THEORY OF WAVES THAT ARE CLOSE TO EXACT SOLUTIONS OF NONLINEAR ELECTRODYNAMICS AND OPTICS

V. M. ELEONSKIĬ and V. P. SILIN

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Bearing in mind the instability of one-dimensional wave distributions in non-linear electrodynamics and in optics, it seems essential, in analogy with the nonlinear theory of stability of hydrodynamic flows, to study two-dimensional and three-dimensional steady-state distributions of the electromagnetic field. To construct such solutions of the nonlinear field equations, a general scheme of asymptotic expansions is proposed, representing a generalization of the theory of weakly-nonlinear natural oscillations of system with many degrees of freedom to the case of distributed systems. A large number of concrete two-dimensional solutions of nonlinear electrodynamics is constructed.

1. In many recent papers[1-3] devoted to nonlinear electrodynamics and optics, the exact one-dimensional steady-state distribution of the electromagnetic field in the nonlinear medium were investigated in sufficient detail (for example, solutions of the type of a plane wave of finite amplitude or a plane waveguide layer). It was shown in some cases[4] that the steady-state one-dimensional distributions mentioned above are unstable against vanishingly small perturbations. Using the analogy with the theory of nonlinear stability of hydrodynamic flows[5,6], we can expect that under certain conditions, two-dimensional (or three-dimensional) steady-state distributions, which may turn out also to be stable, will exist near the unstable one-dimensional distributions of the electromagnetic field in a nonlinear medium. Moreover, regardless of whether the two-dimensional distributions of the field are stable or not, the answer to the question of the existence and characteristic features of the latter is, in our opinion, of undisputed interest. We recall that an analysis of hydrodynamic flows under conditions when the laminar flow becomes unstable and turbulent motion sets in, shows that in many cases the development of the linear instability for flows with a one-dimensional distribution of the velocity field leads to the occurrence of a two-dimensional (or three-dimensional) flow which is close to the initial one-dimensional flow. At small amplitudes of the perturbations, the resultant two-dimensional flow has in a number of cases a certain periodic structure. At small but finite amplitudes of the perturbations, the occurrence of a certain periodic structure in the distributions of the hydrodynamic quantities may be connected with the excitation of not only the fundamental but also of higher two-dimensional (or three-dimensional) modes. The fundamental mode, which predetermines at small amplitudes the character of the perturbed flow, is revealed in an analysis of the linear approximation. However, the influence of the nonlinearity leads not only to the excitation of higher spatial modes in the steady-state (or time-averaged) flows, but also to a difference between the average distribution of the hydrodynamic quantities across the flow, compared with the corresponding distributions in the unperturbed one-dimensional flow.

A complete analysis of analogous processes in nonlinear electrodynamics should be based on an investigation of two-dimensional (and three-dimensional) nonstationary distributions of the electromagnetic field in the nonlinear medium, which would make it possible to trace the evolution of the linear perturbations and to solve completely the problem of establishment of two-dimensional (or three-dimensional) field distributions. As the first step, we investigate below the question of the construction and characteristic features of two-dimensional steady-state field distributions in a nonlinear medium. We use here the following general scheme of constructing asymptotic expansions in terms of the small amplitude; these expansions represent steady-state two-dimensional field distributions in the nonlinear medium. We consider the linear approximation of one of the exact steady-state one-dimensional electromagnetic-field distributions in a nonlinear medium. Out of all the admissible solutions of the linear approximation, we single out those that lead to solutions that are bounded and are periodic in the "new" spatial variable. The latter is reached as a result of an analysis of the eigenvalue problem arising in the linear approximation and making it possible to separate the fundamental wave number and the fundamental two-dimensional mode, which at low amplitudes characterize the structure of the perturbed two-dimensional field distribution. During the course of the construction of the asymptotic expansion it becomes necessary to exclude the secular terms. In analogy with the theory of weakly-linear natural oscillations[7], the secular terms are excluded by using the assumption that the fundamental wave number depends on the perturbation amplitude. In the construction of the asymptotic solution, there arise not only higher two-dimensional modes and not only the aforementioned distortion of the field distribution averaged over the "new" variable compared with the initial one-dimensional distribution, but also a distortion of the fundamental two-dimensional mode compared with that arising in the linear approximation. Further, the process of eliminating the secular terms gives rise to an asymptotic expansion of the fundamental wave number in terms of the perturbation amplitude (more accurately, the total amplitude of the fundamental two-dimensional mode), which should be regarded as an asymptotic expansion of the relation that determines in implicit manner the wave number of the two-dimensional steady-state distribution.

The performed investigation shows that near certain one-dimensional steady-state distributions of the electromagnetic field in a nonlinear medium there exist two-dimensional steady-state field distributions that have, at small but finite amplitudes of the fundamental two-dimensional mode, a certain periodic structure. In their character, these field distributions are analogous to weakly-nonlinear single-frequency stationary eigenstates that occur under certain conditions in nonlinear oscillating systems with many degrees of freedom[7]. In spite of the fact that the analysis of the two-dimensional field distributions has been carried out in this case for a planar geometry, we can state that the fundamental conclusions concerning the character of the two-dimensional field distributions that are close to the exact one-dimensional distributions in the nonlinear medium, will be valid also in the case of a more complicated field geometry.

Writing the transverse electric field in the form

$$E(\mathbf{r}, t) = E_+(\mathbf{r}) \sin \omega t + E_-(\mathbf{r}) \cos \omega t \tag{1.1}$$

and using the usual notion wherein the nonlinear dielectric constant of the medium at the frequency ω does not lead to the occurrence of higher harmonics of the electromagnetic-field frequency[3], we can show that the system of equations for the functions $E_\pm(\mathbf{r})$, which determine the steady-state distribution of the electric field in the nonlinear medium, is[3,8]

$$\Delta E_\pm + [k_\omega^2 - \varkappa^2 + \varkappa^2 N(E_+^2 + E_-^2)]E_\pm = 0. \tag{1.2}$$

Here $k_\omega^2 \equiv \omega^2/c^2$, and

$$-\varkappa^2[1 - N(E^2)] \equiv k_\omega^2[\varepsilon(\omega; E^2) - 1], \tag{1.3}$$

where the nonlinear dielectric constant $\epsilon(\omega; E^2)$ is assumed to be real, corresponding to neglect of the dissipative processes in the medium. Substitution of $E_+ = E \cos \Psi$ and $E_- = E \sin \Psi$ leads to a system of equations that determines the amplitude E and the phase Ψ of the electric field:

$$\Delta E + [k_\omega^2 - (\operatorname{grad} \Psi)^2 - \varkappa^2 + \varkappa^2 N(E^2)]E = 0,$$
$$\operatorname{div}[E^2 \operatorname{grad} \Psi] = 0. \tag{1.4}$$

We consider below the case of a planar geometry when the functions E and Ψ depend only on two spatial variables—"transverse" x and "longitudinal" z. Moreover, we confine ourselves to the case when

$$N(E^2) = E^2 / E_c^2,$$

where E_c is a certain critical field. This form of the nonlinearity is sufficient for the understanding of a wide gamut of phenomena in nonlinear electrodynamics and optics. In addition, for a given form of nonlinearity there are known[1,3] explicit analytic expressions for the exact one-dimensional solutions of the system (1.4). We call attention to the fact that the system of nonlinear equations (1.4) admits of several types of exact solutions, namely solutions with zero field, solutions of the type of plane wave with finite constant amplitude, and finally exact one-dimensional solutions of the form $E \equiv E(x)$ and $\Psi \equiv -k_\infty z$, where k_∞ = const. We shall consider below two-dimensional steady-state solutions that are close to the exact solutions of all the aforementioned types.

2. To clarify the characteristic features of two-dimensional steady-state distributions of the electromagnetic field in a nonlinear medium and their asymptotic representations, we consider first the simpler case Ψ = const. The system (1.4) for the nonlinearity assumed by us degenerates into one equation for the amplitude of the electric field

$$\Delta E + (k_\omega^2 - \varkappa^2 + \varkappa^2 E^2 / E_c^2)E = 0. \tag{2.1}$$

Let $k_\omega^2 > \kappa^2$, corresponding to transparency of the medium in the linear approximation. The transformation $E = ae(x, z)$, a = const leads to the equation

$$\Delta e + (k_\omega^2 - \varkappa^2)e = -\varkappa^2(a / E_c)^2 e^3, \tag{2.2}$$

which degenerates when $a \to 0$ into the linear equation

$$\Delta e^{(0)} + (k_\omega^2 - \varkappa^2) e^{(0)} = 0. \tag{2.3}$$

One of the solutions of (2.3), bounded and periodic in each of the variables, is

$$e^{(0)} = \cos(k_\perp x) \cos(k_\parallel z). \tag{2.4}$$

Here $k_{\perp,\parallel}$ are the projections of the wave vector, such that

$$k_\perp^2 + k_\parallel^2 = k_\omega^2 - \varkappa^2 > 0. \tag{2.5}$$

When $(a/E_c)^2 \ll 1$ we can expect the weakly-linear two-dimensional solutions of (2.1), close to the solution of the linear approximation (2.4), to be represented in the form of an asymptotic expansion in integer powers of the power $(a/E_c)^2$. We note that the weakly nonlinear solutions of the equation for the field amplitude (2.1), close to the solution of the linear approximation, make it possible to understand many properties possessed also by other similar solutions.

