INFRARED DIVERGENCES IN QUANTUM ELECTRODYNAMICS AND STATES OF DEFINITE CHARGE-CONJUGATION PARITY

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It is shown that in processes like pair annihilation from states of definite charge-conjugation parity in the continuous spectrum, the infrared divergences do not compensate each other in each order of perturbation theory. In this connection, the annihilation probability, taking into account all radiative corrections and the emission of all real soft photons, no longer ceases to depend on the regularization parameter (the photon mass λ) for $(\lambda/m)^2 \ll 1$, as usually happens, but for $(\lambda/m)^{2\alpha A} \ll 1$, where $\alpha = 1/137$ and $A \sim \ln(E/m)$. Therefore, although in the limit $\lambda \to 0$ the probability does not exhibit an infrared divergence, in a real experiment, where the role of λ^{-1} is played by the dimensions of the laboratory or of the installation, the average number of emitted photons turns out to be of the order of unity, and the probability for annihilations from states with C = +1 and C = −1 will depend on the size of the laboratory and will differ substantially from the corresponding expressions of the Born approximation.

As is well known (cf., e.g.,[1]) the infrared divergences in quantum electrodynamics appear as a consequence of the inapplicability of perturbation expansions with respect to the parameter $\alpha = e^2/\hbar c$ in the region of low frequencies, since (as already follows from classical electrodynamics) the number of emitted photons, which is of the order $\alpha \ln(m/\omega)$ becomes large, i.e., for $\omega \to 0$ the probability of emission of any finite number of photons tends to zero, and only the probability for the emission of an infinite number of quanta is finite. Therefore, for the elimination of infrared divergences in radiative corrections to any process, one considers the same process with emission of additional real photons of total energy not surpassing a certain value ΔE. It is proved in the literature that in the sum of the cross section including radiative corrections and the cross section corresponding to the emission of additional soft photons the infrared divergences coming from the radiative corrections and those from the emission of real soft photons compensate each other in each order of perturbation theory.

Thus, the differential cross section for Compton scattering including radiative corrections to order α has the form[2]:

$$d\sigma_0 = d\sigma_B\left\{1-\frac{\alpha}{\pi}\left[2(2\varphi\,\mathrm{cth}\,2\varphi-1)\ln\frac{m}{\lambda}+f\right]\right\},\quad \lambda\to 0, \qquad (1)$$

where $d\sigma_B$ is the cross section in the lowest (Born) approximation; $\cosh 2\varphi = |pp'|m^{-2}$; p, p′ are the initial and final electron momenta; λ is the virtual photon mass, introduced for the regularization of the infrared divergence; f is an invariant function which does not depend on λ for $\lambda \to 0$. On the other hand, the differential cross section for Compton scattering with the emission of one real additional photon of energy (in the system where p = 0) not exceeding ΔE, $\lambda \ll \Delta E \ll m$ is

$$d\sigma_1 = d\sigma_B\frac{\alpha}{\pi}\left[2(2\varphi\,\mathrm{cth}\,2\varphi-1)\ln\frac{2\Delta E}{\lambda}+2\varphi\,\mathrm{cth}\,2\varphi[1-2h(2\varphi)]+1\right]. \qquad (2)$$

Here

$$h(\varphi)=\varphi^{-1}\int_0^{\varphi} du\, u\,\mathrm{cth}\, u.$$

It is obvious that the sum $d\sigma_0 + d\sigma_1$ does not contain λ, i.e., the infrared divergences which are present in (1) and (2) compensate one another. A similar compensation occurs in other cases considered in the literature[1)][3-5].

We would like to call attention to the fact that the compensation of infrared divergences in each order of perturbation theory, which occurred in all cases so far considered in the literature is not a general rule[2)].

In the process to be considered below of positron-electron annihilation from a state of definite charge-conjugation parity of the continuous spectrum, infrared divergences are present in each order of perturbation theory and disappear only after summing all orders, i.e., only after an infinite number of both virtual and real soft quanta has been taken into account. A definite C-parity of the state of the electron-positron pair means that the wave function of the system is either antisymmetric (C = +1) or symmetric (C = −1) with respect to the substitution $e^+ \rightleftarrows e^-$[1]. Such a state can annihilate only into an even (C = +1) or odd (C = −1) number of photons.

