# SELF-ACTION OF WAVES IN RAREFIED PLASMA LOCATED IN A MAGNETIC FIELD

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An expression is derived for the four-photon interaction tensor in a plasma located in a magnetic field. The cubic current is found by means of the expression, and general nonlinear equations are derived for the self-action of waves in the absence of collisions. It is shown that self-action constitutes a change in polarization and an appearance of amplitude modulation of the wave. These processes are of an oscillatory nature. The oscillation frequency is proportional to the wave energy. Solutions of the self-action equations are obtained for waves propagating perpendicular or parallel to the magnetic field. Resonances are investigated. The results of numerical calculations are presented.

THE field of a powerful electromagnetic wave propagating in a plasma produces strong disturbances in the medium, and changes, in particular, its conductivity and its dielectric constant. This in turn changes the conditions of propagation of the disturbance-causing wave itself. This is how the nonlinear self-action of a wave is realized.

In a dense plasma, where collisions are significant, the principal role in the self-action of a plane wave is played by the change of its absorption[1]. We shall consider here a rarefied plasma, where the collisions are infrequent enough to be neglected. In this case the absorption is negligible[1)]. This leads to conservation of the energy and entropy of the wave, and consequently to a reversibility of the nonlinear self-action. This characteristic property distinguishes the phenomenon considered here from other nonlinear interactions such as decay and nonlinear scattering. Self-action of a plane wave reduces to a change of its polarization and to the occurrence of amplitude modulation. In the absence of a magnetic field, nonlinear rotation of the polarization ellipse was investigated by Tsytovich and one of the authors[2]. In the presence of a magnetic field, the wave polarization depends on its frequency, thus greatly changing the character of the phenomena, in that the polarization ellipse experiences a complicated deformation in addition to rotating. This produces amplitude modulation of the wave. In addition, at definite frequencies, the effect becomes resonantly amplified in a magnetic field. The present paper is devoted to a theoretical investigation of self-action of waves in a magnetoactive plasma.

## 1. FUNDAMENTAL EQUATIONS

The propagation of a plane electromagnetic wave in a plasma is described jointly by the Maxwell equations for the field and by the kinetic equations for the electrons and ions. We assume that the wave has a sufficiently high frequency $\omega$:

$$\omega \gg \omega_{He}\sqrt{m_e/M_i}.$$

Here $\omega_{He} = eH/mc$ is the electron gyromagnetic frequency, and $m_e$ and $M_i$ are respectively the electron and ion masses. In this case we can neglect the motion of the ions and the thermal spread of the electron velocities (if we disregard regions close to resonances of the linear theory). The motion of electrons in the absence of collisions is then described by the hydrodynamic equations for their concentration n and for their average directional velocity **v**:

$$\partial n/\partial t = -\nabla(n\mathbf{v}),$$
$$\frac{\partial \mathbf{v}}{\partial t} = -(\mathbf{v}\nabla)\mathbf{v} + \frac{e}{m}\left\{\mathbf{E} + \frac{1}{c}[\mathbf{vH}]\right\} + \frac{e}{mc}[\mathbf{vH}_0]. \quad (1)^*$$

Here $\mathbf{H}_0$ is the external magnetic field; **E** and **H** are the intensities of the electric and magnetic fields of the wave and are connected by Maxwell's equations.

We change over to the Fourier representation of the unknown functions n, **v**, **E**, and **H**. Equations (1) are then rewritten in the form

$$n_{\mathbf{k}\omega} = \frac{\mathbf{k}}{\omega}\int d\lambda_2 n_{\mathbf{k}_1\omega_1}\mathbf{v}_{\mathbf{k}_2\omega_2}, \quad (2)$$

$$\mathbf{v}_{\mathbf{k}\omega} = \frac{1}{\omega}\int\left\{(\mathbf{v}_{\mathbf{k}_1\omega_1}\mathbf{k}_2)\mathbf{v}_{\mathbf{k}_2\omega_2} - \frac{ie}{mc}[\mathbf{v}_{\mathbf{k}_1\omega_1}\mathbf{H}_{\mathbf{k}_2\omega_2}]\right\}d\lambda_2 - \frac{ie}{mc\omega}\mathbf{E}_{\mathbf{k}\omega} - \frac{ie}{mc\omega}[\mathbf{v}_{\mathbf{k}\omega}\mathbf{H}_0], \quad (3)$$

with

$$(k^2\delta_{ij} - k_ik_j - \omega^2\varepsilon_{ij})E_j = 4\pi i\omega j_i.$$

Here $\epsilon_{ij}$ is the dielectric tensor in the hydrodynamic approximation[3]:

$$\varepsilon_{11} = \varepsilon_{22} = 1 - \frac{p^2}{1-u^2}, \quad \varepsilon_{12} = -\varepsilon_{21} = -\frac{iup^2}{1-u^2},$$
$$p = \omega_{0e}/\omega, \quad u = \omega_{He}/\omega,$$

$\omega_{0e}$ is the Langmuir frequency for electrons, and $d\lambda_2$ will be defined below. In the right side of (3) we have separated the "extraneous current," which is due to the nonlinear effect.

