DAMPING OF VORTEX OSCILLATIONS IN TYPE II SUPERCONDUCTORS

A. M. FINKEL'SHTEĬN

Institute for Physical Problems, USSR Academy of Sciences

Submitted July 18, 1968

Zh. Eksp. Teor. Fiz. 56, 333–339 (January, 1969)

The motion of vortices in type II superconductors is considered on the basis of the hydrodynamic theory. Energy dissipations for oscillations of the vortices and the damping which arises as a consequence of the nonlinearity of the equations describing the changes in the shape of the vortices are estimated.

INTRODUCTION

IN the discovery of oscillations of vortex filaments in type II superconductors by resonance methods, the absorption line width is determined by the damping of the oscillations. In the present work, damping is considered ($\kappa = \lambda/\xi$, where ξ is the coherence length, λ the penetration depth) for the case in which the Ginzburg-Landau parameter is $\kappa \gg 1$, while the magnetic field satisfies the condition $h_{c1} < h \ll h_{c2}$ are the lower and upper critical fields, respectively, within the limits of which the superconductor is in a mixed state[1]). The latter inequality means that the distance between the vortices $d \gg \xi$. Under these conditions, the London model is applicable. This model describes the mixed state as a combination of vortices and a charged ideal liquid.

In the London model, to the ordinary equations of magnetohydrodynamics—the equation of continuity $\operatorname{div} \mathbf{v}_s = 0$, conservation of momentum[2]

$$\frac{\partial \mathbf{v}_s}{\partial t} + (\mathbf{v}_s \nabla)\mathbf{v}_s = \frac{e}{m}\left(\mathbf{E} + \frac{[\mathbf{v}_s \mathbf{h}]}{c}\right) - \nabla\mu, \qquad (1)^*$$

and also the Maxwell equations $\operatorname{curl} \mathbf{E} = -c^{-1}\partial \mathbf{h}/\partial t$ and $\operatorname{rot} \mathbf{h} = 4\pi c^{-1}\mathbf{j}$, where $\mathbf{j}$ - $N_s e \mathbf{v}_s$ (we neglect the displacement current), is added the condition of flux quantization

$$\operatorname{rot} \mathbf{v}_s + \frac{e\mathbf{h}}{mc} = \sum_i \gamma \int d\mathbf{s}_i \delta(\mathbf{R} - \mathbf{R}_i). \qquad (2)$$

Here $\mathbf{v}_s$ is the velocity of the superfluid; N_s the concentration of superconducting electrons; $\gamma = \pi\hbar/m$ is the circulation about each of the vortices; $\mathbf{R}_i$ is the vector function giving the location of the nucleus of the i-th vortex. Integration is carried out along the i-th vortex, and summation is taken over all the vortices.

On the basis of these equations, the following results have been obtained in a number of researches:[3-6]

a) The nucleus of the vortex moves with the local velocity of the liquid

$$\mathbf{v}_c = \mathbf{v}_{si}, \qquad (3)$$

where $\mathbf{v}_c$ is the transport velocity of the vortex nucleus and $\mathbf{v}_{si}$ is the velocity of the superfluid in which the vortex is immersed:

b) The velocity field is described by the expression

$$\mathbf{v}_s = -(4\pi)^{-1}\gamma \sum_i \int [d\mathbf{s}_i \operatorname{grad}]\left(|\mathbf{R} - \mathbf{R}_i|^{-1} \exp\left(-\frac{|\mathbf{R} - \mathbf{R}_i|}{\lambda}\right)\right) \qquad (4)$$

where the penetration depth is $\lambda = (mc^2/4\pi N_s e^2)^{1/2}$.

c) The total interaction energy is equal to the sum of the interaction energy of all pairs of vortices

$$E = \frac{1}{2}\sum_{i \neq j} E_{ij},$$

where E_{ij} is the interaction energy of a single pair:

$$E_{ij} = \left(\frac{\varphi_0}{4\pi\lambda}\right)^2 \int\int d\mathbf{s}_i\, d\mathbf{s}_j |\mathbf{R}_i - \mathbf{R}_j|^{-1} \exp\left(-\frac{|\mathbf{R}_i - \mathbf{R}_j|}{\lambda}\right), \qquad (5)$$

$\varphi_0 = 2\pi\hbar c/2e$ is the magnetic-flux quantum.