If C = +1, the cross section $d\sigma_{+0}$ for annihilation into two photons[3)] including radiative corrections to order α will have the structure (1), where the infrared-divergent term $\sim \alpha d\sigma_{+B}$ comes from the interference of the basic (Born) matrix element with infrared-divergent radiative corrections. On the other hand, owing to C-parity conservation, the additional soft photons can be emitted only in even numbers. Therefore the cross section $d\sigma_{+1}$

[1)]Equations (1), (2) with the appropriate $d\sigma_B$ and f are valid for all processes containing one electron line.

[2)]Cf. the situation for Coulomb divergences [6].

[3)]The indices indicate the C-parity of the state and the number of additionally emitted soft photons.

will vanish and the cross section $d\sigma_{+2}$ is obviously $\sim \alpha^2 d\sigma_{+B}$. Consequently in the sum $d\sigma_{+0} + d\sigma_{+1} + d\sigma_{+2}$ the infrared divergence of the term $d\sigma_{+B}$ remains uncompensated. We show that in this case the infrared divergences remain uncompensated in every order of perturbation theory, but the total sum of the radiative corrections and of the cross sections involving the emission of real soft photons do not contain infrared divergences.

Considering the annihilation of an electron-positron pair one usually selects as the initial state

$$\psi = \frac{1}{\sqrt{2}}(\psi_+ + \psi_-), \tag{3}$$

where $\psi_\pm$ are states of the pair with even or odd charge-conjugation parity. The differential cross section for annihilation from the state ψ into two hard photons and an arbitrary number n of soft photons, taking into account all radiative corrections, will be written in the form

$$d\sigma = \sum_{n=0}^{\infty} d\sigma_n = \sum_{n\,\mathrm{even}\geq 0}^{\infty} d\sigma_n + \sum_{n\,\mathrm{odd}\geq 1}^{\infty} d\sigma_n. \tag{4}$$

It is clear that the sum over even n represents the cross section for annihilation from the state $\psi_+/\sqrt{2}$ and the sum over odd n comes from the state $\psi_-/\sqrt{2}$. Consequently

$$d\sigma = {}^1\!/_2 d\sigma_+ + {}^1\!/_2 d\sigma_-, \tag{5}$$

where $d\sigma_\pm$ are the cross sections for annihilation from the states $\psi_\pm$. Thus, if $d\sigma$ does not contain infrared divergences then $d\sigma_+$ and $d\sigma_-$ should not contain such divergences. Moreover, for $\lambda \to 0$ the two cross sections $d\sigma_+$ and $d\sigma_-$ should be equal, since in that limit an infinite number of soft quanta is emitted, and the difference of one emitted soft photon becomes inessential.

In order to show how these assertions follow from the formalism of quantum electrodynamics, we utilize the method of eliminating infrared divergences developed by Yennie, Frautschi, and Suura[7]. These authors have given a convenient representation for the cross section of a process accompanied by the emission of n real photons of total energy ΔE, including all radiative corrections (Eq. (2.13) of that paper). Making use of that formula it is not difficult to obtain the cross section for the process, taking into account all radiative corrections and the emission of an arbitrary number of real photons of energy smaller than ΔE in the form[4)]

$$d\sigma = \int_0^{\Delta E} d\varepsilon \sum_{n=0}^{\infty} \frac{d\sigma_n}{d\varepsilon} = e^{2\alpha B} \int_0^{\Delta E} d\varepsilon \frac{1}{2\pi} \int_{-\infty}^{\infty} dy\, e^{iy\varepsilon} \sum_{r=0}^{\infty} \frac{1}{r!} \int \prod_{i=1}^{r} \frac{d^3k_i e^{-iyk_{i0}}}{\sqrt{\mathbf{k}_i^2+\lambda^2}} \times \tilde{\beta}_r(k_1, k_2, \ldots, k_r) \sum_{n\geq r}^{\infty} \frac{J^{n-r}}{(n-r)!}, \tag{6}$$

