ON THE CONNECTION BETWEEN THE SCHWARZSCHILD AND TOLMAN COORDINATE SYSTEMS

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Transformations connecting the Schwarzschild and Tolman coordinate systems in the case of a dust-like sphere with uniform volume density are considered. The Schwarzschild metric within matter is found for this case. The singularities of the coordinate transformations are discussed.

In the general theory of relativity, two exact solutions for spherically symmetric gravitational fields are known: 1) the Schwarzschild solution[1,2] for an external observer which describes the field in the empty space around the material body, and 2) the Tolman solution[1,2] for a comoving observer, which moves together with the grain-like matter (p = 0).

This leads naturally to the question of the connection between these solutions. In the present paper we find a coordinate transformation which connects both solutions in the case of a sphere with uniform volume density and zero pressure (part of the Friedmann world), and we also continue the Schwarzschild metric into the material body for this case. The analogous problem for the Kruskal metric has been considered by Murai.[3]

1. The generalized Schwarzschild interval, in which $g_{0\alpha} = 0$ by definition, and in which the radius is defined such that the circumference of a circle is equal to $2\pi r$, is equal to[1]

$$ds^2 = e^{\nu(t,r)}c^2dt^2 - e^{\lambda(t,r)}dr^2 - r^2d\sigma, \quad d\sigma = d\theta^2 + \sin^2\theta d\varphi^2, \tag{1}$$

we transform this into the Tolman interval[1,2]

$$ds^2 = c^2d\tau^2 - e^{\omega(\tau,R)}dR^2 - r^2(R,\tau)d\sigma, \tag{2}$$

where in the parametrized form[1]

$$r = R\sin^2\frac{\eta}{2}, \; c\tau = \psi(R)S(\eta), \; S(\eta) = \int\limits_{\pi/2}^{\eta/2}\sin^2\xi\, d\xi = \frac{1}{4}(\eta - \sin\eta) - \frac{\pi}{4}. \tag{3}$$

In the case of an anti-collapse which goes over into a collapse the parameter η varies within the limits $0 \le \eta \le 2\pi$. The Tolman coordinate system was defined such that $\tau = 0$ and $r = R$ for $\eta = \pi$ (the moment of largest expansion). For a sphere with constant volume density, the function $\psi(R)$ has the form[4]

$$\psi(R) = \begin{cases} 2R_0\sqrt{R_0/r_g}, & \psi'(R) = 0, \quad R \leqslant R_0, & \text{(4a)} \\ 2R\sqrt{R/r_g}, & \psi'(R) = {}^3/_2\,\psi/R, \quad R \geqslant R_0, & \text{(4b)} \end{cases}$$

where R_0 is the maximal radius of the sphere in the accompanying coordinate system at $\tau = 0$, and $r_g = 2kMc^{-2}$ is the gravitational radius of the mass M. The derivative $\psi'(R)$ has a cut at the boundary of matter. The e^{ω} component of the metric tensor in the Tolman interval (2) is equal to

$$e^{\omega} = \left(1 - \frac{4R^2}{\psi^2}\right)^{-1}\left(\sin^2\frac{\eta}{2} - 2RS\operatorname{ctg}\frac{\eta}{2}\frac{\psi'}{\psi}\right)^2, \quad 0 < 1 - \frac{4R^2}{\psi^2} \leqslant 1. \tag{5}$$

In (5), $S\cot(\eta/2) < 0$, and $e^{\omega} > 0$ everywhere.

Let us express the interval (2) in terms of the variables R and η:

$$cd\tau = \psi' SdR + \frac{\psi}{2}\sin^2\frac{\eta}{2}d\eta,$$

$$ds^2 = \left[\psi'^2S^2 - \left(1 - \frac{4R^2}{\psi^2}\right)^{-1}\left(\sin^2\frac{\eta}{2} - 2RS\operatorname{ctg}\frac{\eta}{2}\frac{\psi'}{\psi}\right)^2\right]dR^2 + \psi'\psi S\sin^2\frac{\eta}{2}dR\,d\eta + \frac{\psi^2}{4}\sin^4\frac{\eta}{2}d\eta^2 - r^2d\sigma. \tag{6}$$

We note that (6) contains two terms which are discontinuous at the boundary of matter. The presence of cuts in the metric of the type (6) has been pointed out by Khrapko.[5] It can be verified by direct calculation that the cuts in the metric (6) do not lead to the appearance of δ like singularities in the curvature tensor R_{iklm}.

