NONCANONICAL CHANGE FROM ONE HAMILTONIAN TO ANOTHER

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A general theory of noncanonical transformations is constructed, which offers a possibility of finding essentially new invariance properties of physical systems. A connection is established between noncanonical change from one Hamiltonian to another and the replacement of the time by other parameters. It is shown that in quantum theory such a change has corresponding to it a method of quantization of physical quantities such as charge and mass, for example. As an example we consider the application of the general theory to the Coulomb interaction of two nonrelativistic particles. There is a unique relation of the new Hamiltonian to a 15-parameter group C_{15} which is different from the Malkin-Man'ko group. The group so introduced changes the interaction constant and makes it possible to proceed without solving the equations of motion for the two particles, and find the solutions by means of transformations from the known solutions for noninteracting particles.

1. INTRODUCTION

THE existing methods, based on the use of the usual Poisson brackets, for studying the commutation relations of observables with the Hamiltonian do not offer the possibility of discovering many relations characterizing invariance properties of physical systems. For example, this is indicated by the fact, not explainable by these methods, that the majority of the special functions applied in the exact solutions of physical problems have been found[1–4] to have their natural origins in groups. The theory of noncanonical changes to new Hamiltonians and new Poisson brackets, considered in this paper, allows us to find previously unknown invariance properties of physical systems and greatly extends the applicability of group methods to the solution of physical problems.

The invariant relations hold both in classical and in quantum theory, and they can be treated in the framework of a unified theory[5,6] of classical and quantum systems. Instead of this we shall verify them independently for corresponding classical and quantum cases. The examples treated in this paper are merely illustrative and do not exhaust the new approach to the problems touched upon.

2. INVARIANCE AND GROUP PROPERTIES OF PHYSICAL SYSTEMS AND THE GENERAL METHOD OF NONCANONICAL CHANGE TO NEW POISSON BRACKETS

The invariance properties of a physical system with respect to an r-parameter continuous group are characterized by the prescription of r observables $F_\lambda = F_\lambda(p, q)$ which form a Lie algebra:

$$[F_{\lambda_1}, F_{\lambda_2}] = \sum_{\lambda_3} c_{\lambda_1\lambda_2}^{\lambda_3} F_{\lambda_3}. \tag{2.1}$$

Here $c_{\lambda_1\lambda_2}^{\lambda_3}$ are the structure constants of the group, and [ ] are Poisson brackets; the respective formulas for these in the classical and quantum cases are

$$[F_1, F_2] = \sum_k \left(\frac{\partial F_1}{\partial p_k}\frac{\partial F_2}{\partial q_k} - \frac{\partial F_1}{\partial q_k}\frac{\partial F_2}{\partial p_k}\right), \tag{2.2}$$

$$[F_1, F_2] = \frac{i}{\hbar}(F_1F_2 - F_2F_1). \tag{2.3}$$

If $[H, F_\lambda] = 0$, then the group defined by (2.1) characterizes the invariance properties of the Hamiltonian H. In the more general case when H is one of the elements F_λ, and does not commute with the other elements, the relations (2.1) characterize the dynamical invariance of the physical system. If we introduce canonical parameters ξ^λ for the group and set $X = \sum_\lambda \xi^\lambda F_\lambda$, the dependence of an observable on the parameters ξ^λ is given by the equation

$$F(\xi^\lambda, p, q) = F(0, p, q) + \frac{1}{1!}[X, F(0, p, q)] + \frac{1}{2!}[X, [X, F(0, p, q)]] + \ldots \tag{2.4}$$

In the special case when $X = tH$, Eq. (2.4) gives the dependence of the classical and quantum observables on the time t.

The transition from the commutation relations (2.1) to the group transformations (2.4) is, as we know, not associated with the specific form of the Poisson brackets. All that is necessary is that the conditions

$$[F_1, F_2] \equiv -[F_2, F_1], \tag{2.5}$$

$$[F_1, [F_2, F_3]] + [F_2, [F_3, F_1]] + [F_3, [F_1, F_2]] \equiv 0. \tag{2.6}$$

be true identically for arbitrary F_1, F_2, F_3. It follows that if we can find new Poisson brackets, which are canonically nonequivalent to (2.2) and (2.3) (i.e., which change the canonical relations between the observables), but which satisfy the identities (2.5) and (2.6), then we get essentially new ways of studying the invariance properties of physical systems. In fact, in this case the possibility of satisfying (2.1) will depend on the choice of the Poisson brackets; that is, to each new choice there will in general correspond the emergence of different invariance-group properties of the physical system.