Such a description corresponds to a temperature $T = 0$, since all the liquid is assumed to be superfluid. Moreover, the friction of the normal electrons of the vortex nucleus against the lattice is not taken into account. In the work of Nozières and Vinen,[7] it is shown that this is valid for sufficiently pure superconductors, when

$$\omega_{c2}\tau \gg 1, \qquad (6)$$

where $\omega_{c2} = eh_{c2}/mc$ is the cyclotron frequency in the field h_{c2}; τ is the relaxation time.

We shall assume in the following that $T \ll T_c$ (T_c is the temperature of transition in the superconducting state) and that the condition (6) is satisfied. On this basis, it is assumed that the perturbation due to the normal electrons is small and the motion of the vortices is described by the unperturbed equations (3)–(5). Energy dissipation and damping associated with such motion are considered below.

1. ENERGY DISSIPATION IN VORTEX NUCLEI IN THE PRESENCE OF OSCILLATIONS

We shall estimate the dissipation associated with the friction of the vortex nuclei against the lattice in the presence of oscillations of the vortex filaments. The vortex nucleus is regarded as a cylinder of normal metal of radius a, where $a \approx \xi$. The energy dissipation reduces here to ohmic losses in the nucleus. According to the model suggested in[7], the energy loss per unit length of filament per unit time will be

$$w = 2N_c m v_{nc}^2 \pi a^2 / \tau, \qquad (7)$$

where v_{nc} is the transport velocity of the normal electrons of the nucleus and N_c is their concentration.

Assuming that v_{nc} is identical with the velocity of motion of the nucleus, we obtain the following for the total energy dissipation per unit time:

*$[\mathbf{v}_s \mathbf{h}] \equiv \mathbf{v}_s \times \mathbf{h}$.

$$w_c = N_v L \cdot 2N_c m \langle v_c^2 \rangle \pi a^2 / \tau. \quad (8)$$

Here N_V is the total number of vortices, L the length of the sample, and $\langle v_c^2 \rangle$ the mean square velocity of the vortex nuclei in the oscillations. For oscillations with frequency ω:

$$\langle v_c^2 \rangle \approx \omega^2 \langle u^2 \rangle, \quad (9)$$

where $\langle u^2 \rangle$ is the mean square deviation of the vortex filaments from their equilibrium position. According to Fetter,[5]

$$\langle u^2 \rangle = \hbar (L N_v m N_s \gamma)^{-1} f(p_\parallel, p_\perp) 2n, \quad (10)$$

where

$$f(p_\parallel, p_\perp) = \frac{\Omega(p_\parallel, p_\perp) + (1 - \lambda^2 p_\parallel^2)(1 + p_\parallel^2 \lambda^2)^{-1} \eta(p_\parallel, p_\perp)}{\omega(p_\parallel, p_\perp)}$$

is some function expressed in terms of the lattice sums Ω, η, and ω, determined in[5]; $p_\parallel$ is the component of the wave vector of propagation of the oscillations **p**, directed along the magnetic field, while $p_\perp$ is the modulus of the projection of the vector **p** on the plane perpendicular to the magnetic field; n is the number of quanta of oscillations.

Taking it into account that $\pi a^2 h_{c2} \approx \pi^2 h_{c2} = \varphi_0$, we get from (8)—(10) the ratio of the energy losses in a period to the energy of the oscillations:

$$\frac{2\pi w_c \omega^{-1}}{\varepsilon} \sim \frac{1}{\omega_{c2}\tau} f(p_\parallel, p_\perp), \quad (11)$$

where ϵ is the energy of the oscillations ($\epsilon = n\hbar\omega$).

