1$ and $T_2$ is determined by the relations between $L_x$, $L_y$, and $L_z$ in the limiting transition. Several cases take place:

1. $\dfrac{L_x}{L_y\ln(L_y/\lambda_2)} = \text{const}; \quad \dfrac{L_z}{\ln(L_y/\lambda_2)} = \text{const}.$

In this case the ratios $L_x/L_y$ and $L_y/L_z$ increase without limit, and $T_1 \neq T_2 \neq T_3$.

The temperature dependences of $L_0$, $N_1$, $N_2$, and $N_3$ are shown in Fig. 2 by solid lines for the case $T_3 = 2T_2 = 3T_1$. The dashed lines in the same figure show the

[2)]After the first draft of this paper was submitted for publication, the author has learned of a paper by Krueger [[12]], who also obtained expressions for the number of particles at different relations between $L_x$, $L_y$, $L_z$, and $\lambda$.

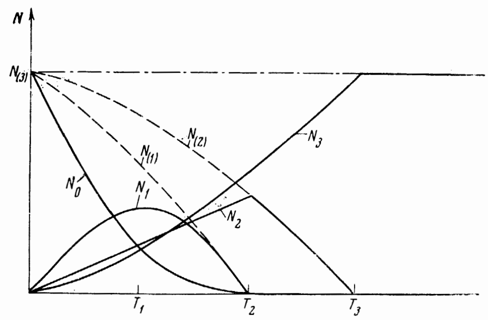


FIG. 2.

numbers of the particles of one-dimensional and two-dimensional gas $N_{(1)}$ and $N_{(2)}$.

$$2.\ \frac{L_x}{L_y} = \text{const}; \quad \frac{L_y}{L_z} = \text{const}.$$

Here $T_1 = T_2 = T_3$, and $N_1$ and $N_2$ are small throughout, i.e., $N_{(2)} = N_0$

$$3.\ \frac{L_x}{L_y} = \text{const}; \quad \frac{L_z}{\ln(L_y/\lambda_2)} = \text{const}.$$

In this case $T_1$ and $T_2$ but $T_2 \neq T_3$. No one-dimensional phase appears.

$$4.\ \frac{L_x}{L_y^2} = \text{const}; \quad \frac{L_y}{L_z} = \text{const}.$$

Here $T_3 = T_3$ but $T_1 \neq T_2$. There is no two-dimensional phase, and $N_{(2)} = N_{(1)}$.

Since the quantities $N_0$, $N_1$, and $N_2$ depend not only on the volume and the three-dimensional density but also on the form of the surface bounding the volume, they cannot be regarded as additive extensive thermodynamic quantities.

It should be noted that the one-dimensional and two-dimensional gases considered above experience Bose condensation by virtue of the fact that the one-dimensional and accordingly two-dimensional density increase without limit in the thermodynamic limit. Bose condensation in a two-dimensional gas was previously considered in [13] and [14].

The foregoing results can be used to calculate $N_{0c}$, $N_{1c}$, and $N_b$ in a cylindrical ring of height h, radius R, and wall thickness $d \ll R$, which, neglecting the curvature of the surface, can be regarded as a parallelepiped with edges h, $2\pi R$, and d. The quantization of the magnetic flux in different cases can take place either below $T_3$ or below $T_2$, or else below $T_1$.3)

It follows therefore that the temperature of the transition of the superfluid and superconducting states can depend on the dimensions. Similar effects, namely the decrease of the temperature of the transition with decreasing thickness of the film of a material, actually take place for superfluid helium [14] and superconducting tin [15] films.

The author is grateful to A. I. Ansel'm, A. G. Aronov, E. K. Kudinov, Yu. N. Obraztsov, and G. E. Pikus for a discussion of questions touched upon in the article.

3) A criterion weaker than ODLRO is proposed in [12] for the presence of superfluidity and superconductivity (see also [3,4]). It is satisfied for all $T < T_3$. For the quantization of the magnetic flux, such a weaker criterion is not suitable.

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Translated by J. G. Adashko
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