The general method of noncanonical change to new Poisson brackets, applicable to both the classical and the quantum cases, is as follows. Let T be a linear transformation (homomorphism) of the infinite-dimensional space formed from all possible observables F which are under consideration. We introduce new brackets by the formula

$$[F_1, F_2]_T = [TF_1, TF_2]. \tag{2.7}$$

The transformation T with inverse will be canonical if for arbitrary observables

$$[TF_1, TF_2] = T[F_1, F_2]. \tag{2.8}$$

In particular, let us examine the meaning of the transformation T for the quantum case, in which one takes for the observables F arbitrary Hermitian operators in a Hilbert space.[1)] If F_1 and F_2 are two observables, their linear combinations $c_1F_1 + c_2F_2$ (where c_1 and c_2 are real numbers) will also be observables, and therefore the set of all quantum-mechanical observables considered forms an infinite-dimensional linear space. The canonical transformation characterized by a unitary operator S takes an observable F over into $T_SF = SFS^{-1}$; such operators are also Hermitian and therefore belong to the linear space of observables. The canonical transformation T_S has an inverse (so that from $T_SF = 0$ it follows that $F = 0$) and is characterized by the property that if a Poisson bracket was not equal to zero before a transformation it will also be different from zero after the transformation. In contrast with this, a homomorphism or automorphism of (2.7) can be a noncanonical transformation, and in this case, as will be seen from later examples, it can take nonzero Poisson brackets over into zero.

Let us return to the general case. Let T be chosen so that the transformation is not canonical. We shall show that the identities (2.5) and (2.6) are satisfied.

The relation (2.5) for $[\,]_T$ follows from the definition (2.7). With the old brackets the Jacobi identity (2.6) is valid for arbitrary observables, and therefore it is still valid when F_1, F_2, F_3 are replaced by TF_1, TF_2, TF_3. Using the definition (2.7), we see that the new brackets satisfy (2.6), i.e., it is in fact permissible to take them as the new Poisson brackets.

3. NONCANONICAL CHANGE FROM ONE HAMILTONIAN TO ANOTHER IN CLASSICAL MECHANICS

Let us consider a noncanonical change of a special type in classical mechanics, which will be important for what follows. This change, as we shall see, allows us to discover new invariance properties of the Newtonian and Coulomb interactions, and furthermore the results so obtained can be carried over into quantum theory by means of the correspondence principle (associated with replacement of classical Poisson brackets by quantum Poisson brackets).

Let the classical Hamiltonian be a function of some constant K_0:

$$H = H(p, q, K_0). \tag{3.1}$$

We shall assume that (3.1) can be uniquely solved for K_0:

$$K_0 = \Phi(p, q, H), \tag{3.2}$$

and that conversely (perhaps with some supplementary restrictions) (3.1) follows from (3.2). We define the automorphism T as the transformation under which observables $F(p, q, K_0)$ go over into

$$TF\{p, q, K_0\} = F\{p, q, \Phi(p, q, H_0)\}, \tag{3.3}$$

where H_0 is a fixed number, the numerical value of the energy. The constant K_0 accordingly goes over into the function

$$K = \Phi(p, q, H_0). \tag{3.4}$$

Under the inverse transformation T^{-1} the number H_0 is again replaced by the function (3.1). Since with the change from (3.2) to (3.4) the Poisson bracket of H_0 and F becomes zero, the transformation (2.7) is here a noncanonical change to new Poisson brackets.

