FINITE-AMPLITUDE DRIFT WAVES IN AN UNSTABLE PLASMA

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The nonlinear equations that describe drift waves in a collision-dominated inhomogeneous plasma in a strong magnetic field are investigated. It is shown that an asymptotic stationary drift wave can be established in such a plasma as a result of the competition between nonlinear damping and linear excitation.

1. A number of recent experiments carried out with Q-machines[1] indicate the existence of stationary drift waves whose frequency is in agreement with the predictions of the linear theory. However, the relative amplitudes of the drift wave are by no means infinitesimally small. For this reason it is of interest to investigate nonlinear stationary drift waves that can propagate in an unstable plasma.

Drift waves in an ideal plasma have been investigated by Petviashvili[2] who obtained a solution in the form of a simple Riemann wave. One of the characteristics of such a wave is the fact that the curvature of the leading edge increases in the course of time. This phenomenon can be thought of as the enhancement of the higher harmonics associated with the initially sinusoidal profile.[3] If these nonlinear distortions of the wave are rapid enough, when viscosity is taken into account the damping of the finite amplitude perturbations will be strongest at the higher harmonics. This damping will be called nonlinear damping.

It is well known (cf. for example[4]) that when friction between electrons and ions is taken into account an inhomogeneous plasma can be unstable against excitation of low-amplitude drift waves. One expects that under these conditions the competition between the nonlinear damping and the linear excitation in an inhomogeneous plasma will lead to the establishment of an asymptotic stationary drift wave of finite amplitude with a definite spectral composition.

The spectral composition of a stationary wave, that is to say its profile, will be determined, in particular, by the magnitude of the viscosity. If the viscosity is strong enough, the majority of the higher harmonics will be highly damped so that the profile is determined primarily by the first few harmonics.

In the present work, making certain simplifying assumptions we present a solution for the two-fluid hydrodynamic equations which corresponds to the limiting case of high viscosity mentioned above.

2. We start with the two-fluid model of a fully ionized gas, which is described by the following equations:

$$m_i \frac{d_i \mathbf{v}_i}{dt} = -e\nabla\varphi + \frac{e}{c}[\mathbf{v}_i, \mathbf{H}] + \nu_\perp \nabla_\perp^2 \mathbf{v}_{i\perp} + \nu_{ei} m_e (\mathbf{v}_e - \mathbf{v}_i), \quad (1)^*$$

$$0 = e\nabla\varphi - \frac{e}{c}[\mathbf{v}_e, \mathbf{H}] - \frac{T_e}{n}\nabla n - \nu_{ei} m_e (\mathbf{v}_e - \mathbf{v}_i), \quad (2)$$

$$\frac{\partial n}{\partial t} + \operatorname{div} n\mathbf{v}_e = 0, \quad (3)$$

$$\frac{\partial n}{\partial t} + \operatorname{div} n\mathbf{v}_i = 0, \quad (4)$$

where φ is the electrostatic potential; **H** is the fixed uniform magnetic field; n is the number density of the electrons or ions; $\mathbf{v}_{i,e}$ and $m_{i,e}$ are the mean velocity and the mass of the ions or electrons; $v_{i\perp}$ is the ion velocity component perpendicular to the magnetic field; ν_{ei} is the electron-ion collision frequency, $\nu_\perp = (3/10)T_i/\Omega_i^2 T_i$ is the coefficient of transverse ion viscosity (T_i is the ion temperature, Ω_i is the ion-cyclotron frequency, τ_i is the mean ion-ion collision time); T_e is the electron temperature; e is the electron charge; c is the velocity of light; $d_i/dt \equiv \partial/\partial t + \mathbf{v}_i\nabla$ is the Lagrangian derivative for the ion gas and t is the time.

Equations (1)–(4) describe a neutral plasma under conditions such that the electron inertia, the ion pressure, and the distortion of the magnetic field can be neglected. Of the large number of dissipative mechanisms contained in Eqs. (1)–(4) we shall be interested only in the electron-ion friction and the transverse ion viscosity. The absence of a longitudinal viscosity in Eq. (1) is associated with the fact that we are only considering perturbations that are highly elongated along the magnetic field, that is to say, perturbations that satisfy the condition $k_z^2/k_\perp^2 < (\Omega_i\tau_i)^2$ where $k_\perp$ and k_z are the characteristic wave numbers transverse to and parallel to H.

If all dissipative terms and transverse ion inertia are neglected in Eqs. (1)–(4), we obtain a system of equations that describe a simple Riemann wave.[2] An important feature of the system of equations is the fact that they can describe the nonlinear distortion; we will call this set of equations the zeroth approximation. For reasons of simplicity, all other effects will be considered only in the first-order approximation. It is found, however, that this procedure allows us to find stationary periodic solutions.