According to the table given in[5], we have, for oscillations propagating along the magnetic field ($p_\perp = 0$),

$$f(p_\parallel, 0) = 1, \quad (12)$$

and for oscillations propagating transverse to the magnetic field ($p_\parallel = 0$),

$$f(0, p_\perp) \sim 1 / p_\perp d \gg 1. \quad (13)$$

It follows from (11) and (12) that the energy dissipation of oscillations propagating along the magnetic field is small in comparison with the energy of oscillations in the case when the initial inequality (6) is satisfied.

For the possibility of the existence of oscillations propagating transverse to the magnetic field, it is required, according to (11) and (13), that there be a stronger condition on τ than the condition (6), namely,

$$\omega_{c2}\tau p_\perp d \gg 1, \quad (14)$$

therefore, there exists a region

$$1 / \omega_{c2} p_\perp d \gtrsim \tau \gg 1 / \omega_{c2}, \quad (15)$$

in which oscillations will be observed that propagate along the magnetic field, while oscillations which propagate transversely will not exist.

Such a situation arises because the amplitude of the oscillations which propagate transversely to the magnetic field is greater than the amplitude of the oscillations which propagate along it; therefore, the energy dissipation of these oscillations ($p_\parallel = 0$) becomes larger, in spite of the smallness of the force of friction.

2. NONDISSIPATIVE DAMPING

Besides the damping causing the dissipation in the oscillations, there is a damping of a nondissipative character, connected with the nonlinearity of the equations describing the shape of the vortex. The nonlinearity of the system leads to the result that for the presence of oscillations with a definite frequency, excitation of oscillations occurs at other frequencies, as a consequence of which the damping of the oscillations of the initial frequency takes place.

In the unperturbed state the vortices are parallel to the magnetic field and form a regular lattice. The direction of the magnetic field is taken to be the z axis. Then the position of the nucleus of the ith vortex is described by the vector function

$$\mathbf{R}_i(z) = (\mathbf{r}_i + \mathbf{u}_i(z), z), \quad (16)$$

where $\mathbf{R}_i^0(z) = (\mathbf{r}_i, z)$ corresponds to the unperturbed vortex, while the function $\mathbf{u}_i(z)$ gives the deviation from the equilibrium position. The vector $\mathbf{u}_i(z)$ lies in the (xy) plane.

Since the nucleus of the vortex is fixed in the liquid, the rate of change of the shape of the ith vortex $\mathbf{u}_i(z)$, is equal to

$$\dot{\mathbf{u}}_i(z) = \mathbf{v}_s^{(xy)}(\mathbf{r}_i + \mathbf{u}_i(z), z) - \frac{d\mathbf{u}_i}{dz} v_s^{(z)}(\mathbf{r}_i + \mathbf{u}_i(z), z), \quad (17)$$

where $\mathbf{v}_s^{(xy)}$ is the projection of the velocity on the (xy) plane, and $v_s^{(z)}$ on the z axis.

By varying the interaction energy of the vortices, and using (5), (4) and (17), we establish the fact that

$$\frac{\delta}{\delta \mathbf{u}_i(z)} \frac{1}{2} \sum_{i \neq j} E_{ij} = (m N_s \gamma)[\hat{\mathbf{z}} \dot{\mathbf{u}}_i(z)]. \quad (18)$$

Here $\hat{z}$ is the unit vector in the direction of the magnetic field. Introducing the new variables

$$q_i(z) = (N_s m \gamma)^{1/2} u_{xi}(z), \quad p_i(z) = (N_s m \gamma)^{1/2} u_{yi}(z), \quad (19)$$

we find that the equations of the change in shape of the vortices have canonical form, while q_i and p_i are conjugate variables, and $\mathcal{H} = \frac{1}{2}\sum_{i \neq j} E_{ij}$ is the Hamiltonian.