We shall attempt to construct an asymptotic expansion of the form

$$e = e^{(0)} + \left(\frac{a}{E_c}\right)^2 e^{(1)} + \left(\frac{a}{E_c}\right)^4 e^{(2)} + \ldots, \tag{2.6}$$

in which all the $e^{(n)}$ are bounded and periodic in each of the variables. To eliminate the secular terms that arise in the perturbation-theory series, it is necessary to use the assumption that the wave vector of the sought solution depends on the parameter $(a/E_c)^2$. It is thus assumed that the sought weakly-nonlinear solution, which is close at $(a/E_c)^2 \ll 1$ to the solution of the linear problem (2.4), is periodic in each of the variables and can be characterized by a definite fundamental wave vector, the modulus of which depends on the parameter $(a/E_c)^2$. It will become obvious from the succeeding calculations that the projections of the fundamental wave vectors are subject to the condition

$$k_\perp^2 + k_\parallel^2 = k_\omega^2 - \varkappa^2 + f[k_\perp^2, k_\parallel^2, (a / E_c)^2], \tag{2.7}$$

which is the nonlinear analog of relation (2.5). To some degree, such a weakly-nonlinear solution of the field equation (2.2) can be regarded as the analog of the weakly-nonlinear single-frequency natural oscillations in systems with many degrees of freedom[7].

Introducing new independent variables, namely the phases $\varphi_\perp = k_\perp x$ and $\varphi_\| = k_\| z$, we rewrite (2.2) in the form

$$\left(k_\perp^2 \frac{\partial^2}{\partial\varphi_\perp^2} + k_\|^2 \frac{\partial^2}{\partial\varphi_\|^2} + k_\omega^2 - \varkappa^2\right) e = -\varkappa^2 \left(\frac{a}{E_c}\right)^2 e^3. \quad (2.8)$$

Here $k_{\perp,\|}$ should be regarded as a function of the parameter $(a/E_c)^2$, which should be determined during the course of constructing the asymptotic expansion (2.6). When $a \to 0$, we get from (2.8) the linear equation

$$\left(k_\perp^2 \frac{\partial^2}{\partial\varphi_\perp^2} + k_\|^2 \frac{\partial^2}{\partial\varphi_\|^2} + k_\omega^2 - \varkappa^2\right) e^{(0)} = 0, \quad (2.9)$$

which by virtue of relation (2.5) admits of the solution $e^{(0)} = \cos\varphi_\perp \cos\varphi_\|$. Representing $k^2_{\perp,\|}$ in the form of the expansions

$$k^2_{\perp,\|}(a) = k^2_{\perp,\|} + \omega^{(1)}_{\perp,\|}(a/E_c)^2 + \omega^{(2)}_{\perp,\|}(a/E_c)^4 + \ldots \quad (2.10)$$

and using (2.6), we find that the nonlinear equation (2.8) leads to the following sequence of linear inhomogeneous equations:

$$\hat{L}e^{(1)} = -\varkappa^2 (e^{(0)})^3 - \hat{l}^{(1)} e^{(0)},$$

$$\hat{L}e^{(2)} = -3\varkappa^2 (e^{(0)})^2 e^{(1)} - \hat{l}^{(1)} e^{(1)} - \hat{l}^{(2)} e^{(0)}, \quad (2.11)$$

$$\hat{L} \equiv k_\perp^2 \frac{\partial^2}{\partial\varphi_\perp^2} + k_\|^2 \frac{\partial^2}{\partial\varphi_\|^2} + k_\omega^2 - \varkappa^2,$$

$$\hat{l}^{(n)} \equiv \omega_\perp^{(n)} \frac{\partial^2}{\partial\varphi_\perp^2} + \omega_\|^{(n)} \frac{\partial^2}{\partial\varphi_\|^2}. \quad (2.12)$$

Using the solutions of (2.11) and (2.12), we find that the first terms of the sought asymptotic expression (2.6) can be represented in the form

$$e = \cos\varphi_\perp \cos\varphi_\| + \frac{3}{128}\left(\frac{a}{E_c}\right)^2 \frac{\varkappa^2}{k_\perp^2}\left[1 + \left(\frac{a}{E_c}\right)^2 \frac{\varkappa^4 \alpha_\perp}{k_\perp^2 k_\|^2}\right] \cos 3\varphi_\perp \cos\varphi_\|$$
$$+ \frac{3}{128}\left(\frac{a}{E_c}\right)^2 \frac{\varkappa^2}{k_\|^2}\left[1 + \left(\frac{a}{E_c}\right)^2 \frac{\varkappa^4 \alpha_\|}{k_\perp^2 k_\|^2}\right] \cos\varphi_\perp \cos 3\varphi_\|$$
$$+ \frac{1}{128}\left(\frac{a}{E_c}\right)^2 \frac{\varkappa^2}{k_\omega^2 - \varkappa^2}\left[1 + \left(\frac{a}{E_c}\right)^2 \frac{\varkappa^4 \alpha}{k_\perp^2 k_\|^2}\right] \cos 3\varphi_\perp \cos 3\varphi_\|$$
$$+ \frac{(a/E_c)^4}{16324}\left[\frac{\varkappa^4}{k_\perp^4}\beta_\perp \cos 5\varphi_\perp \cos\varphi_\| + \frac{\varkappa^4}{k_\|^4}\beta_\| \cos\varphi_\perp \cos 5\varphi_\|\right.$$
$$+ \frac{\varkappa^4 \gamma_\perp}{k_\perp^2 (k_\omega^2 - \varkappa^2)} \cos 5\varphi_\perp \cos 3\varphi_\| + \frac{\varkappa^4 \gamma_\|}{k_\|^2 (k_\omega^2 - \varkappa^2)} \cos 3\varphi_\perp \cos 5\varphi_\|$$
$$\left. + \frac{\varkappa^4}{(k_\omega^2 - \varkappa^2)^2} \cos 5\varphi_\perp \cos 5\varphi_\|\right] \ldots \quad (2.13)$$

Here the functions α, $\alpha_{\perp,\|}$, $\beta_{\perp,\|}$, and $\gamma_{\perp,\|}$, do not have any singularities at all the admissible values of $k_{\perp,\|}$. We note that the functions $\alpha_{\perp,\|}$, unlike α, $\beta_{\perp,\|}$ and $\gamma_{\perp,\|}$ depend not only on $k_{\perp,\|}$, but also on the quantities $\omega^{(1)}_{\perp,\|}$. In the calculation of the first terms of (2.13), we eliminated the secular terms. This results in

$$\omega_\perp^{(1)} + \omega_\|^{(1)} = 9\varkappa^2/16,$$

$$\omega_\perp^{(2)} + \omega_\|^{(2)} = \frac{3\varkappa^4}{2048}\left(\frac{1}{k_\omega^2 - \varkappa^2} + \frac{9}{k_\perp^2} + \frac{9}{k_\|^2}\right). \quad (2.14)$$

Thus, the process of eliminating the secular terms resulting from the fact that the right-hand sides of the sequence of linear equations (2.11) and (2.12) and their like contain a series of terms proportional to the fundamental two-dimensional mode $\cos\varphi_\perp \cos\varphi_\|$, leads to a determination of a sum $\omega_\perp^{(n)} + \omega_\|^{(n)}$ in each in the parameter $(a/E_c)^2 \ll 1$. Consequently, the process of constructing the asymptotic expansion for a weakly nonlinear two-dimensional distribution of the field leads not only to a determination of the amplitude functions for all the higher two-dimensional modes, corresponding to multiple values of the projections of the fundamental wave vector, but also to an asymptotic expansion for the square of the modulus of the fundamental wave vector

$$k_\perp^2(a) + k_\|^2(a) = k_\omega^2 - \varkappa^2 + {}^9/_{16}\varkappa^2 (a/E_c)^2$$
$$+ \frac{3\varkappa^4}{2048}\left(\frac{a}{E_c}\right)^4 \left(\frac{1}{k_\omega^2 - \varkappa^2} + \frac{9}{k_\perp^2} + \frac{9}{k_\|^2}\right) \quad (2.15)$$

The latter expression should be regarded as an asymptotic expansion of (2.7), which determines in implicit form the connection between the projections of the fundamental wave vector of the weakly-nonlinear two-dimensional solution. By solving Eq. (2.15) with respect to one of the projections of the wave vector for a given value of the other projection, we can verify that the approximation corresponding to retaining the first three terms in the asymptotic expansion (2.15) leads to a one-to-one connection between the projections of the fundamental wave vector. We note that during the course of constructing the asymptotic expansion, the parameter a/E_c acquires the meaning of the total amplitude of the fundamental two-dimensional mode. Indeed, the elimination of the secular terms is carried out in such a way, that small divisors arise in the asymptotic expression (2.13) in any order in the parameter of the expansion $(a/E_c)^2 \ll 1$ the amplitude of the fundamental two-dimensional mode remains unchanged and equal to a/E_c. As $k_\perp \to 0$ (or $k_\| \to 0$). The parameter a/E_c then loses the meaning of the total amplitude of the fundamental mode, since not only the two-dimensional mode, but also all the higher two-dimensional modes of the form $\cos n\varphi_\perp \cos\varphi_\|$, or respectively $\cos\varphi_\perp \cos n\varphi_\|$, degenerate into one of the fundamental one-dimensional modes ($\cos\varphi_\|$ or $\cos\varphi_\perp$). Consequently, when $k \to 0$ (or $k \to 0$), the method of eliminating the secular terms must also be changed.