To find the current $j_i$ it is necessary to solve Eqs. (2) and (3) simultaneously using, as is customary, the method of successive approximations. The expansion is in powers of the dimensionless small parameter

[1)]The generation of harmonic waves leads to a loss of energy of the fundamental wave. However, for non-induced processes, this effect is quadratic in the wave power W, and consequently is small compared with the processes considered here, which are linear in W.

*$[\mathbf{vH}] \equiv \mathbf{v} \times \mathbf{H}$.

$eE/m\omega c \ll 1$. We have

$$j_i = j_i^{(2)} + j_i^{(3)} + \ldots;$$

$$j_i^{(2)} = \int d\lambda_2 \tilde{S}_{ijl} E_{\mathbf{k}_1 j} E_{\mathbf{k}_2 l}, \quad j_i^{(3)} = \int d\lambda_3 \tilde{\Sigma}_{ijls} E_{\mathbf{k}_1 j} E_{\mathbf{k}_2 l} E_{\mathbf{k}_3 s};$$

$$d\lambda_2 = \delta(\omega - \omega_1 - \omega_2)\,\delta(\mathbf{k} - \mathbf{k}_1 - \mathbf{k}_2)\,d\omega_1 d\omega_2 d\mathbf{k}_1 d\mathbf{k}_2.$$
$$d\lambda_3 = \delta(\omega - \omega_1 - \omega_2 - \omega_3)\,\delta(\mathbf{k} - \mathbf{k}_1 - \mathbf{k}_2 - \mathbf{k}_3)\,d\omega_1 d\omega_2 d\omega_3 d\mathbf{k}_1 d\mathbf{k}_2 d\mathbf{k}_3.$$

The tensor $\tilde{S}_{ijl}$, describing the "quadratic current," was obtained earlier in a number of papers (see, for example,[4]). Allowance for $\tilde{S}_{ijl}$ makes it possible to describe three-photon interaction processes (decay and coalescence of waves). However, the quadratic current makes no contribution to the effect of self-action of the wave, in which we are interested in the present article.

The expression for the tensor $\tilde{\Sigma}_{ijls}$, which gives the cubic current, can also be obtained by simultaneously solving Eqs. (2) and (3). This expression is given in[5], and describes various processes of four-photon interaction of waves in a plasma in a magnetic field, namely the scattering of a pair of photons by each other with production of a new pair of photons, frequency tripling, and self-action of the wave. Using Feynman diagrams, we can represent the amplitude of the four-photon process described by $\tilde{\Sigma}_{ijls}$ in the form of a sum of amplitudes corresponding to the diagrams shown in Fig. 1.

The external photon lines correspond here to the wave fields $E_{\mathbf{k}\omega}$, the circles correspond to the nonlinear elements $\int d\lambda_2 \tilde{S}_{ijl}$ and $\int d\lambda_3 \tilde{\Sigma}_{ijl}$, and the wavy line corresponds to a virtual photon described by the reciprocal Maxwell operator.

We are interested here only in the particular case of self-action of a mono-chromatic wave. In the linear approximation, its spectrum is given by

$$E_{\mathbf{k}\omega} = \sum_\sigma \mathbf{E}\,\delta(\mathbf{k} - \mathbf{k}_0)\,\delta(\omega - \omega_\sigma(\mathbf{k})), \tag{4}$$

where $\omega_\sigma(\mathbf{k})$ is the solution of the nonlinear dispersion equation $|k^2\delta_{ij} - k_i k_j - \omega^2 \epsilon_{ij}| = 0$, corresponding to the polarization $\sigma$.

When account is taken of the nonlinearity, the amplitude of the wave field varies in time. Let us assume that it varies slowly compared with the fundamental frequency of the wave. The spectral density (4), however, no longer has a $\delta$-function character; it is "smeared" near the frequency $\omega_\sigma(\mathbf{k})$. If the wave amplitude is weakly inhomogeneous in space, then the spectral density (4) is also weakly smeared. Taking this into account, we expand the slowly varying amplitudes in (3), assuming in first approximation that the wave is monochromatic. In the expansion, following the Van der Pol method[6], we place $\omega$ and $\mathbf{k}$ in the left side of (3) by the operators

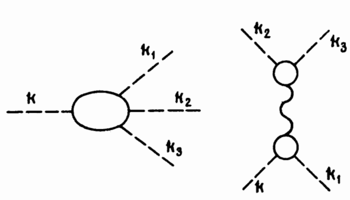


FIG. 1.

$$\omega \to \omega - \frac{1}{i}\frac{d}{dt}, \quad \mathbf{k} \to \mathbf{k} + \frac{1}{i}\frac{d}{d\mathbf{r}}.$$