In the Schwarzschild interval (1) we also go over to the variables R and η:

$$r = R\sin^2\frac{\eta}{2}, \quad t = t(R,\eta), \quad dt = t_RdR + t_\eta d\eta,$$

$$ds^2 = \left(e^{\nu}c^2t_R^2 - e^{\lambda}\sin^4\frac{\eta}{2}\right)dR^2 + 2\left(e^{\nu}c^2t_Rt_\eta - e^{\lambda}R\sin^3\frac{\eta}{2}\cos\frac{\eta}{2}\right)dR\,d\eta + \left(e^{\nu}c^2t_\eta^2 - e^{\lambda}R^2\sin^2\frac{\eta}{2}\cos^2\frac{\eta}{2}\right)d\eta^2 - r^2d\sigma. \tag{7}$$

Equating the coefficients in (6) and (7), we obtain three equations for the unknown functions r_R, t_η, e^{λ}, e^{ν} of the arguments R and η:

$$\psi'^2S^2 - \left(1 - \frac{4R^2}{\psi^2}\right)^{-1}\left(\sin^2\frac{\eta}{2} - 2RS\operatorname{ctg}\frac{\eta}{2}\frac{\psi'}{\psi}\right)^2 + e^{\lambda}\sin^4\frac{\eta}{2} = e^{\nu}c^2t_R^2, \tag{8a}$$

$$\frac{1}{2}\psi'\psi S\sin^2\frac{\eta}{2} + e^{\lambda}R\sin^3\frac{\eta}{2}\cos\frac{\eta}{2} = e^{\nu}c^2t_Rt_\eta, \tag{8b}$$

$$\frac{\psi^2}{4}\sin^4\frac{\eta}{2} + e^{\lambda}R^2\sin^2\frac{\eta}{2}\cos^2\frac{\eta}{2} = e^{\nu}c^2t_\eta^2. \tag{8c}$$

For the right-hand sides of the equations (8) we have the identity (8a) × (8c) = (8b)^2. An analogous equation must also hold for the left-hand sides. After some algebraic transformations we obtain from this equation a simple expression for λ:

$$e^{-\lambda} = 1 - \frac{4R^2}{\psi^2\sin^2(\eta/2)} = \begin{cases} 1 - r_g/r, & R \geqslant R_0, \\ 1 - \dfrac{R^2r_g}{R_0^3\sin^2(\eta/2)}, & R \leqslant R_0. \end{cases} \tag{9}$$

2. Outside matter, one must of course obtain the Schwarzschild solution in empty space. Let us show

this. Dividing (8a) by (8b) and using (4b), we find

$$t_R = \frac{3S\,[(R/r_g)\sin^2(\eta/2) - 1] + \sin^3(\eta/2)\cos(\eta/2)}{R\,[(R/r_g) - 1]\sin^4(\eta/2)}\, t_\eta = \frac{Q_R}{Q_\eta} t_\eta, \tag{10}$$

where we have introduced the notation

$$Q_R = \frac{R/r_g}{c\,(R/r_g - 1)^{1/2}}\left[3S + \frac{\sin^3(\eta/2)\cos(\eta/2)}{(R/r_g)\sin^2(\eta/2) - 1}\right],$$

$$Q_\eta = \frac{(R^2/r_g)\sin^4(\eta/2)\,(R/r_g - 1)^{1/2}}{c\,[(R/r_g)\sin^2(\eta/2) - 1]}. \tag{11}$$