This property was already known to Kirchhoff for motion of a strictly linear vortex in a neutral liquid. Fetter[5,8], by setting $\mathbf{u} = \mathbf{v}_s$, obtained the result that the equations have a canonical form in first order in **u**, which corresponds to the harmonic approximation. The relation (18), which is valid for any order in **u**, allows us to pass on to the consideration of anharmonic interaction. We shall show, by following Fetter, that the motion can be quantized if we assume q_i and p_i to be quantum mechanical operators, which obey the Heisenberg equation of motion

$$i\hbar \dot{q}_i(z) = [q_i(z), \mathcal{H}], \quad i\hbar \dot{p}_i(z) = [p_i(z), \mathcal{H}]. \quad (20)$$

Here the correspondence principle requires that the operators satisfy the canonical commutation relations

$$[q_i(z_i), p_j(z_j')] = i\hbar \delta_{ij} \delta(z_i - z_j'). \quad (21)$$

We expand the Hamiltonian $\mathcal{H}$ in powers of **u**:

$$\mathcal{H} = \frac{1}{2} \sum_{i \neq j} E_{ij} = \mathcal{H}^{(0)} + \mathcal{H}^{(2)} + \mathcal{H}^{(3)} + \ldots, \quad (22)$$

where the upper index denotes the order of expansion in **u**. The term $\mathcal{H}^{(1)}$ is absent, since we are considering oscillations relative to the equilibrium location of the vortex lattice. The term $\mathcal{H}^{(2)}$ corresponds to harmonic oscillations considered by Fetter.[5] In order to take into account the anharmonic interaction, we consider $\mathcal{H}^{(3)}$:

$$\mathcal{H}^{(3)}=\frac{1}{2}\sum_{i\neq j}E_{ij}^{(3)}=\frac{1}{2}(4\pi\lambda)^{-1}mN_s\gamma^2\left(\frac{2}{\pi}\right)^{1/2}$$
$$\times\sum_{i\neq j}\iint dz_i\,dz_j\Big[-\frac{1}{6}\frac{(\mathbf{r}_{ij}\mathbf{u}_{ij})^3}{\lambda^6}\left(\frac{R_{ij}^0}{\lambda}\right)^{-7/2}K_{7/2}\left(\frac{R_{ij}^0}{\lambda}\right)$$
$$+\frac{1}{2}\frac{(\mathbf{r}_{ij}\mathbf{u}_{ij})\mathbf{u}_{ij}^2}{\lambda^4}\left(\frac{R_{ij}^0}{\lambda}\right)^{-5/2}K_{5/2}\left(\frac{R_{ij}^0}{\lambda}\right)$$
$$-\frac{(\mathbf{r}_{ij}\mathbf{u}_{ij})}{\lambda^2}\frac{d\mathbf{u}_i}{dz_i}\frac{d\mathbf{u}_j}{dz_j}\left(\frac{R_{ij}^0}{\lambda}\right)^{-3/2}K_{3/2}\left(\frac{R_{ij}^0}{\lambda}\right)\Big]. \qquad (23)$$

Here we have introduced the notation $\mathbf{r}_{ij}=\mathbf{r}_i-\mathbf{r}_j$; $\mathbf{u}_{ij}=\mathbf{u}_i(z_i)-\mathbf{u}_j(z_j)$, and $R_{ij}^0=((z_i-z_j)^2+\mathbf{r}_{ij}^2)^{1/2}$; K is the modified Bessel function of the second kind.

In order to account for the eigenenergy of the vortex filament, we assume that the vortex nucleus consists of a set of identical elementary filaments with vanishing circulations (but such that the total circulation around all the filaments is equal to γ) and with identical form $\mathbf{u}_{i\,el}=\mathbf{u}_{j\,el}$. Then, to obtain the eigenenergy, which reduces to the sum of the interaction energies of all the elementary filaments, it is necessary to average $E_{ij}^{(3)}$ over the area of the cross section of the nucleus. Assuming that the filaments are distributed in centrally symmetric fashion, we obtain the result that $(E_{ij}^{(3)})_{av}=0$, and, consequently, $\mathcal{H}_{eig}^{(3)}=0$.