Thus, at a specified frequency ω and at $k_\omega^2 > \kappa^2$, corresponding to the transparency of the medium in the linear approximation, it becomes possible for steady-state weakly-nonlinear two-dimensional field distributions, characterized by a definite value of the wave vector, to exist in the nonlinear medium; the modulus of this wave vector depends on the amplitude of the fundamental two-dimensional mode. The asymptotic series representing such a field distribution contains higher two-dimensional modes $\cos n\varphi_\perp \cos m\varphi_\|$ (where n and m are odd numbers), which correspond to spatial oscillations of the field with wave-vector projections that are multiples of the projections of the fundamental wave vector. At small but finite amplitude of the fundamental two-dimensional mode, a weakly-nonlinear distribution of the field retains certain attributes characteristic of the linearized problems. Namely, the weakly-nonlinear two-dimensional solution is periodic in each of the spatial variables, and the projections of the fundamental wave vector are connected by relations that are the analog of relation (2.5). We note that averaging the field distribution over the longitudinal and transverse oscillations leads to a zero average field value.

3. We proceed to construct two-dimensional solutions that are close to the exact one-dimensional periodic solution of the nonlinear equation (2.2). Let

$k_\perp$ be the transverse wave number characterizing the one-dimensional periodic solution, and then (2.2) leads us to the ordinary differential equation

$$\left(k_\perp^2 \frac{d^2}{d\varphi_\perp^2} + k_\omega^2 - \varkappa^2\right) E_\perp = -\varkappa^2 \left(\frac{E_\perp}{E_c}\right)^2 E_\perp. \quad (3.1)$$

The latter, as is well known[7], admits of the solution

$$E_\perp = \frac{a}{E_c} e_\perp(\varphi_\perp) = \frac{a}{E_c} \operatorname{cn}(\tilde{\varphi}_\perp), \quad (3.2)$$

where cn(z) is the Jacobi elliptic cosine, and the dependence of the transverse wave number on the amplitude a is given by

$$\frac{k_\perp}{\sqrt{k_\omega^2 - \varkappa^2}} = \frac{\pi}{2K}\left[1 + \frac{1}{2}\frac{\varkappa^2}{k_\omega^2 - \varkappa^2}\left(\frac{a}{E_c}\right)^2\right]^{1/2}, \quad (3.3)$$

where K is the complete elliptic integral of the first kind and $\tilde{\varphi}_\perp = (2/\pi)K\varphi_\perp$. When $a/E_c \to 0$, relations (3.2) and (3.3) lead to $e_\perp \to \cos\varphi_\perp$ and $k_\perp \to \sqrt{k^2 - \kappa^2}$. When $(a/E_c)^2 \ll 1$, the solution of (3.1) close to the linearized problem can be represented in the form

$$E_\perp = \frac{a}{E_c}\left[\cos\varphi_\perp + \frac{1}{32}\frac{\varkappa^2}{k_\omega^2 - \varkappa^2}\left(\frac{a}{E_c}\right)^2 \cos 3\varphi_\perp \quad \ldots\right],$$

$$k_\perp^2 = k_\omega^2 - \varkappa^2 + \frac{3}{4}\varkappa^2\left(\frac{a}{E_c}\right)^2 + \frac{3}{128}\frac{\varkappa^4}{k_\omega^2 - \varkappa^2}\left(\frac{a}{E_c}\right)^4 \quad \ldots \quad (3.4)$$

Putting in (2.2)

$$e = \operatorname{cn}\tilde{\varphi}_\perp + \alpha h(\varphi_\perp, \varphi_\parallel), \quad (3.5)$$

where α is a constant and $\operatorname{cn}\tilde{\varphi}_\perp$ is the exact one-dimensional periodic solution of (2.2), we arrive at the following equation for the function h:

$$\left[k_\perp^2 \frac{\partial^2}{\partial\varphi_\perp^2} + k_\parallel^2 \frac{\partial^2}{\partial\varphi_\parallel^2} + k_\omega^2 - \varkappa^2 + 3\varkappa^2\left(\frac{a}{E_c}\right)^2 \operatorname{cn}^2\tilde{\varphi}_\perp\right] h$$
$$= -3\alpha\varkappa^2\left(\frac{a}{E_c}\right)^2 \operatorname{cn}\tilde{\varphi}_\perp h^2 - \alpha^2\varkappa^2\left(\frac{a}{E_c}\right)^2 h^3. \quad (3.6)$$

In writing down the last equation we took into account the fact that when $\alpha \to 0$ the linear approximation admits of a solution that is periodic in the longitudinal variable. Consequently when $\alpha \ll 1$ we can expect the two-dimensional solutions to have a periodic structure in the longitudinal direction, characterized by a fundamental longitudinal wave number $k_\parallel$. We call attention to the fact that the transverse wave number $k_\perp$ depends only on the parameter $(a/E_c)^2$, which in the general case is not assumed to be small, whereas the sought fundamental longitudinal wave number depends not only on $(a/E_c)^2$ but also on the small parameter α. When $\alpha \ll 1$, the solution of (2.2) that is close to the exact one-dimensional solution (3.2) is represented in the form of the expansion

$$e(\tilde{\varphi}_\perp, \varphi_\parallel) = \operatorname{cn}\tilde{\varphi}_\perp + \alpha[h^{(0)} + \alpha h^{(1)} + \alpha^2 h^{(2)} + \ldots]. \quad (3.7)$$

It is natural to assume here that $k_\parallel^2$ admits of an expansion in the form

$$k_\parallel^2(\alpha) = k_\parallel^2 + \alpha\omega_\parallel^{(1)} + \alpha^2\omega_\parallel^{(2)} \quad \ldots \quad (3.8)$$

The linear approximation leads to the equation

$$\left[k_\perp^2 \frac{\partial^2}{\partial\varphi_\perp^2} + k_\parallel^2 \frac{\partial^2}{\partial\varphi_\parallel^2} + k_\omega^2 - \varkappa^2 + 3\varkappa^2\left(\frac{a}{E_c}\right)^2 \operatorname{cn}^2\tilde{\varphi}_\perp\right] h^{(0)} = 0, \quad (3.9)$$

which admits of a solution $h^{(0)} = \Psi(\varphi_\perp)\cos\varphi_\parallel$, which is longitudinal in the periodic variable. Longitudinal wave number $k_\parallel(0)$ is determined in the linear approximation by the solution of the eigenvalue problem

$$\hat{L}_\perp \Psi(\varphi_\perp) = k_\parallel^2 \Psi(\varphi_\perp) \quad (3.10)$$

for the self-adjoint periodic operator

$$\hat{L}_\perp \equiv k_\perp^2 \frac{d^2}{d\varphi_\perp^2} + k_\omega^2 - \varkappa^2 + 3\varkappa^2\left(\frac{a}{E_c}\right)^2 \operatorname{cn}^2\tilde{\varphi}_\perp. \quad (3.11)$$

Consequently, in the linear approximation, the square of the longitudinal wave number should be equal to one of the non-negative eigenvalues of the operator. If the amplitude of the exact one-dimensional distribution is small, then the periodic operator (3.11) is close to the Mathieu operator $l_\perp$

$$\hat{L}_\perp \sim (k_\omega^2 - \varkappa^2)\hat{l}_\perp = (k_\omega^2 - \varkappa^2)\left[\frac{d^2}{d\varphi_\perp^2} + 1 + \frac{3\varkappa^2}{k_\omega^2 - \varkappa^2}\left(\frac{a}{E_c}\right)^2 \cos^2\varphi_\perp\right], \quad (3.12)$$

whose eigenfunction and eigenvalues were investigated in sufficient detail[9]. When $(a/E_c)^2 \ll 1$, it is easy to verify that longitudinal field oscillations result only from three eigenfunctions of the Mathieu operator, namely

$$\mathrm{Ce}_0\left[\varphi_\perp, \left(\frac{a}{E_c}\right)^2\right], \ \mathrm{Ce}_1\left[\varphi_\perp, \left(\frac{a}{E_c}\right)^2\right], \ \mathrm{Se}_1\left[\varphi_\perp, \left(\frac{a}{E_c}\right)^2\right], \quad (3.13)$$

which degenerates as $(a/E_c)^2 \to 0$ into the functions 1, $\cos\varphi$, and $\sin\varphi$. The eigenfunctions (3.13) correspond to the following eigenvalues:

$$k_\parallel^2[\mathrm{Ce}_0] \sim k_\omega^2 - \varkappa^2 \quad k_\parallel^2[\mathrm{Ce}_1, \ \mathrm{Se}_1] \sim O\left[\left(\frac{a}{E_c}\right)^2\right]. \quad (3.14)$$

Sufficiently complete asymptotic expansions for the latter are shown in[9]. We note that when $(a/E_c)^2 \ll 1$ and $\alpha \ll 1$ there arises, essentially, a two-parameter asymptotic expansion of the two-dimensional weakly-nonlinear field distribution, and in this case the parameter α determines the relation between the one-dimensional and two-dimensional fundamental modes. A similar situation arises also in the theory of nonlinear hydrodynamic stability[5].