Making this substitution in the left side of Maxwell's equations, we obtain for a wave of arbitrary polarization

$$\left(\frac{2k_z}{\omega}\frac{d}{dz} + M\frac{d}{dt}\right)E_x - N\frac{dE_y}{dt} - \left(k_x\frac{d}{dz} + k_z\frac{d}{dx}\right)\frac{E_z}{\omega} = -4\pi j_x^{(3)},$$
$$N\frac{dE_x}{dt} + \left(\frac{2k_z}{\omega}\frac{d}{dz} + \frac{2k_x}{\omega}\frac{d}{dx} + M\frac{d}{dt}\right)E_y = -4\pi j_y^{(3)},$$
$$-\left(k_z\frac{d}{dx} + k_x\frac{d}{dz}\right)\frac{E_x}{\omega} + \left(\frac{2k_x}{\omega}\frac{d}{dx} + 2\frac{d}{dt}\right)E_z = -4\pi j_z^{(3)}. \tag{5}$$

Here

$$M = 2[1 + p^2u^2(1 - u^2)^{-2}], \quad N = -up^2(1 + u^2)(1 - u^2)^{-2}; \tag{6}$$

The z axis is directed along $\mathbf{H}_0$, and the x axis is perpendicular to $\mathbf{H}_0$ and lies in the $(\mathbf{H}_0, \mathbf{k})$ plane. The right side of (5) contains the components of the cubic current, calculated with the aid of the tensor $\tilde{\Sigma}_{ijls}$ (see[5]) for fields having a spectrum (4). In the calculation of the current $j^{(3)}$, it is necessary to take into account the fact that $\mathbf{E}^*_{\mathbf{k}} = \mathbf{E}_{-\mathbf{k}}$ and $\omega(-\mathbf{k}) = -\omega(\mathbf{k})$.

The general solution of the system (5), as can be readily seen, is given by functions of the type

$$M = 2[1 + p^2u^2(1 - u^2)^{-2}], \quad N = -up^2(1 + u^2)(1 - u^2)^{-2}; \tag{7}$$

Here $\mathbf{v} = d\omega/d\mathbf{k}$ is the group velocity. The dependence of f on t is determined by the nonlinear equations (5) without the spatial derivatives. The $f(\xi)$ dependence is arbitrary and is determined by the initial and boundary conditions of the problem. Thus, the weak spatial inhomogeneity of the initial distribution does not influence the nonlinear process: the amplitude of the field is simply transported with a constant velocity $\mathbf{v}$. In the case of the stationary boundary value problem, $f(\xi, t) = \text{const}$, a stationary distribution $f(\xi, t) = f(0, = f(0, |\mathbf{r}|/|\mathbf{v}|)$ is produced. On the other hand, if the boundary distribution changes slowly in time (for example, a high-frequency wave modulated by a low frequency propagates from the boundary into the plasma, and the period of the low-frequency modulation is larger than the characteristic time of the nonlinear process), then the stationary distribution is transported in space.

Thus, the general solution of (5), for arbitrary initial and boundary conditions, can be expressed with the aid of relation (7) in terms of the solution of the spatially homogeneous problem $\mathbf{E} = \mathbf{E}(t)$ with initial conditions corresponding to the linear approximation. We shall henceforth consider only the spatially-homogeneous case.

## 2. PROPAGATION OF EXTRAORDINARY WAVE ACROSS THE MAGNETIC FIELD

We consider an extraordinary propagating across a magnetic field. It has only components $E_x$ and $E_y$, so that the system (5) takes the form

$$M\frac{dE_x}{dt} - N\frac{dE_y}{dt} = -4\pi j_x^{(3)},$$
$$N\frac{dE_x}{dt} + M\frac{dE_y}{dt} = -4\pi j_y^{(3)}. \tag{8}$$

We seek a solution of (8) in the form

$$E_x = ae^{i\varphi_1}, \quad E_y = be^{i\varphi_2}, \quad \varphi = \varphi_2 - \varphi_1, \tag{9}$$

where $a = |E_x|$, $b = |E_y|$, and $\varphi$ is the phase shift between $E_y$ and $E_x$. We substitute (9) in (8) and separate the real and imaginary parts. Separating, in addition, the derivatives $da/dt$, $db/dt$, and $d\varphi/dt$, we obtain:

$$V\frac{da}{dt} = -M\{Aab^2\sin 2\varphi - ub[a^2(B-QY) + b^2(C-QZ)]\cos\varphi\} + N[ba^2(QX - A - E)\cos\varphi - ab^2u(C+QZ)\sin 2\varphi - b^3D_2\cos\varphi], \tag{10}$$

$$V\frac{db}{dt} = -M\{-Aba^2\sin 2\varphi + ua[a^2(B-QY) + b^2(C-QZ)]\cos\varphi\} + N[ab^2(QX + A + E)\cos\varphi + ba^2u(B+QY)\sin 2\varphi + a^3D_1\cos\varphi], \tag{11}$$

$$\begin{aligned} V\frac{d\varphi}{dt} = \frac{M}{ab}\{&Aa^3b\cos 2\varphi - Aab^3\cos 2\varphi + ab^3(D_2 - E - QX) \\ &+ ba^3(E - D_1 - QX) + u[a^4(B-QY) - b^4(C-QZ) \\ &+ 3(C-B)a^2b^2 + Q(Z-Y)a^2b^2]\sin\varphi\} \\ &+ \frac{N}{ab}\{2abu[b^2(C+QZ) - a^2(B+QY)]\sin^2\varphi \\ &+ [b^4D_2 - a^4D_1 - 2a^2b^2QX]\sin\varphi\}, \end{aligned} \tag{12}$$