where

$$J = \int \frac{d^3k}{\sqrt{\mathbf{k}^2+\lambda^2}} e^{-iyk_0} \tilde{S}(k) = 2\alpha\tilde{B} + D, \quad 2\alpha\tilde{B} = \int_0^{k_0\leq\varepsilon} \frac{d^3k}{\sqrt{\mathbf{k}^2+\lambda^2}} \tilde{S}(k),$$

$$D = \int_0^{k_0\leq\varepsilon} \frac{d^3k\, \tilde{S}(k)}{k_0}(e^{-iyk_0} - 1) + \int_\varepsilon^\infty \frac{d^3k}{k_0} \tilde{S}(k) e^{-iyk_0} \tag{7}$$

n is the total number of emitted photons, r is the number of non-infrared photons, $k_0 = (\mathbf{k}^2 + \lambda^2)^{1/2}$.

All infrared divergences coming from the radiative corrections and the emission of real photons are concentrated respectively in the quantities B and $\tilde{B}$, defined as follows[5)]:

$$B = \frac{i}{(2\pi)^3} \int \frac{d^4k}{k^2+\lambda^2} \left(\frac{2p_\mu' - k_\mu}{2p'k - k^2} - \frac{2p_\mu - k_\mu}{2pk - k^2}\right)^2$$
$$= -\frac{1}{2\pi}\left\{2(2\varphi \operatorname{cth} 2\varphi - 1)\ln\frac{m}{\lambda} - \varphi \operatorname{cth}\varphi + 4\varphi \operatorname{cth} 2\varphi[h(2\varphi) - h(\varphi)] + 1\right\},$$

$$\tilde{B}(\varepsilon) = \frac{1}{8\pi^2} \int_0^{k_0=\varepsilon} \frac{d^3k}{\sqrt{\mathbf{k}^2+\lambda^2}} \left(\frac{p_\mu'}{p'k} - \frac{p_\mu}{pk}\right)^2$$
$$= \frac{1}{2\pi}\left\{2(2\varphi \operatorname{cth} 2\varphi - 1)\ln\frac{2\varepsilon}{\lambda} + 2\varphi \operatorname{cth} 2\varphi[1 - 2h(2\varphi)] + 1\right\}. \tag{8}$$

This means that in (6), (7) everywhere, except in B, $\tilde{B}$ one may set $\lambda = 0$. As regards the function D, it becomes for $\lambda \to 0$:

$$D = -\alpha AC - \alpha A \ln(i\varepsilon y), \quad \alpha A \equiv \mathbf{k}^2 \int d\Omega \tilde{S}(k) = \frac{2\alpha}{\pi}(2\varphi \operatorname{cth} 2\varphi - 1), \tag{9}$$

where C is Euler's constant. The functions of y defined by the integrals over k in (6) and (7) are analytic in the lower complex y half-plane and on the real axis, except the point y = 0, where they are singular. Therefore, in integrating over y, the point y = 0 should be avoided from below. We note that the positive quantity αA is important for the sequel and is small (of the order of α) in a wide range of energies, and approaches unity only for extremely high energies, when $-pp'/m^2 \approx e^{\pi/2\alpha} \sim 10^{94}$. For the annihilation cross section from the state ψ (cf. (3)) the sum over n in (6) is taken over all integers $n \geq 0$, therefore

$$\sum_{n\geq r}^{\infty} \frac{J^{n-r}}{(n-r)!} = e^J,$$

so that

$$d\sigma = e^{2\alpha B} \int_0^{\Delta E} d\varepsilon \frac{1}{2\pi} \int_{-\infty}^{\infty} dy\, e^{iy\varepsilon + J} \sum_{r=0}^{\infty} \frac{1}{r!} \int \prod_{i=1}^{r} \frac{d^3k_i}{\sqrt{\mathbf{k}_i^2}} e^{-iyk_i} \tilde{\beta}_r(k_1, \ldots, k_r). \tag{10}$$