Using the symbol $k_\parallel(n)$ for the longitudinal wave number corresponding to the eigenvalue Λ_n and the eigenfunction Ψ_n of the operator $\hat{L}_\perp$, and considering the next higher approximation in the parameter $\alpha \ll 1$, we arrive at the equation

$$\left[\hat{L}_\perp + k_\parallel^2(n)\frac{\partial^2}{\partial\varphi_\parallel^2}\right] h^{(1)} = \omega_\parallel^{(1)} \Psi_n \cos\varphi_\parallel$$
$$- \frac{3}{2}\varkappa^2\left(\frac{a}{E_c}\right)^2 \Psi_n^2 \operatorname{cn}\tilde{\varphi}_\perp - \frac{3}{2}\varkappa^2\left(\frac{a}{E_c}\right)^2 \Psi_n^2 \operatorname{cn}\tilde{\varphi}_\perp \cos 2\varphi_\parallel. \quad (3.15)$$

Putting in the latter

$$h^{(1)} = h_0^{(1)} + h_1^{(1)}\cos\varphi_\parallel + h_2^{(1)}\cos 2\varphi_\parallel,$$

we obtain the following system of equations

$$[\hat{L}_\perp - k_\parallel^2(n)]h_1^{(1)} = \omega_\parallel^{(1)}\Psi_n, \quad (3.16)$$

$$\hat{L}_\perp h_0^{(1)} = -{}^3/_2\varkappa^2(a/E_c)^2\Psi_n^2\operatorname{cn}\tilde{\varphi}_\perp, \quad (3.17)$$

$$[\hat{L}_\perp - 4k_\parallel^2(n)]h_2^{(1)} = -{}^3/_2\varkappa^2(a/E_c)^2\Psi_n^2\operatorname{cn}\tilde{\varphi}_\perp. \quad (3.18)$$

Since $k_\parallel^2(n)$ is one of the eigenvalues of the operator $\hat{L}_\perp$, the condition under which the inhomogeneous equation (3.16) has a solution consists of orthogonality of the right-hand side of the equation to the eigenfunction Ψ_n, and leads to the relation

$$\omega_\parallel^{(1)} \int d\varphi_\perp \Psi_n^2 = 0.$$

Consequently, $\omega_{\parallel}^{(1)} = 0$, and we can assume $h_1^{(1)} = 0$.

Let us turn to Eq. (3.17). Since the eigenfunction of the operator $\hat{L}_\perp$ corresponding to the zero eigenvalue is the derivative of the exact solution of the one-dimensional problem $d\,\mathrm{cn}\,\tilde{\varphi}_\perp/d\varphi_\perp$, the solvability condition of the inhomogeneous equation (3.17) can be written in the form

$$\int d\varphi_\perp \,\mathrm{cn}\,\tilde{\varphi}_\perp \frac{d\,\mathrm{cn}\,\tilde{\varphi}_\perp}{d\varphi_\perp}\Psi_n^2 = 0. \qquad (3.19)$$

The latter relation is satisfied by virtue of the fact that the functions $\mathrm{cn}\,\tilde{\varphi}_\perp$ and $d\,\mathrm{cn}\,\tilde{\varphi}_\perp/d\varphi_\perp$ have different parities. Finally, the inhomogeneous equation (3.18) has a solution if for all $n' \neq n$ and for a given value of the parameter a/E_c none of the eigenvalues of the operator $\hat{L}_\perp$ coincides with the square of double the wave number, $\Lambda_{n'} \neq 4k_{\parallel}^2(n)$.

Thus, the first terms of the asymptotic expansion for the two-dimensional solution close to the exact one-dimensional periodic solution can be written in the form

$$e = \mathrm{cn}\,\tilde{\varphi}_\perp + \alpha\Big\{\Psi_n(\varphi_\perp)\cos\varphi_\parallel - \frac{3}{2}\alpha\left(\frac{\varkappa a}{E_c}\right)^2 \hat{L}_\perp^{-1}\Psi_n^2(\varphi_\perp)\,\mathrm{cn}\,\tilde{\varphi}_\perp$$
$$-\frac{3}{2}\alpha\left(\frac{\varkappa a}{E_c}\right)^2[\hat{L}_\perp - 4k_\parallel^2(n)]^{-1}\Psi_n^2(\varphi_\perp)\,\mathrm{cn}\,\tilde{\varphi}_\perp\cos 2\varphi_\parallel + \ldots\Big\}. \qquad (3.20)$$

Let us consider the next approximation in terms of the parameter α. For the function $h^{(2)}$ we get

$$\left[\hat{L}_\perp + k_\parallel^2(n)\frac{\partial^2}{\partial\varphi_\parallel^2}\right]h^{(2)} = \Big\{\omega_\parallel^{(2)}\Psi_n - \frac{3}{4}\left(\frac{\varkappa a}{E_c}\right)^2\Psi_n^3$$
$$+9\left(\frac{\varkappa a}{E_c}\right)^4 \mathrm{cn}\,\tilde{\varphi}_\perp\Psi_n\hat{L}_\perp^{-1}\Psi_n^2\,\mathrm{cn}\,\tilde{\varphi}_\perp$$
$$+\frac{1}{2}9\left(\frac{\varkappa a}{E_c}\right)^4 \mathrm{cn}\,\tilde{\varphi}_\perp\Psi_n[\hat{L}_\perp - 4k_\parallel^2(n)]^{-1}\Psi_n^2\,\mathrm{cn}\,\tilde{\varphi}_\perp\Big\}\cos\varphi_\parallel$$
$$+\Big\{-\frac{1}{4}\left(\frac{\varkappa a}{E_c}\right)^2\Psi_n^3 + \frac{1}{2}9\left(\frac{\varkappa a}{E_c}\right)^2\Psi_n\,\mathrm{cn}\,\tilde{\varphi}_\perp$$
$$\times[\hat{L}_\perp - 4k_\parallel^2(n)]^{-1}\Psi_n^2\,\mathrm{cn}\,\tilde{\varphi}_\perp\Big\}\cos 3\varphi_\parallel. \qquad (3.21)$$

The substitution

$$h^{(2)} = h_1^{(2)}\cos\varphi_\parallel + h_3^{(2)}\cos 3\varphi_\parallel$$

leads to the following equation for the function $h_1^{(2)}$:

$$[\hat{L}_\perp - k_\parallel^2(n)]h_1^{(2)} = \Psi_n\Big\{\omega_\parallel^{(2)} - \frac{3}{4}\left(\frac{\varkappa a}{E_c}\right)^2\Psi_n^2$$
$$-9\left(\frac{\varkappa a}{E_c}\right)^4\mathrm{cn}\,\tilde{\varphi}_\perp\hat{L}_\perp^{-1}\Psi_n^2\,\mathrm{cn}\,\tilde{\varphi}_\perp$$
$$+\frac{1}{2}9\left(\frac{\varkappa a}{E_c}\right)^4\mathrm{cn}\,\tilde{\varphi}_\perp[\hat{L}_\perp - 4k_\parallel^2(n)]^{-1}\Psi_n^2\,\mathrm{cn}\,\tilde{\varphi}_\perp\Big\}. \qquad (3.22)$$

It is obvious that the condition for the solvability of the obtained equation determines the quantity $\omega_\parallel^{(2)} \neq 0$ and makes it possible to write down the first terms of the asymptotic expansion of the square of the fundamental wave number for $\alpha \ll 1$ in the form

$$k_\parallel^2 = k_\parallel^2(n) + 3\left(\frac{\alpha a}{E_c}\right)^2\Big\{\frac{1}{4}\int d\varphi\,\Psi_n^4$$
$$+3\left(\frac{\varkappa a}{F}\right)^2\int d\varphi\,\Psi_n^2\,\mathrm{cn}\,\tilde{\varphi}\hat{L}_\perp^{-1}\Psi_n^2\,\mathrm{cn}\,\tilde{\varphi}$$
$$-\frac{3}{2}\left(\frac{\varkappa a}{E_c}\right)^2\int d\varphi\,\Psi_n^2\,\mathrm{cn}\,\tilde{\varphi}[\hat{L}_\perp - 4k_\parallel^2(n)]^{-1}\Psi_n^2\,\mathrm{cn}\,\tilde{\varphi}\Big\} \qquad (3.23)$$