where

$$V = \left[\frac{e^2p^2[1 - p^2(1-p^2)/(1-p^2-u^2)]}{m_e^2(4-u^2)(1-u^2)^4(M^2-N^2)\omega}\right]^{-1}$$

and $A, B, C, D_1, D_2, E, Q, X, Y, Z, M, N$ — functions of p and u only:

$$A = 2u^4 + 16u^2 - 6, \quad B = 5u^2 + 7, \quad C = 6u^4 + 11u^2 - 5,$$

$$D_1 = -2u^2 - 10, \quad D_2 = -2u^6 - 24u^4 + 16u^2 - 2,$$

$$E = -u^6 - 14u^4 - 9u^2, \quad Q = -\frac{2(4-u^2-p^2)(1-u^2-p^2)}{3[u^2p^2 + (1-p^2)(4-p^2)]},$$

$$X = -8u^4 + 4u^2 + 4 + 2\delta'(-u^6 - 4u^4 + 5u^2),$$

$$Y = -u^4 + 8u^2 - 7 - 6\delta'(1-u^2),$$

$$Z = -3u^4 + 6u^2 - 3 - \delta'(8u^4 - 10u^2 + 2),$$

$$\delta' = p^2/2(4 - u^2 - p^2);$$

Expressions for M and N were given earlier (see (6)).

The system (10)–(12) describes the self-action of an extraordinary wave propagating in a plasma across a magnetic field. This system has a first integral:

$$M(a^2 + b^2) + 2Nab\sin\varphi = \varepsilon_0. \tag{13}$$

From the formula for the wave energy density W[7] it is seen that $\epsilon_0 = 8\pi W$. The integral (13) denotes consequently the conservation of the energy density.

In the linear theory, the wave is elliptically polarized. The axis ratio, the phase shift, and the constant $\epsilon_0$ are then given by

$$\frac{a_0^2}{b_0^2} = \frac{u^2p^4}{(1-u^2-p^2)^2}, \quad \varphi_0 = \frac{\pi}{2}, \quad \varepsilon_0 = M(a_0^2 + b_0^2) + 2Na_0b_0. \tag{14}$$

Relations (14) constitute the system of initial conditions for Eqs. (10)–(12). Using the integral (13), we can calculate the angle $\varphi$, and then also the time t. This makes it possible to find the second integral and to analyze qualitatively the solution of (10)–(12). This is done in the Appendix. In the general case the amplitudes a and b oscillate with frequency $\Omega \sim e^2\epsilon_0 s(u, p)/m^2c^2\omega$, where f(u, p) is a certain function of the parameters. The phase shift $\varphi$ can either decrease monotonically or oscillate with the same frequency $\Omega$. We shall stop to discuss in detail only the phenomena occurring in the vicinities of the resonances.

From an analysis of the structure of the coefficients of the system (10)–(12) we see that there are three resonant values of the wave frequency:

$$\text{I. } u^2 = 4, \tag{15}$$

$$\text{II. } u^2 = (4 - p^2)(p^2 - 1)p^{-2}, \tag{16}$$

$$\text{III. } u^2 = 1. \tag{17}$$

We shall call these resonances I, II, and III, respectively. Resonance I is well known from the theory of nonlinear oscillators, and occurs at a frequency equal to half the natural frequency of the oscillator, $\omega = \omega_{He}/2$. The wave propagation condition is $n_e^2 > 0$, where $n_e$ is the refractive index of the extraordinary wave

$$n_н^2 = 1 - \frac{p^2(1-p^2)}{1-p^2-u^2}.$$

This condition limits the values of the parameter p for the resonance in question: $0 < p < \sqrt{3}$.

Equations (10)–(12) were solved numerically. The characteristic oscillations of the polarization components $a_1 = a/E_0$ and $b_1 = b/E_0$ and the changes of the polarization ellipse (for $p = 1$, $u \to 2$) are shown in Figs. 2 and 3. Here $E_0 = (8\pi W)^{1/2}$. The angular frequency of the nonlinear oscillations is in this case $\Omega = 79.2\,\Omega_0$, where $\Omega_0 = e^2E_0^2/m^2c^2\omega(4 - u^2)$.