This expression corresponds to Eq. (2.17) in[7]. We now note that the sum over only even (odd) n in (6) corresponds to the contribution to the annihilation cross section from the state $\psi_+/\sqrt{2}$ ($\psi_-/\sqrt{2}$). Since

$$\sum_{n\,\mathrm{odd(even)}\geq r} \frac{J^{n-r}}{(n-r)!} = \begin{cases} \operatorname{ch} J \ (\operatorname{sh} J), & \text{if r is even,} \\ \operatorname{sh} J \ (\operatorname{ch} J), & \text{if r is odd,} \end{cases}$$

then, using these expressions in (6) and rewriting the hyperbolic functions in terms of exponentials, we obtain for $d\sigma_\pm$

$$d\sigma_\pm = e^{2\alpha B} \int_0^{\Delta E} d\varepsilon \frac{1}{2\pi} \int_{-\infty}^{\infty} dy\, e^{iy\varepsilon} \left\{e^J \sum_{r=0}^{\infty} \frac{1}{r!} \int \prod_{i=1}^{r} \frac{d^3k_i}{k_i} e^{-iyk_i} \tilde{\beta}_r(k_1, \ldots, k_r)\right.$$

[4)] All notations and the meaning of the quantities are the same as in [7]; since we consider annihilation rather than scattering, p′ denotes the momentum of the positron.

[5)] The notations are the same as in eqs. (1), (2), with $\lambda \to 0$, and the latter equality for $\tilde{B}$ is valid in the system where $\mathbf{p} = 0$.

$$\pm e^{-J} \sum_{r=0}^{\infty} \frac{(-1)^r}{r!} \int \prod_{i=1}^{r} \frac{d^3k_i}{k_i} e^{-iyk_i} \tilde{\beta}_r(k_1, \ldots, k_r) \Big\}. \quad (11)$$

The infrared divergences in (10), (11) are contained only in B and $\tilde{B}$, and (10) depends on B and $\tilde{B}$ only through $\exp\{2\alpha(B + \tilde{B})\}$ which does not contain infrared divergences. Therefore the infrared divergence is absent not only from the exact expression for $d\sigma$, but also from each order of perturbation theory. On the other hand (11) depends not only on $\exp\{2\alpha(B + \tilde{B})\}$ but also on $\exp\{2\alpha(B - \tilde{B})\}$. This means that each term of the expansion in powers of α diverges owing to the divergence of $B - \tilde{B}$ for $\lambda \to 0$, whereas the total cross sections $d\sigma_\pm$ are finite, since

$$\exp\{2\alpha(B - \tilde{B})\} = \exp\{-2\alpha(B + \tilde{B})\} \exp\{4\alpha B\} \sim (\lambda/m)^{2\alpha A} \to 0.$$

Thus, for sufficiently small λ when $(\lambda/m)^{2\alpha A} \ll 1$, only the first term in (11) survives in this limit

$$d\sigma_+ = d\sigma_- = d\sigma, \quad (12)$$

where $d\sigma = d\sigma_B(1 + O(\alpha A))$, $d\sigma_B = \tilde{\beta}_0$.

In distinction from $d\sigma$ the cross sections $d\sigma_\pm$ are essentially different from the corresponding Born approximation cross sections, for which

$$d\sigma_{+B} = 2d\sigma_B \sim \alpha^2, \quad d\sigma_{-B} \sim \alpha^3. \quad (13)$$

Usually the cross section which takes into account radiative corrections and the emission of soft photons differs from the Born cross section by terms of the order of $\sim \alpha$. As can be seen from (11), the cross sections $d\sigma_\pm$ go over into their Born expressions for $|2\alpha B|$, $2\alpha\tilde{B}(\Delta E) \ll 1$, i.e., when λ is not too small. It is obvious that the compensation of infrared divergences in each order of α does not occur not only for the initial states $\psi_\pm$ but also for each superposition with unequal weights of such states.