Substituting the expansion of $\mathbf{u}_i$ in plane waves in (23) and writing the result in terms of the creation and annihilation operators with the help of the relations given in[5], we obtain the interaction Hamiltonian

$$\mathcal{H}^{(3)}=\sum_{s+t+u=0}\{a_s a_t a_u\mathcal{H}_0^{(3)}(\mathbf{s},\mathbf{t},\mathbf{u})+a_{-s}^{+}a_t a_u\mathcal{H}_1^{(3)}(\mathbf{s},\mathbf{t},\mathbf{u})+\mathrm{h.c.}\}. \quad (24)$$

We do not write out the explicit form of $\mathcal{H}_0^{(3)}$ and $\mathcal{H}_1^{(3)}$ because of their cumbersome nature.

To find the damping, we use the complete one-particle Green's function, which is defined in the interaction representation as

$$G(x-x')=-i\sum_{n=0}^{\infty}\frac{(-i)^n}{n!}\int_{-\infty}^{\infty}\ldots\int_{-\infty}^{\infty}dt_1\ldots dt_n\langle T[\psi(x)\psi^+(x')\mathcal{H}_{int}(t_1)\ldots$$
$$\ldots\mathcal{H}_{int}(t_n)]\rangle_c,$$

where

$$\psi=\frac{1}{\sqrt{V}}\sum_{\mathbf{p}}a_{\mathbf{p}}e^{i(\mathbf{pr}-\omega t)}. \qquad (25)$$

Using the anharmonic interaction of the excitations $\mathcal{H}^{(3)}$ found above as $\mathcal{H}_{int}$, we obtain, in second-order perturbation theory,

$$G^{-1}(\mathbf{p},\omega)=G^{0-1}(\mathbf{p},\omega)-\frac{2}{\hbar^2}\sum_{s}\frac{|\mathcal{H}_1^{(3)}(-\mathbf{s},\mathbf{s}-\mathbf{p},\mathbf{p})|^2}{\omega-\omega(\mathbf{s})-\omega(\mathbf{p}-\mathbf{s})+i\delta}, \quad (26)$$

where $G^0(\mathbf{p},\omega)=(\omega-\omega(\mathbf{p})+i\delta)^{-1}$.

The damping Γ is found from the imaginary correction to the pole of the Green's function. We consider the damping for the case of a dense vortex lattice, when $d\ll\lambda$. The lattice sums are replaced here by the corresponding integrals. It is seen from (26) that the damping is determined by pulses satisfying the decomposition condition

$$\omega(\mathbf{p})=\omega(\mathbf{s})+\omega(\mathbf{p}-\mathbf{s}). \qquad (27)$$

Substituting the explicit form of $\mathcal{H}_1^{(3)}$ in relation (26) and carrying out the necessary calculations, we obtain the following results.

For oscillations propagating along the magnetic field we get for the ratio of the imaginary correction Γ to the frequency of the oscillations: for the case of small momenta, when $p_{\parallel}\lambda\ll 1$,

$$\Gamma/\omega\sim d^2p_{\parallel}^5/N_s; \qquad (28)$$

in the intermediate case $(p_{\parallel}d\ll 1\ll p_{\parallel}\lambda)$

$$\Gamma/\omega\sim p_{\parallel}^2/N_s d; \qquad (29)$$

for large momenta $(p_{\parallel}d\gg 1)$

$$\Gamma/\omega\sim 1/N_s d^6 p_{\parallel}^3. \qquad (30)$$

For propagation transverse to the magnetic field: in the case of small momenta $(p_{\perp}\lambda\ll 1)$

$$\Gamma/\omega\sim p_{\perp}/N_s d^2; \qquad (31)$$

for the case of large momenta $(p_{\perp}\lambda\gg 1)$

$$\Gamma/\omega\sim 1/N_s\lambda^3 p_{\perp}d. \qquad (32)$$

In all these cases, the ratio of the imaginary correction to the frequency is extremely small and therefore the damping as a consequence of the nonlinearity can be distinguished against the background of dissipation damping only for extremely pure superconductors.