The obtained expansion must again be regarded as an expansion of the relation that determines in implicit fashion the dependence of the fundamental longitudinal wave number on the parameter α. Here, however, $\omega \neq 0$, since the nondiagonal matrix element of the right-hand side of (3.22) differ from zero. Consequently, the elimination of the secular terms in this case leads not only to a dependence of the longitudinal wave number on the amplitude of the fundamental two-dimensional mode, but also to a difference between the transverse distribution in the fundamental mode and the distribution $\Psi_n(\varphi_\perp)$ which arises in the linear approximation. Indeed, the fact that the nondiagonal matrix element of the right-hand side of (3.22) do not equal to zero leads to a distortion of the fundamental two-dimensional mode

$$\Psi_n(\varphi_\perp)\cos\varphi_\parallel \to [\Psi_n(\varphi_\perp) + \delta\Psi_n(\varphi_\perp)]\cos\varphi_\parallel, \qquad (3.24)$$

where

$$\delta\Psi_n = [\hat{L}_\perp - k_\parallel^2(n)]^{-1}\Big\{k_\parallel^2 - k_\parallel^2(n) - \frac{3}{4}\left(\frac{\varkappa\alpha a}{E_c}\right)^2\Psi_n^2$$
$$-9\alpha^2\left(\frac{\varkappa a}{E_c}\right)^4\mathrm{cn}\,\tilde{\varphi}_\perp\hat{L}_\perp^{-1}\Psi_n^2\,\mathrm{cn}\,\tilde{\varphi}_\perp$$
$$+\frac{9}{2}\alpha^2\left(\frac{\varkappa a}{E_c}\right)^4\mathrm{cn}\,\tilde{\varphi}_\perp[\hat{L}_\perp - 4k_\parallel^2(n)]^{-1}\Psi_n^2\,\mathrm{cn}\,\tilde{\varphi}_\perp\Big\}\Psi_n \qquad (3.25)$$

It is obvious that the change of the transverse distribution in the fundamental two-dimensional mode, noted above, can be connected with the change of the "potential" part of the operator $\hat{L}_\perp$, namely with the transition from the "potential" function $3(\kappa a/E_c)^2\mathrm{cn}^2\varphi_\perp$ and a certain perturbed potential function equal to $3(\kappa a/E_c)^2\mathrm{cn}^2\tilde{\varphi}_\perp + \delta U(\varphi_\perp, a^2)$. In the lowest order in $\alpha \ll 1$, the diagonal matrix element of the perturbations of the potential $\delta U(\varphi_\perp \alpha^2)$ coincides in essence with the quantity $\omega_\parallel^{(2)}$. The condition that determines completely the perturbed potential in the construction of the asymptotic expansion is a requirement that the transverse distribution in the fundamental two-dimensional mode remain unchanged in any order in the parameter α. We note that in the construction of asymptotic expansions for two-dimensional solutions close to the exact one-dimensional solution, besides the terms corresponding to the longitudinal oscillations with wave numbers that are multiples of the fundamental longitudinal wave number, terms arise that are independent of the longitudinal variables. Consequently, the averaging of the two-dimensional distribution of the field over the longitudinal oscillations leads to a one-dimensional distribution of the field different from the distribution of the field for the initial one-dimensional solution

$$\langle e\rangle_\parallel = \mathrm{cn}\,\tilde{\varphi}_\parallel - \frac{3}{2}\left(\frac{\alpha a}{E_c}\right)^2\varkappa^2\hat{L}_\perp^{-1}\Psi_n^2(\varphi_\perp)\,\mathrm{cn}\,\tilde{\varphi}_\perp. \qquad (3.26)$$

4. Let us assume that $k_\omega^2 < \kappa^2$. When the latter inequality is satisfied, the medium is opaque in the linear approximation. The Eq. (2.1) admits of an exact solution corresponding to a field amplitude $E = \sqrt{1-(k_\omega/\kappa)^2}E_c$ which is constant in all of space. The substitution

$$E = \sqrt{1-(k_\omega/\varkappa)^2}E_c[1 + ae(x,z)] \qquad (4.1)$$

leads to the following equation:

$$\Delta e - 2(k_\omega^2 - \varkappa^2)e = -(\varkappa^2 - k_\omega^2)(3ae^2 + a^2e^3). \qquad (4.2)$$

In the linear approximation, the latter admits of the solution

$$e^{(0)} = \cos(k_\perp x)\cos(k_\parallel z),$$
$$k_\perp^2 + k_\parallel^2 = 2(\varkappa^2 - k_\omega^2) > 0. \quad (4.3)$$

Transforming to dimensionless spatial variables

$$\frac{1}{\sqrt{2}}\xi = \sqrt{\varkappa^2 - k_\omega^2}\, x, \quad \frac{1}{\sqrt{2}}\zeta = \sqrt{\varkappa^2 - k_\omega^2}\, z$$

and accordingly to dimensionless wave numbers

$$\sqrt{2}\,\chi_{\perp,\parallel} = k_{\perp,\parallel}/\sqrt{\varkappa^2 - k_\omega^2}$$

and introducing the phase variables $\varphi_\perp = \chi_\perp\xi$ and $\varphi_\parallel = \chi_\parallel\zeta$, we rewrite (4.2) in the form

$$\left(\chi_\perp^2\frac{\partial^2}{\partial\varphi_\perp^2} + \chi_\parallel^2\frac{\partial^2}{\partial\varphi_\parallel^2} + 1\right)e = -\frac{3}{2}ae^2 - \frac{1}{2}a^2e^3. \quad (4.4)$$

Here $\chi_{\perp,\parallel}$ is the projection of the sought fundamental wave vector, characterizing the periodic structure of the two-dimensional distribution of the field that is close to the exact solution with constant amplitude, and should be regarded as functions of the parameter $a \ll 1$. Simple calculations show that the first terms of the asymptotic expansion of such a two-dimensional solution are

$$e = \cos\varphi_\perp\cos\varphi_\parallel - \frac{3a}{8}\left[1 + \frac{\cos 2\varphi_\perp}{1-4\chi_\perp^2} + \frac{\cos 2\varphi_\parallel}{1-4\chi_\parallel^2} - \frac{1}{3}\cos 2\varphi_\perp\cos 2\varphi_\parallel\right]$$
$$+ \frac{a^2}{64}\cos 3\varphi_\perp\cos 3\varphi_\parallel + \frac{3}{128}\left[\frac{a}{\chi_\perp}\right]^2\left[1 - \frac{3/2}{1-4\chi_\perp^2}\right]\cos 3\varphi_\perp\cos\varphi_\parallel$$
$$+ \frac{3}{128}\left[\frac{a}{\chi_\parallel}\right]^2\left[1 - \frac{3/2}{1-4\chi_\parallel^2}\right]\cos\varphi_\perp\cos 3\varphi_\parallel. \quad (4.5)$$

Eliminating the secular terms of the expansion by using

$$\chi^2_{\perp,\parallel}(a) = \chi^2_{\perp,\parallel} + a\omega^{(1)}_{\perp,\parallel} + a^2\omega^{(2)}_{\perp,\parallel} + \ldots,$$

where $\chi_\perp^2 + \chi_\parallel^2 = 1$, we get

$$\omega_\perp^{(1)} + \omega_\parallel^{(1)} = 0, \quad \omega_\perp^{(2)} + \omega_\parallel^{(2)} = -\frac{3}{4} - \frac{9}{16}\left[\frac{1}{1-4\chi_\perp^2} + \frac{1}{1-4\chi_\parallel^2}\right]. \quad (4.6)$$

The obtained relations show that the asymptotic expansion of the equation that establishes the connection between the projections of the fundamental wave vector is of the form

$$\chi_\perp^2 + \chi_\parallel^2 = 1 - \frac{3}{4}a^2 - \frac{9}{16}a^2\left[\frac{1}{1-4\chi_\perp^2} + \frac{1}{1-4\chi_\parallel^2}\right]. \quad (4.7)$$

We call attention to the fact that in this case, when the medium is opaque in the linear approximation, the small divisors in the first terms of the asymptotic expansion of the two-dimensional solution occur both when $\chi_{\perp,\parallel} = 0$ and when $\chi_{\perp,\parallel} = 1/4$. It can be shown that in the higher approximations, the small divisors arise only at the following values of $\chi_\perp^2$ or $\chi_\parallel^2$: $1/4$, $1/9$, $1/16 \ldots 0$; $3/4$, $8/9$, $15/16 \ldots 1$. We note, however, that the region of values of $1/4 < \chi^2_{\perp,\parallel} < 3/4$ remains free of small divisors. Consequently, it can be assumed that when $a \ll 1$ the asymptotic expansions (4.5) and (4.7) are two-dimensional field distributions close to the exact solution with a constant field amplitude. Unlike the previously considered case of weakly-nonlinear two-dimensional solutions close to the exact solution with zero field, when the averaging of the two-dimensional distribution over the spatial oscillations has led to a zero average field, in this case the corresponding mean values for the two-dimensional distributions of the field turn out to differ from the value $\sqrt{1-(k_\omega/\kappa)^2}E_c$, since $\langle e\rangle_{\parallel,\perp} = -3a/8$.