It is seen from Fig. 2 that the phase difference between the components of the polarization $\varphi$ decreases monotonically with increasing $\Omega t$. Since the quantity $a^2 + b^2$ and $ab\sin\varphi$ are not conserved, the ellipse becomes deformed. This is seen from Fig. 3. The amplitude of the electric field, averaged over the high frequency, is given by

$$\bar{E} = \frac{1}{2\pi}\int_0 [\tilde{a}^2\sin^2 x + \tilde{b}^2\cos^2(x+\varphi)]^{1/2}dx = \frac{2\tilde{a}}{\pi}E\left(\left[1 - \frac{\tilde{b}^2}{\tilde{a}^2}\right]^{1/2}\right).$$

Here E is a complete elliptic integral of the second kind; $\tilde{a}$ and $\tilde{b}$ are the values of the ellipse semiaxes referred to the principal axes[8]; it is assumed that

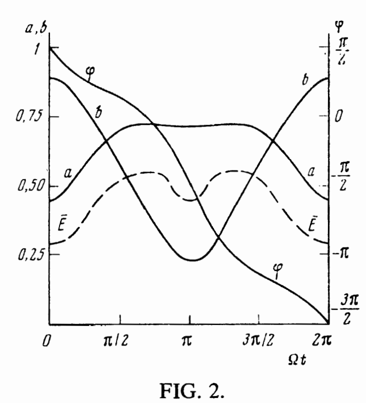


FIG. 2.

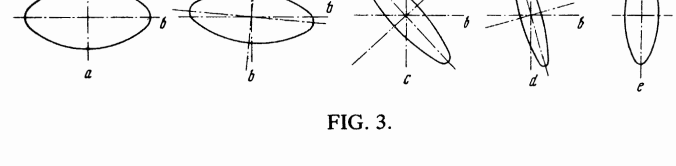


FIG. 3.

$\tilde{a} \geq \tilde{b}$. The amplitude $\overline{E}$ is shown in Fig. 2 (dashed curve). The field amplitude, which is a periodic function of the time with period $2\pi/\Omega$, can be represented in the form

$$\overline{E} = E[1 + \mu_\Omega \cos(\Omega t + \varphi_\Omega) + \mu_{2\Omega} \cos(2\Omega t + \varphi_{2\Omega}) + \ldots],$$

where the modulation depth $\mu$ and the modulation phase $\varphi$ at the frequencies $\Omega, 2\Omega, \ldots$ are determined by the coefficients of the Fourier expansion of the field $\overline{E}$. The dependence of the frequency $\Omega$ and of the maximum and minimum components $a_1$ and $b_1$ in the resonance I on the parameter p is shown in Fig. 4. It is seen from the figure that the oscillation frequency increases with decreasing p.

Unlike resonance I, resonance II is a phenomenon peculiar to waves: it appears when the phase velocities of the extraordinary waves of frequencies $\omega$ and $2\omega$, propagating in the same direction, become equal. This indeed leads to the condition (16). The propagation conditions $n_e^2 > 0$ limit the parameter p in the resonance II to the value $1 < p < \sqrt{2}$. The frequency of the nonlinear oscillations in resonance II decreases rapidly with decreasing p. The amplitude of the oscillation also decreases. It is important that if $p < 1.18$, then the phase shift does not increase monotonically, but merely oscillates about the value $\varphi = \pi/2$ ($\varphi = \pi/2$ is the phase difference between the polarization components in the linear theory, see (14)). This is seen from Fig. 5, where the polarization components and the phase difference are plotted against $\Omega t$ for p = 1.14. Here

$$\Omega = 6.6 \frac{e^2 E_0^2}{m^2 c^2 \omega (u^2 - 0.623)}$$

is the angular frequency of the nonlinear oscillations. When $p > 1.18$ the phase varies monotonically. The frequency $\Omega$ increases exceptionally rapidly as p approaches $\sqrt{2}$. We denote $\sqrt{2} - p = \epsilon$, where $\epsilon \to 0$, and obtain

$$\Omega = \frac{\text{const}}{\varepsilon^7} \frac{e^2 E_0^2}{m^2 c^2 \omega}.$$

The reason for this growth lies in the fact that as $p \to \sqrt{2}$ we have simultaneously $u \to 1$, as seen from (16); we consequently have an intersection of the resonances II and III. We note, however, that when $u \to 1$ an increase in the collision absorption of the extraordinary wave[3] takes place, but is not taken into account here. By virtue of this, the resonance III at the electron gyrofrequency, which appears also in the linear theory, will not be considered here.

We note that there exist such values of the wave frequency and plasma parameter (p, u), at which the state of the polarizations does not change and remains the same as in the linear theory. Indeed, it follows from (10) and (11) that when $\varphi = \pi/2$ we get $da/dt = 0$ and $db/dt = 0$. If it turns out that in this case also $d\varphi/dt = 0$, then the state of the polarization is stationary. As seen from (12), this occurs under the condition

$$M\Big\{-A + \frac{A}{K_0} + \frac{1}{K_0}(D_2 - E - QX) + u\sqrt{K_0}\Big[B - QY + \frac{1}{K_0^2}(C - QZ) + \frac{3}{K_0}(C - B) + \frac{Q}{K_0}(Z - Y)\Big] + (E - D_1 - QX)\Big\} + N\Big\{2u\Big[\frac{1}{K_0}(C + QZ) - (B + QY)\Big] + \frac{D^2}{K_0^{3/2}} - D_1 - \frac{2}{K_0^{1/2}} QX\Big\} = 0. \tag{18}$$

Here $K_0^2 = a_0^2/b_0^2$ is given by formula (14). The dependence of $p^2$ on u, defined by Eq. (18), is shown in Fig. 6 (the region $u \gtrsim 10$ is separated in Fig. 6b). The propagation region $n_e^2 > 0$ is bounded in the figure by the dashed curve.