It follows from (3.2) that since

$$[H, F] = \sum_k \left(\frac{\partial F}{\partial p_k} \frac{dp_k}{dt} + \frac{\partial E}{\partial q_k} \frac{dq_k}{dt} \right) = \frac{dF}{dt}, \tag{3.5}$$

for an arbitrary observable F we have the identically true relation

$$[K_0, F] = \sum_k \left(\frac{\partial \Phi}{\partial p_k} \frac{\partial F}{\partial q_k} - \frac{\partial \Phi}{\partial q_k} \frac{\partial F}{\partial p_k} \right) + \frac{\partial \Phi}{\partial H} \frac{dF}{dt} \equiv 0. \tag{3.6}$$

On the other hand, according to (3.4),

$$[K, F] = \sum_k \left(\frac{\partial \Phi}{\partial p_k} \frac{\partial F}{\partial q_k} - \frac{\partial \Phi}{\partial q_k} \frac{\partial F}{\partial p_k} \right). \tag{3.7}$$

It follows from this that if $TF = F$, i.e., if F does not depend on K_0, then

$$[K_0, F]_T = [K, F] = -\frac{\partial K}{\partial H_0} \frac{dF}{dt}. \tag{3.8}$$

We now introduce instead of the time t an independent parameter τ, for which

$$d\tau = dt \Big/ \left(-\frac{\partial K}{\partial H_0} \right). \tag{3.9}$$

It then follows from (3.8) that

$$\frac{dF}{d\tau} = [K, F] = [K_0, F]_T, \tag{3.10}$$

so that it is possible to regard K as a new Hamiltonian corresponding to the new time τ. Therefore the transformation T, which corresponds to replacement of K_0 by K and H by the constant H_0, can be interpreted physically as a noncanonical change from one Hamiltonian to another.

As an example let us consider the case of a free relativistic particle, for which, as is well known,

$$H^2/c^2 - \mathbf{p}^2 = m_0^2c^2. \tag{3.11}$$

For the constant K_0 we here take $-mc^2$. Then from (3.11) we have

$$K = -(H_0^2 - c^2\mathbf{p}^2)^{1/2}, \quad \frac{\partial K}{\partial H_0} = -\frac{H_0}{(H_0^2 - c^2\mathbf{p}^2)^{1/2}}. \tag{3.12}$$

Using the fact that $c^2p^2/H_0^2 = v^2/c^2$, we find from (3.9)

$$d\tau = dt\sqrt{1 - v^2/c^2}, \tag{3.13}$$

so that the new parameter τ is the proper time of the relativistic particle.

This noncanonical transformation from one Hamiltonian to another can be generalized in various ways.

[1)] In the general case it is not necessary that the set of quantum-mechanical observables include arbitrary Hermitian operators; it suffices to require that this set constitute an algebra with the algebraic operators of Lie and Jordan multiplication.

4. NONCANONICAL CHANGE FROM ONE HAMILTONIAN TO ANOTHER IN QUANTUM THEORY

In quantum theory, where the observables F are Hermitian operators in Hilbert space, one could construct a general theory of noncanonical transformations for Hamiltonians corresponding to the classical theory expounded above. For brevity, however, we shall consider only a special form of noncanonical change, in which one can trace all of the main features of the new approach to the problem of studying the invariance properties of quantum systems.

Let the Hamiltonian operator H depend linearly on a constant K_0:

$$H = A(p, q) - K_0 B^{-1}(p, q), \tag{4.1}$$

where A and B^{-1} are Hermitian operators. If the system is in a state belonging to an eigenvalue H_0 of the operator H, then

$$H\psi(H_0, K_0) = H_0\psi(H_0, K_0). \tag{4.2}$$

Let us introduce a new function φ which depends on the parameters K_0 and H_0 and is connected with the wave function $\psi(H_0, K_0)$ by the formula

$$\psi(H_0, K_0) = B\varphi(H_0, K_0). \tag{4.3}$$

Then, when we use (4.1), Eq. (4.2) can be rewritten as a new eigenvalue equation:

$$K\varphi(H_0, K_0) = K_0\varphi(H_0, K_0), \tag{4.4}$$

where the operator K is defined by the formula

$$K = AB - H_0 B. \tag{4.5}$$

With respect to the usual definition of the norm the operator K is not Hermitian. The spectral theory of this sort of nonhermitian operators is known (see, for example, the monograph[7]), but if the operator B^{-1} is positive definite, so that $\langle\psi|B^{-1}\psi\rangle \geq 0$ for arbitrary ψ, it is possible to regard operators of the form (4.5) as Hermitian operators in a Hilbert space with a new definition of the norm. In fact, let us introduce the new norm with the definition

$$\langle\langle\varphi_1|\varphi_2\rangle\rangle = \langle\varphi_1|B\varphi_2\rangle = \langle B^{-1}\psi_1|\psi_2\rangle,$$