5. It is known[3,8] that Eq. (2.1) when $k_\omega^2 < \kappa^2$ has an exact one-dimensional solution

$$E(x) = \sqrt{2}E_c\sqrt{1-(k_\omega/\varkappa)^2}/\mathrm{ch}\sqrt{\varkappa^2 - k_\omega^2}\,x, \quad (5.1)$$

which corresponds to the occurrence in the medium of a field distribution localized with respect to one of the spatial variables. Putting in (2.1)

$$E = \sqrt{1-(k_\omega/\varkappa)^2}E_c[\sqrt{2}/\mathrm{ch}\,\xi + ae(\xi,\zeta)],$$

we obtain for the function $e(\xi, \zeta)$ an equation in the form

$$(\Delta + 6/\mathrm{ch}^2\xi - 1)e = -\frac{3\sqrt{2}a}{\mathrm{ch}\,\xi}e^2 - a^2e^3. \quad (5.2)$$

Let us consider the linear approximation to the exact one-dimensional solution (5.1), which admits of a solution in the form $e^{(0)} = \Psi(\xi)\cos(\chi_\parallel\zeta)$ and leads to the eigenvalue problem

$$\hat{L}_\perp\Psi = \chi_\parallel^2\Psi, \quad \hat{L}_\perp \equiv \frac{d^2}{d\xi^2} + \frac{6}{\mathrm{ch}^2\xi} - 1. \quad (5.3)$$

when $\Lambda > -1$, the problem $\hat{L}_\perp\Psi_\Lambda = \Lambda\Psi_\Lambda$ leads to discrete eigenvalues $\Lambda_n = (2-n)^2 - 1$ $(n = 0, 1, 2)$, whereas when $\Lambda < -1$ the eigenvalue spectrum is continuous. It is obvious that solutions that are periodic in the longitudinal variables is obtained only from the ground state of the operator $\hat{L}_\perp$, for which $n = 0$ and $\Lambda_0 = 3$. Thus, in the linear approximation, $e^{(0)} = \sqrt{3}\zeta/\cosh^2\xi$. Constructing the two-dimensional solution periodic in the longitudinal variable and close to the exact one-dimensional solution (5.1) by using the asymptotic expansions

$$e = \cos\varphi_\parallel/\mathrm{ch}^2\xi + ae^{(1)} + a^2e^{(2)} + \ldots, \quad \chi_\parallel^2 = 3 + a\omega_\parallel^{(1)} + a^2\omega_\parallel^{(2)} + \ldots, \quad (5.4)$$

we arrive after a number of simple transformations to a system of linear inhomogeneous equations with respect to the functions $e_i^{(1)}(\xi)$, where $i = 0, 1, 2$, having the same form as the system of equations (3.16), (3.17), and (3.18).

It is easy to verify that all the general conclusions concerning the character of the asymptotic expansions, the solvability conditions, and the change of the transverse distribution of the field in the fundamental two-dimensional mode are valid also in this case. In particular, $\omega_\parallel^{(1)} = 0$ and the analog of relation (3.19) is

$$\int_{-\infty}^{+\infty} d\xi\left(\frac{\mathrm{sh}\,\xi}{\mathrm{ch}^2\xi}\right)\frac{1}{\mathrm{ch}\,\xi}\left(\frac{1}{\mathrm{ch}^2\xi}\right)^2 = 0,$$

where $\sinh\xi/\cosh^2\xi$ is the eigenfunction of the operator $\hat{L}_\perp$ corresponding to the zero eigenvalue. Finally, all the eigenvalues of the operator $\hat{L}_\perp$ satisfy the condition $\Lambda_n \neq 4k_\parallel^2 = 12$, a condition essential for the possibility of solving the inhomogeneous equation with respect to the function $e^{(1)}(\xi)$. In the case under consideration, an expression was obtained in explicit form for the function $e_0^{(1)}(\xi)$, which determines in the first approximation the difference between the two-dimensional field distribution averaged over the longitudinal oscillations and the distribution characteristic of the exact one-dimensional solution. It can be shown that the first terms of the asymptotic expansion of the sought two-dimensional solutions are given by

$$e = \frac{\cos\varphi_\parallel}{\mathrm{ch}^2\xi} - \frac{a}{\sqrt{2}}\left(\frac{1}{\mathrm{ch}\,\xi} - \frac{1}{2\,\mathrm{ch}^3\xi}\right) + ae_2^{(1)}(\xi)\cos 2\varphi_\parallel + \ldots \quad (5.5)$$

The function $e_2^{(1)}(\xi)$ admits of a representation in the form of the following indefinite integral:

$$e_2^{(1)}(z\equiv \operatorname{th}\xi)=c_+P_2^{\sqrt{13}}(z)+c_-Q_2^{\sqrt{13}}(z)$$
$$+c\int^z d\zeta\,(1-\zeta^2)^{5/2}[P_2^{\sqrt{13}}(z)\,Q_2^{\sqrt{13}}(\zeta)-Q_2^{\sqrt{13}}(z)\,P_2^{\sqrt{13}}(\zeta)]. \quad (5.6)$$

Here c is a certain numerical constant, P(z) and Q(z) are the associated Legendre functions of first and second order, respectively, and the constants $c_\pm$ are determined from the condition for the vanishing of the function $e_2^{(1)}(z)$ at $z = \pm 1$. Thus, near a one-dimensional field distribution (localized in one of the spatial variables and periodic in the other) the main period of the longitudinal oscillations of the field is comparable with the characteristic dimension of the localization region in the transverse direction. We note that the value of the field on the symmetry axis of the two-dimensional distribution, which is a rigorously defined quantity for the one-dimensional solution, depends continuously in this case on the amplitude of the fundamental two-dimensional mode.

6. Let us consider the case when the phase of the field $\Psi \neq$ const and the field distribution in the nonlinear medium is connected with the field energy flux density. One of the exact solutions of the system (1.4) for the assumed nonlinearity is

$$E_0=E_c\sqrt{\varkappa^2+k_\infty{}^2-k_\omega{}^2}/\varkappa,\qquad \Psi_0=-k_\infty z.$$

This solution corresponds to a plane wave of finite amplitude and leads to the presence of a longitudinal field energy flux density $S_{||} = k_\infty E_0^2$. The substitution

$$E=E_0[1+ae(x,\,z)],\qquad \Psi=\Psi_0+a\psi(x,\,z) \quad (6.1)$$

leads to the following system:

$$\Delta e+2\Big(\frac{\varkappa E_0}{E_c}\Big)^2e+2k_\infty\frac{\partial\psi}{\partial z}=-3a\Big(\frac{\varkappa E_0}{E_c}\Big)^2e^2$$
$$-a^2\Big(\frac{\varkappa E_0}{E_c}\Big)^2e^3+a(\operatorname{grad}\psi)^2+a^2e(\operatorname{grad}\psi)^2-2ak_\infty e\frac{\partial\psi}{\partial z},$$
$$\Delta\psi-2k_\infty\frac{\partial e}{\partial z}=ak_\infty\frac{\partial e^2}{\partial z}-2a\operatorname{div}(e\operatorname{grad}\psi)-a^2\operatorname{div}(e^2\operatorname{grad}\psi). \quad (6.2)$$

In the linear approximation, a solution bounded and periodic in each of the variables is

$$e^{(0)}=\cos(k_\perp x)\cos(k_{||}z),\qquad \psi^{(0)}=\frac{2k_\infty k_{||}}{k_\perp{}^2+k_{||}{}^2}\cos(k_\perp x)\sin(k_{||}z). \quad (6.3)$$

The connection between the linear-approximation wave-vector projections is given by

$$(k_\perp{}^2+k_{||}{}^2)\,[2(\varkappa E_0/E_c)^2-k_\perp{}^2-k_{||}{}^2]+4k_\infty{}^2k_{||}{}^2=0. \quad (6.4)$$

Simple calculations show that when $a \ll 1$ the asymptotic expansion of the two-dimensional solution that is close to a plane wave of finite amplitude is given by

$$E=E_0+aE_0\Big\{\cos\varphi_\perp\cos\varphi_{||}-\frac{3}{8}a$$
$$+a\Big[-\frac{3}{8}+\frac{k_{||}{}^2}{k^2}\Big(\frac{k_{||}{}^2}{k^2}-1\Big)\Big(\frac{k_\infty E_c}{\varkappa E_0}\Big)^2\Big]\frac{(\varkappa E_0/E_c)^2\cos2\varphi_\perp}{(\varkappa E_0/E_c)^2-2k_\perp{}^2}$$
$$+a\Big[-\frac{3}{8}+\Big(\frac{k_{||}{}^2}{k^2}\Big)^2\Big(\frac{k_\infty E_c}{\varkappa E_0}\Big)^2-\frac{1}{4}\Big(\frac{k_\infty E_c}{\varkappa E_0}\Big)^2\Big]\frac{(\varkappa E_0/E_c)^2\cos2\varphi_{||}}{(\varkappa E_0/E_c)^2-2k_{||}{}^2+2k_\infty{}^2}$$
$$+a\Big[\frac{1}{8}+\frac{k_{||}{}^2}{4k^2}\Big(\frac{k_\infty E_c}{\varkappa E_0}\Big)^2\Big]\Big(\frac{\varkappa E_0}{kE_c}\Big)^2\cos2\varphi_\perp\cos2\varphi_{||}\Big\}. \quad (6.5)$$

To eliminate the secular terms, we again use the assumption that the fundamental wave vector depends on the parameter a. Calculations show that

$$\omega_{||}^{(1)}+\omega_\perp^{(1)}=\frac{4k_\infty{}^2k^2}{k^4+4k_\infty{}^2k_{||}{}^2}\omega_{||}^{(1)}. \quad (6.6)$$