Thus, as a result of the nonlinear interaction of the polarization components, an oscillatory energy pumping from the larger component to the smaller one takes place. The initial polarization ellipse then undergoes rotation, as well as deformation whose rate is not constant in time. The frequency $\delta$-function characterizing the spectrum of the weak wave spreads out into a linear spectrum $\omega \pm n\Omega$, where $\Omega$ is the frequency of the nonlinear oscillations. The presence of low-frequency modulation can lead to the occurrence of nonlinear absorption of the high-frequency wave, similar to the resonant absorption of modulated waves indicated in[9]. We note also that a standing wave of nonlinear polarization is produced in the case of stationary boundary conditions. It creates in the plasma a unique period lattice. Indeed, the amplitude of the plasma density oscillations in the wave field will be different in the nodes $n_{k\omega}^{(1)}$ and in the antinodes $n_{k\omega}^{(2)}$ of the polarization standing wave, namely $n_{k\omega}^{(1)}/n_{k\omega}^{(2)} = a^{(1)}/a^{(2)}$, where $a^{(1)}$ and $a^{(2)}$ are the longitudinal polarization components at the node and the antinode. As seen from Fig. 2, the ratio $n^{(1)}/n^{(2)} \sim 1-10$. The

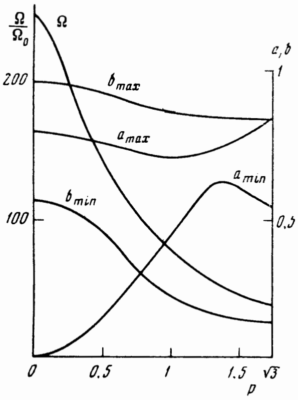


FIG. 4.

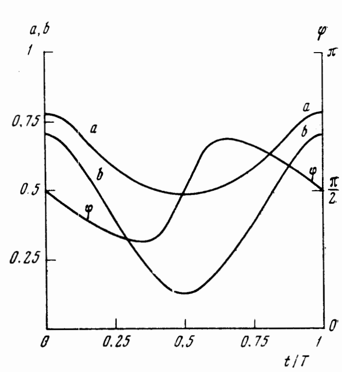


FIG. 5.

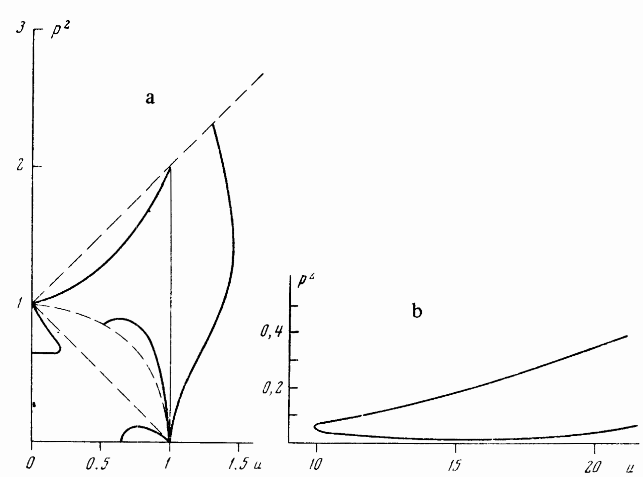


FIG. 6.

spatial period of the "lattice" is $\lambda = 2\pi v/\Omega$, where v is the group velocity of the wave and $\Omega$ is the frequency of the nonlinear oscillations[2)].

## PROPAGATION OF TRANSVERSE WAVES

As is well known from the linear theory, transverse-polarization waves can propagate in a magnetoactive plasma across the magnetic field (ordinary wave, linearly polarized in $H_0$) and along the field (ordinary and extraordinary wave). For such waves, the equations in (5) become much simpler and can be solved analytically.

1. We consider first an ordinary wave propagating across the field. The electric field has in this case one component $E_z$, so that the equations in (5) reduce to a single equation

$$\frac{dE_z}{dt} = i\frac{e^2|E_z|^2p^2(1-p^2)}{m^2c^2\omega(4-u^2)}E_z.$$

We see therefore that self-action leads only to a frequency shift of the ordinary wave, proportional to the wave energy and increasing resonantly as $\omega \to \omega_{He}/2$.

2. We now consider waves propagating along the magnetic field. The equations describing the change of the polarization has as before the form (8). All that change are the expressions for the cubic current $j_x^{(3)}$ and $j_i^{(3)}$. Calculating these currents with the aid of the tensor $\tilde{\Sigma}_{ijls}$ and using again the procedure of separating the real and imaginary parts, we obtain equations for the components a and b and for the phase shift $\varphi$ in the form

$$da/dt = \beta[Mb^2a\sin 2\varphi + Nb(a^2+b^2)\cos\varphi],$$

$$db/dt = -\beta[Ma^2b\sin 2\varphi + Na(a^2+b^2)\cos\varphi],$$

$$\frac{d\varphi}{dt} = -\frac{\beta\sin\varphi}{ab}(b^2-a^2)[2Mab\sin\varphi + N(a^2+b^2)],$$

$$\beta = \frac{e^2p^2k^2}{2mc^2\omega(1-u^2)^2(M^2-N^2)} \qquad (19)$$

When u = 0, these equations go over into equations describing the nonlinear rotation of the polarization ellipse in an isotropic plasma[2].