At the very outset, we shall isolate terms associated with the initial inhomogeneity of the plasma in Eqs. (1)–(4). For this purpose the plasma density is written in the form

$$n(\mathbf{r}, t) = n_0(\mathbf{r}) n'(\mathbf{r}, t), \quad (5)$$

where $n_0(\mathbf{r})$ is the stationary average density, which is taken to be a known function of the coordinates, while $n'(\mathbf{r}, t)$ is a new unknown dimensionless function. Substi-

*$[\mathbf{v}_i, \mathbf{H}] \equiv \mathbf{v}_i \times \mathbf{H}$.

tuting Eq. (5) in Eqs. (1)—(4) we have

$$m_i\frac{d_i\mathbf{v}_i}{dt} = -e\nabla\varphi + \frac{e}{c}[\mathbf{v}_i, \mathbf{H}] + \nu_\perp\nabla_\perp{}^2\mathbf{v}_{i\perp} + \nu_{ei}m_e(\mathbf{v}_e - \mathbf{v}_i), \quad (6)$$

$$0 = e\nabla\varphi - \frac{e}{c}[\mathbf{v}_e, \mathbf{H}] - T_e\nabla\ln n_0 - T_e\nabla\ln n' - \nu_{ei}m_e(\mathbf{v}_e - \mathbf{v}_i), \quad (7)$$

$$\frac{\partial\ln n'}{\partial t} + (\mathbf{v}_e\nabla)\ln n_0 + (\mathbf{v}_e\nabla)\ln n' + \operatorname{div}\mathbf{v}_e = 0, \quad (8)$$

$$\frac{\partial\ln n'}{\partial t} + (\mathbf{v}_i\nabla)\ln n_0 + (\mathbf{v}_i\nabla)\ln n' + \operatorname{div}\mathbf{v}_i = 0. \quad (9)$$

If dissipation is neglected then, as is well known, the plasma density is described by a Boltzmann distribution. When electron-ion friction is introduced we obtain a correction to the density

$$n' = \exp\{e\varphi/T_e\}(1 + n_1), \quad (10)$$

where n_1 is the correction that indicates deviation from the Boltzmann distribution. The equation for the correction n_1 can be obtained without difficulty if one makes use of the linearized electron equation of continuity (8) and the equation for the longitudinal motion (7) and takes account of the transverse electric drift:

$$\left(\frac{\partial^2}{\partial t^2} - D_e\frac{\partial^3}{\partial t\,\partial z^2} - c_s{}^2\frac{\partial^2}{\partial z^2}\right)n_1 = \frac{e}{T_e}\left(-\frac{\partial^2}{\partial t^2} - v_{\text{дp}}\frac{\partial^2}{\partial t\,\partial y} + c_s{}^2\frac{\partial^2}{\partial z^2}\right)\varphi, \quad (11)$$

where

$$c_s{}^2 = \frac{T_e}{m_i}, \quad v_{\text{dr}} = -\frac{c_s{}^2}{\Omega_i}\frac{\partial}{\partial x}\ln n_0; \quad \Omega_i = \frac{eH}{m_ic}; \quad D_e = \frac{T_e}{m_e\nu_{ei}}.$$

In deriving Eq. (11) and in the material below we shall be assuming a plane geometry in which the magnetic field is along the z-axis and in which the density varies along the x-axis. To introduce further generality, in deriving Eq. (11) we have also taken account of the longitudinal ion motion, although in the linear approximation.

Assuming that the magnetic field is very strong we can obtain an expression for the transverse ion velocity:

$$\mathbf{v}_{i\perp} = \frac{1}{\Omega_i{}^2}\left\{[\Omega_i, \nabla\psi] - \nabla_\perp\frac{\partial\psi}{\partial t} + \nu_\perp\nabla_\perp\Delta_\perp\psi\right\}, \quad \psi = \frac{e}{m_i}\varphi. \quad (12)$$

Substituting Eqs. (10) and (12) in Eq. (9) we obtain the following system of equations in addition to (11) and the equation for the z component (6):

$$\frac{\partial\psi}{\partial t} + v_{\text{дp}}\frac{\partial\psi}{\partial y} - \frac{c_s{}^2}{\Omega_i{}^2}\Delta_\perp\left(\frac{\partial}{\partial t} - \nu_\perp\Delta_\perp\right)\psi + v_z\frac{\partial\psi}{\partial z} + c_s{}^2\frac{\partial v_z}{\partial z} + \frac{\partial\psi_1}{\partial t} = 0, \quad (13)$$

$$\frac{\partial v_z}{\partial t} + v_z\frac{\partial v_z}{\partial z} = -\frac{\partial\psi}{\partial z}, \quad (14)$$

$$\left(\frac{\partial^2}{\partial t^2} - D_e\frac{\partial^3}{\partial t\,\partial z^2} - c_s{}^2\frac{\partial^2}{\partial z^2}\right)\psi_1 = \left(-\frac{\partial^2}{\partial t^2} - v_{\text{дp}}\frac{\partial^2}{\partial t\,\partial y} + c_s{}^2\frac{\partial^2}{\partial z^2}\right)\psi, \quad (15)$$

where $\psi_1 = Tn_1/m_i$ while (14) is the equation for the longitudinal ion motion, in which we have neglected the frictional term. When ν and $\psi_1 \to 0$ Eqs. (13) and (14) become the system of equations that describe the nonlinear drift waves with dispersion taken into account.[2]