It can be shown that the equations (11) are consistent and determine the function $Q(R, \eta)$:

$$Q = 2\frac{R^2}{cr_g}\left(\frac{R}{r_g} - 1\right)^{1/2} \int\limits_{\pi/2}^{\eta/2} \frac{\sin^4 \zeta}{(R/r_g)\sin^2\zeta - 1}\, d\zeta. \tag{12}$$

We find from (10) that $t = t(Q)$ [$t(Q)$ is an arbitrary function]. However, we see by a comparison with the corresponding expression of [4], where the motion of the boundary of matter ($R = R_0$) was considered, that $t = Q$. Taking this into account, we find, for example from (8c), that $e^{\nu} = e^{-\lambda} = 1 - r_g/r$ for $R \geq R_0$, i.e., we obtain the Schwarzschild metric.

The formula (12) for t has a singularity: the zero of the denominator of the integrand for $r = R\sin^2(\eta/2) = r_g$. It is not possible to discard the section $R\sin^2(\eta/2) < r_g$, for, as was first shown by Novikov and Ozernoĭ [6] and then by Khrapko, [7] the boundary of matter is observable from outside during the expansion. However, one can continue t analytically into the region $r < r_g$. To this end the singularity in the integral (12) is moved away from the contour of integration into the ζ plane. The necessity of bypassing the singularity has been shown earlier.[8] With this analytic continuation in empty space, t is real for $r > r_g$, and has a constant imaginary part $\pm i\pi r_g/c$ for $r < r_g$. Here dt is always real outside of matter.

3. Let us now find the metric inside matter ($R < R_0$). In this section we set $\psi^2 = 4R_0^3/r_g$, $\psi' = 0$, in accordance with (2) and (4a). Using (9), we obtain from (8c)

$$ct_\eta = \frac{\psi\sin^3(\eta/2)\,(1 - 4R^2/\psi^2)^{1/2}}{2(\sin^2(\eta/2) - 4R^2/\psi^2)^{1/2}}\, e^{-\nu/2}. \tag{13}$$

Analogously, we find from (8b) that

$$ct_R = \frac{2R\sin^2(\eta/2)\cos(\eta/2)}{(1 - 4R^2/\psi^2)^{1/2}(\sin^2(\eta/2) - 4R^2/\psi^2)^{1/2}}\, e^{-\nu/2}. \tag{14}$$

Equations (13) and (14) contain the two unknown functions $t(R, \eta)$ and $\nu(R, \eta)$. In order to eliminate the function t we use the relation

$$\frac{\partial t_R}{\partial \eta} = \frac{\partial t_\eta}{\partial R} = \frac{\partial^2 t}{\partial R\,\partial\eta}, \tag{15}$$

from which we obtain an equation of first order in the partial derivatives of the function ν:

$$-\nu_\eta \operatorname{ctg}\frac{\eta}{2} + \frac{\psi^2 - 4R^2}{4R}\nu_R = \frac{3(1 - 4R^2/\psi^2)}{\sin^2(\eta/2) - 4R^2/\psi^2} - \frac{2}{\sin^2(\eta/2)}. \tag{16}$$

The solution of (16) is constructed from the system of characteristics, the equations for which are

$$2\frac{d\eta}{ds} = \operatorname{ctg}\frac{\eta}{2}, \tag{17}$$

$$\frac{dR}{ds} = -\frac{\psi^2 - 4R^2}{2R}, \tag{18}$$

$$-\frac{1}{2}\frac{d\nu}{ds} = \frac{3(1 - 4R^2/\psi^2)}{\sin^2(\eta/2) - 4R^2/\psi^2} - \frac{2}{\sin^2(\eta/2)}, \tag{19}$$

where s is a parameter defining a point on the characteristic. We note that (13) and (14) imply that $dt/ds = 0$. At the boundary of the sphere the solution of (16) is known: it is the Schwarzschild solution for which

$$e^{\nu(0)} = 1 - \frac{r_g}{r_0} = 1 - \frac{r_g}{R_0\sin^2(\eta_0/2)}. \tag{20}$$