The system of equations (19) has two independent integrals

$$a^2+b^2 = a_0^2+b_0^2 = I_1, \quad ab\sin\varphi = a_0b_0\sin\varphi_0 = I_2. \qquad (20)$$

Here $a_0$, $b_0$, and $\varphi_0$ are the values of the polarization components and of the phase shift in the linear theory; these are initial values for equations (19). The values of $\varphi_0$ can be $\pm\pi/2$ and 0. Using the integrals (20), we can easily find a general solution of (19). When $\varphi_0 = \pm\pi/2$, it is given by

$$a^2 = a_0^2\cos^2\frac{\Omega t}{2} + b_0^2\sin^2\frac{\Omega t}{2},$$

$$b^2 = a_0^2\sin^2\frac{\Omega t}{2} + b_0^2\cos^2\frac{\Omega t}{2},$$

$$\varphi = \arcsin\left\{\left[1+\frac{(a_0^2-b_0^2)^2}{4a_0^2b_0^2}\sin^2\Omega t\right]^{-1/2}\right\}. \qquad (21)$$

The frequency $\Omega$ is different here for the ordinary and extraordinary waves:

$$\Omega_{1,2} = \frac{e^2p^2(2MI_2+NI_1)\omega}{m^2c^2(1-u^2)^2(M^2-N^2)}n_{1,2}^2, \qquad (22)$$

where $n_{1,2}$ is the refractive index for the extraordinary and ordinary waves:

$$n_{1,2}^2 = 1-\frac{p^2}{1\mp u}. \qquad (23)$$

It follows from (21) that, unlike in the case of an extraordinary wave propagating across the magnetic field, in the case considered here the shape of the polarization ellipse does not change, and the ellipse only rotates with a period $T = 2\pi/\Omega$. Therefore the quantities $a^2$ and $b^2$ describe slow time variations of the two Stokes parameters ($S_1$ and $S_2$) of the wave. The running value of the third parameter is

$$S = ab = {}^1/_2(I_1^2 + 4I_2^2\cos\Omega t)^{1/2}.$$

If the phase is elliptically polarized, then each of the two phase-shifted linear polarizations forming the ellipse experience, as is well known, Faraday rotation. When $\varphi_0 = \pi/2$, the electric vector rotates to the left (ordinary wave). When $\varphi_0 = -\pi/2$, the electric vector rotates to the right (extraordinary wave). In contrast, the direction of the nonlinear rotation of the ellipse, determined from (21) and (22), depends on the sign of the quantity $\gamma = 2MI_2 + NI_1$.

If $\gamma < 0$, then the direction of the nonlinear rotation of the ellipse coincides with the direction of rotation of the vector **E**; when $\gamma > 0$, the ellipse rotates oppositely to the rotation of the vector **E**; when $\gamma = 0$ there is no nonlinear effect. It is interesting that in an isotropic plasma the direction of the nonlinear rotation is always opposite to the direction of rotation of the vector **E**.

The Faraday rotation is usually much stronger than the nonlinear rotation. However, it follows from the propagation conditions that there exist frequency bands in which only the ordinary or only the extraordinary wave propagate. In such cases, only one nonlinear rotation determines the slow variation of the polarization (when averaged over a time interval much larger than the period of the wave).

It is seen from (22) that as $u < 1$, and also when $M^2 \to N^2$, the nonlinear effect increases resonantly. The latter resonant occurs under the condition

$$p^2 = 2(1+u)^2/u$$

and is realized only in the extraordinary wave when $u < 1$. The phase velocity of the wave decreases in this resonance, tending to zero when $M^2 \to N^2$.

We note also that in the propagation of a wave linearly polarized at the initial instant we have $\varphi_0 = 0$ and $I_2 = 0$, so that its polarization always remains linear. Self-action, as seen from (21), leads to a shift of the frequency of each of the circular polarizations. Thus, the degeneracy of a wave propagating along the field is lifted as a result of the self-action.

## APPENDIX

The integral (13) makes it possible to simplify Eq.

[2)] It must be emphasized that when the kinetic effects are taken into account the amplitudes of the density oscillations in the stationary lattice can increase by $(c/v_{Te})^2$ times, where $v_{Te}$ is the thermal velocity of the electrons. The time of establishment of these amplitudes, however, is quite large, of the order of $\lambda/v_{Ti}$, where $v_{Ti}$ is the thermal velocity of the ions.