In the limiting case $k_\infty \to 0$, the asymptotic expansion (6.5) coincides with the expansion (4.5) considered above. Elimination of the secular terms in the next-higher approximation leads to the relation

$$\omega_{||}^{(2)}+\omega_\perp^{(2)}=f(k_{||},k_\perp,k_\infty;\,\omega_{||}^{(2)}), \quad (6.7)$$

the right-hand side of which is a linear inhomogeneous function of the quantity $\omega_{||}^{(2)}$. Relation (6.7), unlike the preceding relation (6.6), admits only of solutions with $\omega_{||,\perp}^{(2)} \neq 0$. Let $k_\infty^2 \sim k_\omega^2 \gg \kappa^2$, and then $E_0 \sim E_c$. If at the same time $k_{||}^2 \gg k_\perp^2$, then the asymptotic expansion (6.5) takes the form

$$E\sim E_c+aE_c\Big\{\cos\varphi_\perp\cos\varphi_{||}+3\frac{a}{8}(1+\cos2\varphi_\perp)$$
$$+\frac{3}{4}a\frac{k_\infty{}^2}{\varkappa^2}\cos2\varphi_{||}+\ldots\Big\}, \quad (6.8)$$

provided only that when $k_\infty^2 \gg \kappa^2$ the parameter $a \ll 1$ is such that $ak_\infty^2 \ll \kappa^2$. The obtained asymptotic form corresponds to rapid oscillations in the longitudinal variable and slow ones in the transverse variable. Relation (6.4) shows that when $k_\perp^2 \ll k_{||}^2$ the longitudinal wave number $k_{||} \sim 2k_\infty$. We call attention to the fact that no small divisors appear in the asymptotic expansion when the inequality $ak_\infty^2 \ll \kappa^2$ is satisfied. The amplitude functions of the higher longitudinal modes are small, and the principal terms of the asymptotic expansion for the field distribution in the nonlinear medium are connected with the fundamental two-dimensional mode and with the long-wave distortion of the field distribution in a plane orthogonal to the propagation direction of the plane wave of finite amplitude.

7. In conclusion, let us consider two-dimensional field distributions close to the exact one-dimensional solution of the system (1.4), corresponding to the occurrence in the nonlinear medium of a plane waveguide layer[2,4]. The exact homogeneous solution is given by

$$E_0(x)=\sqrt{2}E_c\frac{\sqrt{\varkappa^2+k_\infty{}^2-k_\omega{}^2}/\varkappa}{\operatorname{ch}\sqrt{\varkappa^2+k_\infty{}^2-k_\omega{}^2}\,x},\qquad \Psi_0(z)=-k_\infty z. \quad (7.1)$$

The substitution $E = E_0 + ae$, $\Psi = \Psi_0 + a\psi$, after introducing the dimensionless variables $\xi = x\sqrt{\kappa^2 + k_\infty^2 - k_\omega^2}$ and $\zeta = z\sqrt{\kappa^2 + k_\infty^2 - k_\omega^2}$ and the symbols

$$\Phi=\sqrt{2}E_c\psi\sqrt{\varkappa^2+k_\infty{}^2-k_\omega{}^2}/\varkappa,\qquad \alpha=\varkappa a/E_c\sqrt{\varkappa^2+k_\infty{}^2-k_\omega{}^2},$$
$$\chi_\infty{}^2=k_\infty{}^2/(k_\infty{}^2+\varkappa^2-k_\omega{}^2), \quad (7.2)$$

leads to the following system of equations

$$\Big(\Delta+\frac{6}{\operatorname{ch}^2\xi}-1\Big)e-2\chi_\infty\partial_\zeta\frac{\Phi}{\operatorname{ch}\xi}=-\sqrt{2}\alpha\chi_\infty e\,\partial_\zeta\Phi$$
$$+\frac{1}{2}\alpha^2e(\partial_\zeta\Phi)^2-3\sqrt{2}\alpha\frac{e^2}{\operatorname{ch}\xi}-\alpha^2e^3+\sqrt{2}\alpha\frac{(\partial_\xi\Phi)^2}{\operatorname{ch}\xi}+\frac{1}{2}\alpha^2(\partial_\xi\Phi)^2,$$
$$\frac{1}{\operatorname{ch}\xi}\Big\{\Big(\Delta+\frac{2}{\operatorname{ch}^2\xi}-1\Big)\frac{\Phi}{\operatorname{ch}\xi}-2\chi_\infty\partial_\zeta e\Big\}=-\sqrt{2}\alpha\partial_\zeta\frac{e\partial_\zeta\Phi}{\operatorname{ch}\xi}$$
$$+\frac{1}{2}\alpha\chi_\infty\partial_\zeta e^2-\frac{1}{2}\alpha^2\partial_\zeta(e^2\partial_\zeta\Phi)-\sqrt{2}\alpha\partial_\xi\frac{e\partial_\xi\Phi}{\operatorname{ch}\xi}-\frac{1}{2}\alpha^2\partial_\xi(e^2\partial_\xi\Phi). \quad (7.3)$$

In the linear approximation, the latter degenerates into the system

$$\Big(\Delta+\frac{6}{\operatorname{ch}^2\xi}-1\Big)e^{(0)}+2\chi_\infty\partial_\zeta\Big(\frac{\Phi^{(0)}}{\operatorname{ch}\xi}\Big)=0,$$

$$-2\chi_\infty\partial_\zeta e^{(0)} + \left(\Delta + \frac{2}{\mathrm{ch}^2\,\xi} - 1\right)\frac{\Phi^{(0)}}{\mathrm{ch}\,\xi} = 0, \tag{7.4}$$

which admits of a solution in the form $e^{(0)} = e_\perp \cos[\chi_{\|}\zeta]$, $\Phi^{(0)} = \Phi_\perp \sin[\chi_{\|}\zeta]$. For the functions $e_\perp$ and $\Phi_\perp$, Eqs. (7.4) lead to an eigenvalue problem for the longitudinal wave number $\chi_{\|}$:

$$\left[\frac{d^2}{d\xi^2} + \frac{6}{\mathrm{ch}^2\,\xi} - 1 - \chi_{\|}^2\right] e_\perp + 2\frac{\chi_\infty \chi_{\|}}{\mathrm{ch}\,\xi}\Phi_\perp = 0,$$

$$2\chi_\infty\chi_{\|} e_\perp + \left[\frac{d^2}{d\xi^2} + \frac{2}{\mathrm{ch}^2\,\xi} - 1 - \chi_{\|}^2\right]\frac{\Phi_\perp}{\mathrm{ch}\,\xi} = 0. \tag{7.5}$$

Eliminating $\Phi_\perp$, we obtain the fourth-order equation

$$\left[\frac{d^2}{d\xi^2} + \frac{2}{\mathrm{ch}^2\,\xi} - 1 - \chi_{\|}^2\right]\left[\frac{d^2}{d\xi^2} + \frac{6}{\mathrm{ch}^2\,\xi} - 1 - \chi_{\|}^2\right] e_\perp - 4\chi_\infty^2\chi_{\|}^2 e_\perp = 0. \tag{7.6}$$

When $\chi_\infty^2 \gg 1$, the proper parameter of the problem is $\chi_{\|}^2 \sim 4\chi_\infty^2$. We call attention to the fact that in the construction of two-dimensional solutions close to a plane wave of finite amplitude, when $k_\infty^2 \gg \kappa^2$, a similar relation was obtained. Let

$$\chi_{\|}^2 = 4\chi_\infty^2 + 2\lambda, \tag{7.7}$$

where $\kappa_\infty^2 \gg 1$ and λ is finite. Equation (7.6) can be rewritten in the form

$$\left\{\left[\frac{d^2}{d\xi^2} + \frac{2}{\mathrm{ch}^2\xi} - 1\right]\left[\frac{d^2}{d\xi^2} + \frac{6}{\mathrm{ch}^2\,\xi} - 1\right] - 4\lambda\left[\frac{d^2}{d\xi^2} + \frac{4}{\mathrm{ch}^2\,\xi} - 1\right] + 4\lambda^2 - 8\chi_\infty^2\left[\frac{d^2}{d\xi^2} + \frac{4}{\mathrm{ch}^2\,\xi} - 1 - \lambda\right]\right\} e_\perp = 0. \tag{7.8}$$

In our case, the latter shows that the parameter λ, which determines the difference between the eigenvalue $\chi_{\|}^2$ and its limiting value $4\chi_\infty^2$ can be obtained by solving the eigenvalue problem for the second order equation

$$(d^2/d\xi^2 + 4/\mathrm{ch}^2\,\xi - 1 - \lambda)e_\perp \sim 0, \tag{7.9}$$

where $\Phi_\perp/\cosh\xi \sim \chi_{\|}e_\perp/2\chi_\infty$. The solution (7.9) leads to two eigenfunctions satisfying the required boundary conditions:

$$e(\xi, \lambda_0) = (1 - \mathrm{th}^2\,\xi)^{(\sqrt{17}-1)/4}, \quad \lambda_0 = \frac{7 - \sqrt{17}}{2};$$

$$e(\xi, \lambda_1) = \mathrm{th}\,\xi(1 - \mathrm{th}^2\,\xi)^{(\sqrt{17}+1)/4}, \quad \lambda_1 = \frac{7 + \sqrt{17}}{2}. \tag{7.10}$$