The point R_0, η_0 is the initial point of the characteristic ($s = 0$):

$$R = R(s), \quad \eta = \eta(s), \quad \nu = \nu(s),$$

$$R_0 = R(0), \quad \eta_0 = \eta(0), \quad \nu_0 = \nu(0). \tag{21}$$

The equations of the characteristics are

$$x\cos\frac{\eta}{2} = \cos\frac{\eta_0}{2}, \quad e^{4s} = x^4 \equiv \frac{1 - \alpha_0 R^2/R_0^2}{1 - \alpha_0}, \quad \alpha_0 = \frac{r_g}{R_0}, \quad x(R_0) = 1. \tag{22}$$

The solution of (16) with the boundary condition (20) has the form

$$e^{\nu} = \frac{(1 - \alpha_0 - x^2\cos^2(\eta/2))^2 x^2 \sin^4(\eta/2)}{(\sin^2(\eta/2) - \alpha_0)\,(1 - x^2\cos^2(\eta/2))^3}. \tag{23}$$

Expressions (9) and (23) determine the Schwarzschild metric inside matter in parametric form. Using (23), we obtain from (13) and (14)

$$ct_\eta = \frac{2x\sin(\eta/2)\,(1 - \cos^2(\eta/2)x^2)^{3/2}}{1 - \alpha_0 - x^2\cos^2(\eta/2)}(\psi^2 - 4R_0^2)^{1/2},$$

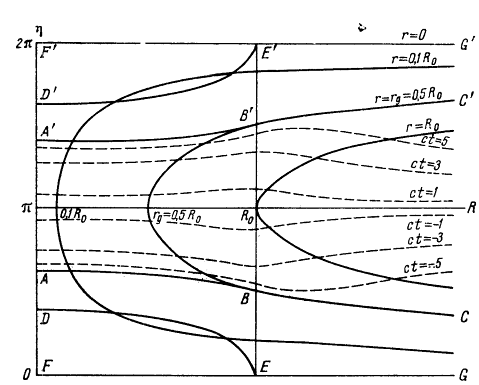
$$ct_R = \frac{2R\cos(\eta/2)\,(1 - x^2\cos^2(\eta/2))^{3/2}}{x^3(1 - \alpha_0 - \cos^2(\eta/2)x^2)\,(\psi^2 - 4R_0^2)^{1/2}}. \tag{24}$$

From (24) we find for t inside matter

$$ct = 2\psi\sqrt{1 - \alpha_0} \int\limits_{\pi/2}^{\arccos(x\cos(\eta/2))} \frac{\sin^4\zeta\, d\zeta}{\sin^2\zeta - \alpha_0}. \tag{25}$$

This result can also be obtained directly from (12) for $R = R_0$, since t inside matter is constant along a characteristic.

A few r, t coordinate lines are shown in an R, η diagram for the case $r_g = 0.5\,R_0$. The boundary of matter in this figure is given by the line E′E. The coordinate t is real in the region bounded by the curves ABC and A′B′C′. The lines AB and A′B′ correspond to singularities of the type $1 - \alpha_0 - x^2\cos^2(\eta/2) = 0$ in (25), which must be bypassed in a similar fashion as in (12); then t acquires a constant imaginary part $\pm i\pi r_g/c$ in the regions ABCGED and A′B′C′G′E′D′. We note that there are regions inside matter from which the characteristics cannot reach the surface. These are the re-



gions FDE and F′D′E′ in the figure. The equations for the lines DE and D′E′ are $x^2 \cos^2(\eta/2) = 1$. It is easily seen from (25) that t is complex in these regions, where now the real part of t is constant, and the modulus is equal to

$$|\mathrm{Re}\, t| = \frac{\pi}{2\sqrt{\alpha_0}} \frac{R_0}{c} \sqrt{1-\alpha_0}\,(1+2\alpha_0).$$

We also note that regardless of the complex value of t, all observable quantities during the expansion are real, as can be seen with the help of the method developed by Khrapko.[7]

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