where $\varphi_1 = B^{-1}\psi_1$, $\varphi_2 = B^{-1}\psi_2$. Let X = YB, where Y and B are Hermitian operators with respect to the old norm. Then

$$\langle\langle\varphi_1|X\varphi_2\rangle\rangle = \langle\varphi_1|BYB\varphi_2\rangle = \langle YB\varphi_1|B\varphi_2\rangle = \langle\langle X\varphi_1|\varphi_2\rangle\rangle,$$

i.e., the operator X is Hermitian with respect to the new norm, and in studying it we can use the ordinary theory of Hermitian operators. The operator K is the result of multiplying the Hermitian operators $A - H_0$ and B, and therefore according to this demonstration it is Hermitian with respect to the new definition of the norm.

The change from (4.1) and (4.2) to (4.4) and (4.5) can be interpreted as a noncanonical change from the Hamiltonian operator H to the new Hamiltonian K. According to (4.3), the wave functions for the two alternative ways of writing the equations are mutually determined in terms of each other. In fact, if the solution of Eq. (4.2) is known, then according to (4.4) $\varphi = B^{-1}\psi$ will be the eigenfunction of the operator K, and conversely, if φ is an eigenfunction of the operator K, then $B\varphi$ will be the eigenfunction of the operator H. Nevertheless there is an essential difference in the statements of the eigenvalue problems. For the first Hamiltonian one regards the constant K_0 as given and looks for the spectrum of energy levels H_0. For the second Hamiltonian the energy H_0 remains fixed and one looks for the constants K_0 and the corresponding eigenfunctions φ. Along with this the existence of a dynamical invariance of the new Hamiltonian will lead to new group-theoretical methods for finding the wave functions and to a simplification of the solution of the ordinary wave equation (4.2). In particular, if by means of an appropriate transformation one can get solutions of (4.4) for $K_0 \neq 0$ from the solutions with $K_0 = 0$, this will provide a possibility of getting solutions of the ordinary wave equation (4.2) for $K_0 \neq 0$ from the solutions with $K_0 = 0$.

If in physical applications constants of the type of K_0 have the meaning of physical constants, for example charge or mass, this method of noncanonical change to a new Hamiltonian will lead to a way of quantizing them.

5. INTRODUCTION OF THE 15-PARAMETER GROUP C_{15} WHICH CHARACTERIZES THE INVARIANCE PROPERTIES OF THE COULOMB INTERACTION IN CLASSICAL AND QUANTUM MECHANICS

We shall apply the method of noncanonical change from one Hamiltonian to another to the case of the Coulomb interaction of two particles with charges e_1 and e_2 and reduced mass $m = m_1 m_2/(m_1 + m_2)$. Setting $K_0 = -e_1 e_2$, we have

$$H = p^2/2m - K_0/r. \tag{5.1}$$

It follows from (3.4) and (4.5) that in both classical and quantum mechanics the new Hamiltonian will be of the form

$$K = p^2 r/2m - H_0 r. \tag{5.2}$$

Taking as generators the new Hamiltonian K and the components r_s (s = 1, 2, 3) of the radius vector, and calculating all possible Poisson brackets expressed in terms of these generators, we find that K and r_s unambiguously generate a Lie algebra of 15 linearly independent elements X_λ. Using (2.4), one gets from this the local definition of a 15-parameter Lie group.

We can write out the generators X_λ of the group so found in explicit form for both classical and quantum theory. To make the forms the same we use the concept of the Jordan product of observables, which will be denoted by a dot:

$$F_1 \cdot F_2 = {}^1/_2 (F_1 F_2 + F_2 F_1). \tag{5.3}$$

It follows from (5.3) that in quantum theory, where the observables are Hermitian operators, the Jordan product of two observables is also a Hermitian operator, and in classical theory the Jordan product is the same as the ordinary product. The indices s take values 1, 2, 3 and indices α, β the values 1, 2, 3, 4, and raising and lowering of indices is done by means of $g_{\alpha\beta}$, where

$$g_{11}=g_{22}=g_{33}=-g_{44}=1,\quad g_{\alpha\beta}=0 \text{ for } \alpha\neq\beta. \qquad (5.4)$$