(10)–(12). Indeed, let us eliminate with the aid of (13) the angle $\varphi$ from (10) and (12), and let us divide both sides of the obtained expressions by the constant quantity $\epsilon_0^2$. We then obtain a system of equations for the dimensionless amplitudes and for the dimensionless time:

$$\frac{da_1^2}{d\tau} = \frac{1}{N^2}\{4N^2a_1^2b_1^2 - [2 - M(a_1^2 + b_1^2)]^2\}^{1/2} \times \{a_1^2(\alpha + \beta) + b_1^2\gamma_1 - \delta_1\}, \tag{A.1}$$

$$\frac{db_1^2}{d\tau} = \frac{1}{N^2}\{4N^2a_1^2b_1^2 - [2 - M(a_1^2 + b_1^2)]^2\}^{1/2} \times \{a_1^2\gamma_2 + b_1^2(\beta - \alpha) + \delta_2\}, \quad \tau = \varepsilon_0 t / V. \tag{A.2}$$

Here $\alpha, \beta, \gamma_1, \gamma_2, \delta_1$, and $\delta_2$ are functions of the dimensionless variables u and p. Eliminating the time $\tau$ from (A.2), we can readily find the second integral of the system (10)–(12) (detailed calculations are given in[5]). The existence of this integral shows that the nonlinear system described by Eqs. (A.1–A.2) is conservative. On the phase plane $(a_1^2, b_1^2)$ this system has a singular point of the center type. It is also the point representing the state of equilibrium of the system, since the common factor in Eqs. (A.1)–(A.2) never tends to infinity. It follows from (A.1)–(A.2) that the states of equilibrium correspond also the points lying on the curve

$$4N^2a_1^2b_1^2 = [2 - M(a_1^2 + b_1^2)]^2. \tag{A.3}$$

The initial state of the system (10)–(12) is represented on the phase plane $(a_1^2, b_1^2)$ by a point having the coordinates

$$a_0^2 = \left\{\frac{M}{2}\left[1 + \frac{(1 - u^2 - p^2)^2}{u^2p^4}\right] + N\frac{|1 - u^2 - p^2|}{up^2}\right\}^{-1},$$
$$b_0^2 = \frac{(1 - u^2 - p^2)^2}{u^2p^4}. \tag{A.4}$$

The phase trajectory representing the variation of the state of the system passes through this initial point. It is shown in Fig. 7 (trajectory 1 ($u \to 2$, $p = 1$)). The curve of the equilibrium states (A.3) also passes through the same point. The motion of the system is along the section of the trajectory 1, shown in Fig. 7 by the solid curve, and limited by the points of intersection with the curve of the equilibrium states (A.3). The point representing the state of the system oscillates, moving over the indicated section of the curve, and being reflected from the equilibrium points. There are no closed trajectories on the $(a_1^2 b_1^2)$ plane. We note that closed trajectories are possible on the

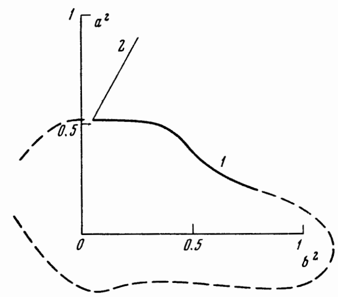


FIG. 7.

(a, $\varphi$) or (b, $\varphi$) plane. An example of such a motion is seen in Fig. 5.

We point out a case when the singular point is located at infinity. The phase trajectory is then a segment of a straight line, also shown in Fig. 7 (trajectory 2). Eqs. (10)–(12) can then be integrated[5]. An example of such a point is $p = 1$, $u \to 1$.

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[1] V. L. Ginzburg and A. V. Gurevich, Usp. Fiz. Nauk **70**, 201 (1960) [Sov. Phys.-Usp. **3**, 115 (1960)].

[2] V. N. Tsytovich and A. B. Shvartsburg, ZhETF Pis. Red. **3**, 105 (1966) [JETP Lett. **3**, 65 (1966); A. P. Kropotkin and V. V. Pustovalov, Zh. Eksp. Teor. Fiz. **49**, 10 (1965) [Sov. Phys.-JETP **22**, 6 (1966)].

[3] V. L. Ginzburg, Rasprostranenie ėlektromagnitnykh voln v plasme (Propagation of Electromagnet Waves in a Plasma), Nauka, 1967.

[4] N. L. Tsintsadze, Dissertation, Institut fiziki AN Gruzinskoĭ SSR, Tbilisi, 1966.

[5] A. V. Gurevich and A. B. Shvartsburg, Preprint FIYaN, No. 121, 1968.

[6] A. N. Akhmanov and R. V. Khokhlov, Nelineĭnya optika (Nonlinear Optics). VINITI, 1965.

[7] V. P. Silin and A. A. Rukhadze, Élektromagnitnye svoĭstva plazmy (Electromagnetic Properties of Plasma), 1961, p. 142.

[8] L. D. Landau and E. M. Lifshitz, Teoriya polya (Field Theory), Gostekhizdat, 1948 [Addison-Wesley, 1951].

[9] A. V. Gurevich and A. B. Shvartsburg, Geomagnetizm i aėronomiya **8**, No. 6, 1968.

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