3. DAMPING OF OSCILLATIONS AT FINITE TEMPERATURES

The energy dissipation considered above for vortex nuclei and the nondissipative part of the damping of the oscillations depend weakly on the temperature. The temperature dependence is determined by the energy dissipation on normal electrons outside of the nucleus, the number density of which depends strongly on the temperature.

Oscillations of the vortex filaments are accompanied by oscillations of the magnetic field. Because of this, a vortex electric field appears, and the induction current of normal electrons associated with it; this leads to dissipation of the energy of the oscillations.

The energy loss per unit time per unit volume will be

$$w=\sigma\mathbf{E}^2\approx\sigma_n e^{-\Delta(T)/T}\mathbf{E}^2, \qquad (33)$$

where $\sigma_n=Ne^2/m$ is the conductivity of the metal in the normal state and $\Delta(T)$ is the width of the gap in the superconductor.

To find E, we take the curl of both sides of one of the Maxwell equations:

$$\mathrm{rot}\,\mathrm{rot}\,\mathbf{E}=-c^{-1}\partial\,\mathrm{rot}\,\mathbf{h}/\partial t. \qquad (34)$$

Combining (34) with the remaining Maxwell equations

$$\mathrm{rot}\,\mathbf{h}=\frac{4\pi}{c}\mathbf{j},\quad \mathrm{div}\,\mathbf{E}=0, \qquad (35)$$

we obtain

$$\Delta\mathbf{E}=\frac{4\pi}{c^2}\frac{\partial\mathbf{j}}{\partial t}. \qquad (36)$$

Hence

$$\mathbf{E}=-\frac{1}{c^2}\int\frac{\partial\mathbf{j}}{\partial t}R^{-1}dV=-\frac{\omega}{c^2}\int\mathbf{j}_1R^{-1}dV. \qquad (37)$$

where R is the length of the vector connecting the points of integration and observation and $\mathbf{j}_1=\omega^{-1}\partial\mathbf{j}/\partial t$.

We note that

$$j_l \to 0, \quad \text{if} \quad p \to 0, \tag{38}$$

since p = 0 corresponds to the unperturbed state, for which there is no oscillation current. From (33) and (37), we find the ratio of the energy dissipated per period of oscillation to the energy of oscillation:

$$\frac{2\pi w_n \omega^{-1}}{\varepsilon} = \tau\omega_1 e^{-\Delta/T}, \tag{39}$$

where ω_1 is some function of the wave vector p, and we obtain from (38) the result that

$$\omega_1(p) \to 0, \quad \text{if} \quad p \to 0. \tag{40}$$

It follows from (39) and (40) that by decreasing the temperature, the dissipation as the result of the induction current can be made small, at least, for long-wave oscillations.

In conclusion, the author takes this occasion to thank I. M. Khalatnikov, A. F. Andreev and S. V. Iordanskiĭ for valuable advice.

[1] A. A. Abrikosov, Zh. Eksp. Teor. Fiz. 32, 1442 (1957) [Sov. Phys.-JETP 5, 1174 (1957)].
[2] A. A. Abrikosov, M. P. Kemoklidze, and I. M. Khalatnikov, Zh. Eksp. Teor. Fiz. 48, 765 (1965) [Sov. Phys.-JETP 21, 506 (1965)].
[3] B. B. Goodman, Phys. Lett. 18, 3 (1965).
[4] A. L. Fetter, P. C. Hohenburg, and P. Pincus, Phys. Rev. 147, 140 (1966).
[5] A. L. Fetter, Phys. Rev. 163, 390 (1967).
[6] B. B. Goodman and J. Matricon, J. de Phys. 27, Suppl. C3, 39 (1966).
[7] P. Nozieres and W. F. Vinen, Phil. Mag. 14, 667 (1966).
[8] A. L. Fetter, Phys. Rev. 162, 143 (1967).

Translated by R. T. Beyer
42