We proceed to construct the two-dimensional solutions (periodic in the longitudinal variable and vanishing at $\xi \to \pm\infty$) that are close to the one-dimensional waveguide solution (7.1). Let $e_\perp$ and $\Phi_\perp$ be the exact eigenfunctions of the linear approximation, and let $\chi_{\|}$ be the corresponding eigenvalue. Assuming that when $\alpha \ll 1$ the distributions of the amplitude and of the phase of the field in the nonlinear medium admit of the asymptotic expansions

$$e = e_\perp(\xi)\cos\varphi_{\|} + \alpha e^{(1)}(\xi, \varphi_{\|}) + \ldots, \quad \Phi = \Phi_\perp(\xi)\sin\varphi_{\|} + \alpha\Phi^{(1)}(\xi, \varphi_{\|}) \tag{7.11}$$

we find that $e^{(1)}$ and $\Phi^{(1)}$ satisfy the system of equations

$$\left(\Delta + \frac{6}{\mathrm{ch}^2\,\xi} - 1\right)e^{(1)} + 2\chi_\infty\partial_\zeta\frac{\Phi^{(1)}}{\mathrm{ch}\,\xi} = -\frac{1}{\sqrt{2}}\chi_\infty\chi_{\|}e_\perp\Phi_\perp - \frac{3e_\perp^2}{\sqrt{2}\,\mathrm{ch}\,\xi} + \frac{(\partial_\xi\Phi_\perp)^2}{\sqrt{2}\,\mathrm{ch}\,\xi} - \left[\frac{1}{\sqrt{2}}\chi_\infty\chi_{\|}e_\perp\Phi_\perp + \frac{3e_\perp^2}{\sqrt{2}\,\mathrm{ch}\,\xi} + \frac{(\partial_\xi\Phi_\perp)^2}{\sqrt{2}\,\mathrm{ch}\,\xi}\right]\cos 2\varphi_{\|}.$$

$$\frac{1}{\mathrm{ch}\,\xi}\left[\left(\Delta + \frac{2}{\mathrm{ch}^2\,\xi} - 1\right)\frac{\Phi^{(1)}}{\mathrm{ch}\,\xi} - 2\chi_\infty\partial_\zeta e^{(1)}\right] = \left[\sqrt{2}\chi_{\|}^2\frac{e_\perp\Phi_\perp}{\mathrm{ch}\,\xi} - \frac{1}{2}\chi_\infty\chi_{\|}e_\perp^2 - \frac{1}{\sqrt{2}}\partial_\xi\frac{e_\perp\partial_\xi\Phi_\perp}{\mathrm{ch}\,\xi}\right]\sin 2\varphi_{\|}. \tag{7.12}$$

In writing down this system, account was taken of the fact that in the first approximation in the parameter $\alpha \ll 1$ the longitudinal wave number retains the value characteristic of the linear approximation. In other words, $\chi_{\|}^2(\alpha) = \chi_{\|}^2 + O(\alpha^2)$. Indeed, assuming the contrary and analyzing the solvability conditions for the fundamental two-dimensional mode, we can verify that the quantity $\omega_{\|}^{(1)}$, which determines the term that is linear in α in the asymptotic expansion of the fundamental longitudinal wave number, is equal to zero.

Using the representations

$$e^{(1)} = e_0^{(1)} + e_2^{(1)}\cos 2\varphi_{\|}, \quad \Phi^{(1)} = \Phi_2^{(1)}\sin 2\varphi_{\|},$$

we find that $e_0^{(1)}$ satisfies the inhomogeneous equation

$$\left(\frac{d^2}{d\xi^2} + \frac{6}{\mathrm{ch}^2\,\xi} - 1\right)e_0^{(1)} = -\frac{1}{\sqrt{2}}\chi_\infty\chi_{\|}e_\perp\Phi_\perp - \frac{3e_\perp^2}{2\,\mathrm{ch}\,\xi} + \frac{(\partial_\xi\Phi_\perp)^2}{\sqrt{2}\,\mathrm{ch}\,\xi}. \tag{7.13}$$

In order for the obtained solution to be solvable it is necessary that the eigenfunction of the operator in the left side of (7.13), corresponding to zero eigenvalue, the orthogonal to the right-hand side. Such a condition is satisfied, since the parity of the functions $e_\perp$ and $\Phi_\perp$ is the same, and the aforementioned eigenfunction, equal to $\sinh\xi/\cosh^2\xi$, is odd. Consequently,

$$e_0^{(1)} = \left(\frac{d^2}{d\xi^2} + \frac{6}{\mathrm{ch}^2\,\xi} - 1\right)^{-1}\left[-\frac{1}{\sqrt{2}}\chi_\infty\chi_{\|}e_\perp\Phi_\perp - \frac{3e_\perp^2}{\sqrt{2}\,\mathrm{ch}\,\xi} + \frac{(\partial_\xi\Phi_\perp)^2}{\sqrt{2}\,\mathrm{ch}\,\xi}\right]. \tag{7.14}$$

Since two linearly independent solutions are known for the homogeneous equation corresponding to (7.13) (one is $\sinh\xi/\cosh^2\xi$, which coincides following the substitution $\tanh\xi = z$ with the associated Legendre function of the first kind $P_2^{(1)}$, and the second is $Q_2^{(1)}$, which is the associated Legendre function of the second kind), the solution of the inhomogeneous equation (7.13) can be written in the form of an indefinite integral similar to (5.6). The system of equations determining the functions $e_2^{(1)}$ and $2^{(1)}$ is of the form

$$\left(\frac{d^2}{d\xi^2} + \frac{6}{\mathrm{ch}^2\,\xi} - 1 - 4\chi_{\|}^2\right)e_2^{(1)} + 4\chi_\infty\chi_{\|}\frac{\Phi_2^{(1)}}{\mathrm{ch}\,\xi} = -\frac{1}{\sqrt{2}}\chi_\infty\chi_{\|}e_\perp\Phi_\perp - \frac{3e_\perp^2}{\sqrt{2}\,\mathrm{ch}\,\xi} - \frac{(\partial_\xi\Phi_\perp)^2}{\sqrt{2}\,\mathrm{ch}\,\xi},$$

$$\frac{1}{\mathrm{ch}\,\xi}\left[4\chi_\infty\chi_{\|}e_2^{(1)} + \left(\frac{d^2}{d\xi^2} + \frac{2}{\mathrm{ch}^2\,\xi} - 1 - 4\chi_{\|}^2\right)\frac{\Phi_2^{(1)}}{\mathrm{ch}\,\xi}\right] = \sqrt{2}\chi_{\|}^2\frac{e_\perp\Phi_\perp}{\mathrm{ch}\,\xi} - \frac{1}{2}\chi_\infty\chi_{\|}e_\perp^2 - \partial_\xi\frac{e_\perp\partial_\xi\Phi_\perp}{\sqrt{2}\,\mathrm{ch}\,\xi}. \tag{7.15}$$

When $\chi_\infty^2 \gg 1$, the latter can be greatly simplified. Indeed, in the right sides of (7.15) it is sufficient to retain only the terms containing the factors $\chi_\infty\chi_{\|}$ or $\chi_{\|}^2$. Moreover, in the left sides of (7.15), in the limiting case under consideration, there appear only differential operators with constant coefficients. Consequently, when $\chi_\infty^2 \gg 1$ the system (7.15) assumes the form

$$\left(\frac{d^2}{d\xi^2} - 4\chi_{\parallel}^2\right) e_2^{(1)} + 4\chi_\infty\chi_{\parallel}\frac{\Phi_2^{(1)}}{\operatorname{ch}\xi} \sim -\frac{1}{\sqrt{2}}\chi_\infty\chi_{\parallel}e_\perp\Phi_\perp,$$

$$\frac{1}{\operatorname{ch}\xi}\left[4\chi_\infty\chi_{\parallel}e_2^{(1)} + \left(\frac{d^2}{d\xi^2} - 4\chi_{\parallel}^2\right)\frac{\Phi_2^{(1)}}{\operatorname{ch}\xi}\right] \sim \sqrt{2}\,\chi_{\parallel}^2\frac{e_\perp\Phi_\perp}{\operatorname{ch}\xi} - \frac{1}{2}\chi_\infty\chi_{\parallel}e_\perp^2. \quad (7.16)$$

For the combinations $e_2^{(1)} \pm \Phi_2^{(1)}$, the system (7.16) breaks up into two independent inhomogeneous equations of second degree, the solution of which entails no difficulty.

Thus, near the field distribution corresponding to the occurrence in the nonlinear medium of a plane waveguide layer, there exist two-dimensional field distributions that are localized in the transverse variable and are periodic in the longitudinal one. The longitudinal wave number, which determines the periodic structure of the perturbed natural waveguide depends on the amplitude (more accurately, the square of the amplitude) of the fundamental two-dimensional mode and the wave number k_∞, which determines the energy flux density in the unperturbed plane waveguide layer. If $k_\infty \ll \kappa$, then the wavelength of the longitudinal spatial oscillations is comparable with the characteristic dimension that determines the region of localization of the field across the waveguide layer. On the other hand, if $k_\infty \gg \kappa$, then, provided the parameter $\alpha\chi_\infty^2 \ll 1$, the wavelength of the longitudinal oscillations of the field is small compared with the characteristic dimensions that determines the transverse dimension of the waveguide field. The principal terms of the asymptotic expansion for the field distribution in the nonlinear medium are the fundamental two-dimensional mode and the term that does not depend on the longitudinal variable and determines the difference between the two-dimensional field amplitude distribution averaged over the longitudinal oscillations, and the corresponding distribution for the unperturbed waveguide. On the other hand, the amplitude functions of the higher two-dimensional modes turn out to be small. We note that the considered limiting case is similar to the previously investigated case of two-dimensional field distributions close to a plane wave of finite amplitude.

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