The absence of compensation of divergences in each order, showing that perturbation theory is inapplicable, reminds one of the situation in perturbation theory for degenerate states (cf.[8], Sec. 39). The states $\psi_\pm$ being eigenfunctions of a doubly degenerate level, are "incorrect" zeroth approximation functions as regards the infrared part of the electromagnetic interaction (i.e., their change under this perturbation is not small), whereas the functions $\psi = (\psi_+ + \psi_-)/\sqrt{2}$ and $\psi' = (\psi_+ - \psi_-)/\sqrt{2}$ form a correct system for the zeroth approximation. Of course, this circumstance does not forbid the existence of the solutions $\psi_\pm$, but only restricts their applicability to perturbation theory; the exact solution (11) does not contain any difficulties.

All that was said above refers to the ideal case when $\lambda = 0$, and when the states are described by plane waves throughout the whole of space. In this limiting case the e^+e^- interaction leads to the emission of an infinite number of infinitely soft photons, so that the states $\psi_\pm$ become physically indistinguishable. This is confirmed, in particular, by the equality of the annihilation cross sections $d\sigma_\pm$ in this limit. However, in a real experiment the motion of the particles is restricted by the dimensions of the laboratory or the experimental installation. This leads to the emission of only a finite number of photons and allows us to prepare physically distinguishable states $\psi_\pm$. Indeed, the emission of a photon of energy ω is achieved over a coherence length[1,9]

$$l \sim Ep/m^2\omega,$$

where E and p are the energy and momentum of the particle. Therefore the spectrum $d\omega/\omega$ can occur only in that region of frequencies, for which the coherence length is small compared with the mean free path L (or the laboratory dimensions), i.e., for

$$\omega \gg \omega_L, \quad \omega_L = Ep/m^2L.$$

For frequencies $\omega \lesssim \omega_L$ the coherence length becomes of the order of, or larger than L and the spectrum stops being infrared-divergent. The conrete form of the spectrum is not important for us for $\omega \lesssim \omega_L$[6]. The qualitative modification of the spectrum in this region from infrared divergence to convergence can be formally taken into consideration by means of introducing a photon mass $\lambda \sim \omega_L$. Then the average number of emitted photons determined by the quantity $2\alpha\tilde{B}$ and depending logarithmically on λ or L, turns out to be of the order of unity or smaller in real experiments, and not infinite as would happen for $\lambda \to 0$. Therefore the annihilation probability with the emission of a definite number of photons depends essentially on λ or on the macroscopic dimensions of the installation. However, for the processes considered usually (such as annihilation from a state ψ) the probability summed over the number of emitted photons does not depend on λ for $\lambda \ll m$. On the other hand, the probability of annihilation from states $\psi_\pm$ (also summed over the number of soft photons) contains the terms $\sim \exp 2\alpha(B - \tilde{B}) \sim (\lambda/m)^{2\alpha A}$, which for real values of λ give an essential contribution (cf. infra), so that (in distinction from $d\sigma$) $d\sigma_\pm$ will depend on the dimensions of the installation.

Thus, the concept of charge-conjugation parity is physically definable only for finite systems, for which the average number of emitted photons is finite, and the parameter $(\lambda/m)^{2\alpha A}$ is not very small.

The preparation of the states $\psi_\pm$ seems to be difficult. One of the sources of such states may be the decay of a neutral system, e.g. π^0-, η-, ρ^0-, ω-mesons with definite C-parity into a pair of charged particles[7]. Each photon emitted during the decay changes the C-parity of the system of charged particles into the opposite one. However, so long as the particles have not separated too far, the number of soft photons will be small and the C-parity of the charged particles will coincide with the C-parity of the decaying system. The state prepared in this manner describes particles which fly away from each other. One may convert such states into states of colliding particles, if one lets the system decay in a magnetic field, in a plane perpendicular to the field. Then after the particles describe a circle, at the decay point we shall have a state of colliding particles with charge-conjugation parity equal to the initial value, if during the time of rotation no photon was emitted. The probability of emission of a photon in a magnetic field

[6] One might think that in this region it will have the form $\omega d\omega$, like the spectrum of a particle effecting a finite motion.