We set

$$M_{12}=p_2r_1-p_1r_2,\ldots,\quad M_{4s}=-p_sr,\quad S=\left(\mathbf{p}\cdot\frac{\mathbf{r}}{r}\right)r,$$

$$Q_s=r_s,\quad Q_4=-r,\quad P_s=-\left(p_s\cdot\left(\mathbf{p}\cdot\frac{\mathbf{r}}{r}\right)\right)r+\frac{1}{2}\left(p^2\cdot\frac{r_s}{r}\right)r,$$
$$P_4=-{}^1\!/_2p^2r \qquad (5.5)$$

and introduce the quantities

$$K_4=-\frac{1}{2}K=\frac{1}{2m}(P_4-mH_0Q_4),\quad K_s=\frac{1}{2m}(P_s-mH_0Q_s), \qquad (5.6)$$

$$L_4=-\frac{1}{2}L=\frac{1}{2m}(P_4+mH_0Q_4),\quad L_s=\frac{1}{2m}(P_s+mH_0Q_s). \qquad (5.7)$$

Then the generators X_λ will be the elements $M_{\alpha\beta}$, K_α, L_α, S, and one easily finds from (5.4)–(5.7) that in both classical and quantum theory they satisfy the following relations:

$$[M_{\alpha_1\alpha_2},M_{\beta_1\beta_2}]=g_{\alpha_1\beta_2}M_{\alpha_2\beta_1}+g_{\alpha_2\beta_1}M_{\alpha_1\beta_2}-g_{\alpha_1\beta_1}M_{\alpha_2\beta_2}-g_{\alpha_2\beta_2}M_{\alpha_1\beta_1},$$
$$[M_{\alpha_1\alpha_2},K_\beta]=g_{\alpha_2\beta}K_{\alpha_1}-g_{\alpha_1\beta}K_{\alpha_2}, \qquad (5.8)$$

$$[K_\alpha,K_\beta]=-\frac{H_0}{2m}M_{\alpha\beta},\quad [S,M_{\alpha\beta}]=0, \qquad (5.9)$$

$$[M_{\alpha_1\alpha_2},L_\beta]=g_{\alpha_2\beta}L_{\alpha_1}-g_{\alpha_1\beta}L_{\alpha_2},\quad [L_\alpha,L_\beta]=\frac{H_0}{2m}M_{\alpha\beta}, \qquad (5.10)$$

$$[K_\alpha,L_\beta]=-\frac{H_0}{2m}g_{\alpha\beta}S,\quad [S,K_\alpha]=-L_\alpha,\quad [S,L_\alpha]=-K_\alpha. \qquad (5.11)$$

It follows from these relations that the group we have introduced, which characterizes the dynamical properties of the new Hamiltonian K, is locally isomorphic to the 15-parameter group C_{15} of the conformal transformations of a four-dimensional pseudoeuclidean space.

At the time when the writer found this 15-parameter group C_{15}, which characterizes new invariance properties of the nonrelativistic Coulomb interaction, a number of papers were published[8-13] devoted to the broadening of the group characterizing the invariance properties of the Coulomb interaction with the usual choice of the Hamiltonian. The most interesting results are those of Malkin and Man'ko,[13] who introduced a 15-parameter group whose generators allow one to raise and lower the energy levels for the Hamiltonian (5.1). But although the Malkin-Man'ko group is also locally isomorphic to the group of conformal transformations C_{15}, it is decidedly different from our group. Whereas our group does not change the energy H_0 but changes the Coulomb interaction constant e_1e_2, i.e., takes the system from states with one value of this constant into states with another value of the charges, the Malkin-Man'ko group does not change the charges, but takes the system into states with different energy values. Furthermore it is not hard to verify that the invariants formed from corresponding generators of our group and the Malkin-Man'ko group are not the same, and therefore the abstract mathematical properties of the two groups are also different.

As an example, for the classical case we take the subgroup with the generators K, L, S, which is locally isomorphic to the group of rotations of a three-dimensional pseudoeuclidean space. We can always choose the one-parameter transformation with the generator S, which appears in this subgroup, so that for $H_0>0$ the numerical value of K is zero, and for $H_0<0$ the numerical value of L is zero. Therefore from the physical quantities characterizing the rectilinear motion of two noninteracting particles we obtain by means of a transformation with the generator S physical quantities which describe the Coulomb interaction of two particles with the same value of H_0.

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