[7] States with definite C-parity can also appear as a consequence of photodisintegration of positronium, a fact which has been brought to our attention by R. J. Glauber.

during one revolution equals $5\pi\alpha p_\perp/\sqrt{3}m$, where $p_\perp$ is the momentum perpendicular to the magnetic field; for $p_\perp/m \lesssim 1$ this probability is negligible.

We consider in more detail the dependence of $d\sigma_\pm/d\epsilon$ on the energy ϵ of the quanta which accompany the process. We note that the second term in the right-hand side of (11) is obtained from the first by a change of the sign in front of α, corresponding to the emission of real photons ($J \sim \alpha$, $\widetilde{\beta}_r \sim \alpha^r$). Each of these terms turns out to be an analytic function of α, which has to be determined. It suffices for this to restrict one's attention to the contribution of the terms r = 0, r = 1, since the contributions of the other terms will be of the order α compared to these two. The term r = 0 contains the integral

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} dy\, e^{iy\varepsilon+D} = \frac{e^{-\alpha AC}}{\varepsilon}\times\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{dz\, e^{iz}}{(iz)^{\alpha A}} = \frac{e^{-\alpha AC}}{\varepsilon\Gamma(\alpha A)} = \frac{e^{-\alpha AC}}{\Gamma(1+\alpha A)}\frac{\alpha A}{\varepsilon}. \quad (14)$$

The integral with respect to z, in which, as stated above, the point z = 0 is avoided from below, coincides with the well known Hankel integral representation for the gamma-function, valid for arbitrary αA (cf.[10], Sec. 12.22).

The term r = 1 reduces to the integral

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} dy\, e^{i\varepsilon y+D}\int\frac{d^3k}{k}\widetilde{\beta}_1(k)e^{-iyk} = \frac{1}{2\pi}\int_{-\infty}^{\infty} dz\, e^{iz+D(z)}\int_0^{\infty} dx\, G_1(\varepsilon x)e^{-izx}, \quad (15)$$

where

$$G_1(k) = |\mathbf{k}|\int d\Omega\widetilde{\beta}_1(\mathbf{k}).$$

Expanding $G_1(\epsilon x)$ in a series around x = 1, and again making use of the Hankel representation, we obtain for (15)

$$\sum_{n=0}^{\infty} G_1^{(n)}(\varepsilon)\frac{\varepsilon^n}{n!}\sum_{k=0}^{n}\frac{(-1)^k n!}{k!}\cdot\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{dz\, e^{iz+D(z)}}{(iz)^{n-k+1}}$$
$$= e^{-\alpha AC}\sum_{n=0}^{\infty} G_1^{(n)}(\varepsilon)\varepsilon^n\sum_{k=0}^{n}\frac{(-1)^k}{k!\,\Gamma(n-k+1+\alpha A)}$$
$$= \frac{e^{-\alpha AC}}{\Gamma(\alpha A)}\sum_{n=0}^{\infty} G_1^{(n)}(\varepsilon)\frac{(-1)^n\varepsilon^n}{n!(n+\alpha A)} = \frac{e^{-\alpha AC}}{\Gamma(1+\alpha A)}$$
$$\times\left[G_1(\varepsilon)+\alpha A\sum_{n=1}^{\infty} G_1^{(n)}(\varepsilon)\frac{(-1)^n\varepsilon^n}{n!(n+\alpha A)}\right] \equiv \frac{e^{-\alpha AC}}{\Gamma(1+\alpha A)}g_1(\varepsilon,\alpha A). \quad (16)$$

Utilizing Eqs. (14) and (16), which are valid for any αA, we obtain

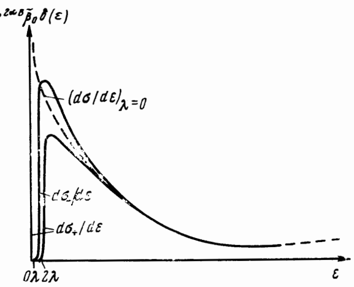
$$\frac{d\sigma_\pm}{d\varepsilon} = \frac{e^{-\alpha AC}}{\Gamma(1+\alpha A)}e^{2\alpha(B+\widetilde{B}(\varepsilon))}$$
$$\times\left[\frac{\alpha A}{\varepsilon}\widetilde{\beta}_0 + g_1(\varepsilon,\alpha A)+\ldots\right]$$
$$\mp\frac{e^{\alpha AC}}{\Gamma(1-\alpha A)}e^{2\alpha(B-\widetilde{B}(\varepsilon))}$$
$$\times\left[\frac{\alpha A}{\varepsilon}\widetilde{\beta}_0 + g_1(\varepsilon,-\alpha A)+\ldots\right], \quad (17)$$

where the dots denote terms which are of order α compared to those written out. Equation (17) is valid for $\epsilon \gg \lambda$. For $\epsilon \sim m$ both terms in the square bracket are of the same order, but for $\epsilon \ll m$ the term $\alpha A\widetilde{\beta}_0/\epsilon$ is the determining one, since in this region $g_1(\epsilon) \sim \alpha\epsilon$.

For energies $\epsilon \sim \lambda$ the spectrum is cut off. Thus for $\epsilon < 3\lambda$ we have

$$\frac{d\sigma_+}{d\varepsilon} = 2e^{2\alpha B}\left\{\widetilde{\beta}_0\delta(\varepsilon)+\frac{1}{2!}\int_\lambda^{\varepsilon-\lambda} dk_{10}|\mathbf{k}_1||\mathbf{k}_2|\right.$$
$$\left.\times\int d\Omega_1 d\Omega_2\,\widetilde{\rho}_2(\mathbf{k}_1,\mathbf{k}_2)\theta(\varepsilon-2\lambda)\right\},\quad k_{10}+k_{20}=\varepsilon,$$
$$\frac{d\sigma^-}{d\varepsilon} = 2e^{2\alpha B}|\mathbf{k}_1|\int d\Omega_1\,\widetilde{\rho}_1(\mathbf{k}_1)\theta(\varepsilon-\lambda),\quad k_{10}=\varepsilon. \quad (18)$$

The qualitative aspect of the distributions $d\sigma_\pm/d\epsilon$ is illustrated in the figure. The distributions $d\sigma_\pm/d\epsilon$ and the difference between these two attain their maximal



values in an energy region ϵ of the order of several λ. The difference between the cross sections $d\sigma_+$ and $d\sigma_-$ is composed of the cross section $2e^{2\alpha B}\widetilde{\beta}_0$ of the process without any accompanying photons, and the area between the $d\sigma_+/d\epsilon$ and $d\sigma_-/d\epsilon$ curves for $\epsilon > \lambda$, which up to terms of the order αA can be obtained by integrating the expressions (17) from $\epsilon = \lambda$ to $\epsilon = \Delta E$. This yields

$$d\sigma_+ - d\sigma_- \approx 2e^{2\alpha B}\,\widetilde{\beta}_0\left\{1-\frac{e^{\alpha AC}}{\Gamma(1-\alpha A)}\left[e^{-2\alpha\widetilde{B}(\lambda)}-e^{-2\alpha\widetilde{B}(\Delta E)}\right]\right\}. \quad (19)$$

From here the results (12), (13) follow immediately, corresponding to large or small values of $2\alpha B$, $2\alpha\widetilde{B}(\Delta E)$. In turn, these values are determined by the magnitude of λ, the particle energy and the maximum energy ΔE carried away by the quanta.

For large energies

$$2\alpha B \approx -\frac{\alpha}{2\pi}\ln\left(-\frac{2pp'}{m^2}\right)\ln\left(-\frac{2pp'm^2}{\lambda^4}\right);$$

for electrons and positrons of energy 10 MeV and a length $\hbar/\lambda c$ = 10 cm, this leads to $2\alpha B \approx -1$, i.e. to a substantial difference both from perturbation theory and from the limiting case $\lambda = 0$, when $d\sigma_+ = d\sigma_-$.

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