We now seek a solution for Eqs. (13)—(15) in the form of a stationary wave as $t \to \infty$:

$$v_z = v_z(\xi), \quad \psi = \psi(\xi), \quad \psi_1 = \psi_1(\xi), \quad \xi = \mathbf{kr} - \omega t, \quad (16)$$

where $\mathbf{k}$ and ω are the mean wave vector and frequency. Since we are only considering perturbations that are elongated along $\mathbf{H}$, in Eqs. (13)—(15) we can neglect the effect of the longitudinal ion motion on the linear growth rate, that is to say, we neglect terms of the form $c_s^2\,\partial/\partial z$. Substitution of Eq. (16) in Eqs. (13)—(15) and the elimination of ψ and ψ_1 then yields

$$\omega D_ek_z{}^2(-\omega + \omega_*)\frac{dv_z}{d\xi} + b\omega^3\frac{d^2v_z}{d\xi^2} + b\omega^2\nu_\perp k_\perp{}^2\frac{d^3v_z}{d\xi^3} + D_ek_z{}^2b\omega^2\frac{d^3v_z}{d\xi^3} + D_ek_z{}^2b\omega\nu_\perp k_\perp{}^2\frac{d^4v_z}{d\xi^4} + \omega[{}^1/_2(2\omega - \omega_*)k_zv_z{}^2 - {}^1/_3k_z{}^2v_z{}^3] + D_ek_z{}^2\frac{d}{d\xi}\left[\frac{1}{2}(2\omega - \omega_*)k_zv_z{}^2 - \frac{1}{3}k_z{}^2v_z{}^3\right] = 0, \quad (17)$$

where $b = c_s^2k_\perp^2/\Omega^2$; $\omega_* = k_yv_{dr}$ is the drift frequency.

Proceeding in the same way as in the linear analysis we can show that the first term on the left side of Eq. (17) describes the linear drift waves, the second term describes the linear excitation, the third is small, the fourth is associated with dispersion effects, the fifth represents the viscous damping and the last two terms are due to the nonlinear deformation of the drift waves. In order to simplify the subsequent analysis we shall assume that the plasma is not very dense (so that the value of D_e is reasonably large); in this case we can neglect the third and sixth terms in Eq. (17). The cubic factor in the seventh term can also be neglected. Furthermore, we shall neglect dispersion effects, that is to say, in Eq. (17) we neglect the fourth term, which is small when $b = k_\perp^2r_H^2 \ll 1$ where r_H is the ion-Larmor radius corresponding to the electron temperature. In this way finally we obtain a nonlinear ordinary differential equation of fourth order:

$$b\nu_\perp k_\perp{}^2\frac{d^4v_z}{d\xi^4} + \frac{b\omega^2}{D_ek_z{}^2}\frac{d^2v_z}{d\xi^2} + \frac{d}{d\xi}\left[(-\omega + \omega_*)v_z + \frac{2\omega - \omega_*}{2\omega}k_zv_z{}^2\right] = 0. \quad (18)$$

3. We now consider the case in which the viscosity is high so that the basic contribution to the profile comes from the first few harmonics. We seek a solution of Eq. (18) in the form of a summation of three harmonics:

$$v_z = v_1\sin\xi + v_2\sin 2\xi + v_3\sin 3\xi. \quad (19)$$

Substituting Eq. (19) in Eq. (18), expanding the nonlinear term in a trigonometric series and equating the coefficient of $\cos\xi$ equal to zero, as the zeroth approximation we find

$$\omega \approx \omega_*. \quad (20)$$

Now, equating the coefficients of $\sin\xi$, $\sin 2\xi$ and $\sin 3\xi$ to zero taking account of Eq. (20), we obtain three algebraic equations that determine the amplitudes v_1, v_2 and v_3:

$$\begin{aligned} &-v_12b\left(\frac{\omega_*{}^2}{D_ek_z{}^2} - \nu_\perp k_\perp{}^2\right) - k_zv_2(v_1 + v_3) = 0, \\ &-v_22b\left(\frac{4\omega_*{}^2}{D_ek_z{}^2} - 16\nu_\perp k_\perp{}^2\right) + k_zv_1(v_1 - 2v_3) = 0, \\ &-v_32b\left(\frac{9\omega_*{}^2}{D_ek_z{}^2} - 81\nu_\perp k_\perp{}^2\right) + 3k_zv_1v_2 = 0. \end{aligned} \quad (21)$$

From Eq. (21) we have

$$k_z v_3 = -\frac{k_z^2 v_1 v_2}{6b(9\nu_\perp k_\perp^2 - \omega_*^2/D_e k_z^2)}, \tag{22}$$

$$k_z^2 v_1^2 = 8b\left(\frac{\omega_*^2}{D_e k_z^2} - 4\nu_\perp k_\perp^2\right) k_z v_2 \left[1 + k_z v_2 \Big/ 3b\left(9\nu_\perp k_\perp^2 - \frac{\omega_*^2}{D_e k_z^2}\right)\right]^{-1} \tag{23}$$

$$k_z v_2 = b\left\{3\left(9\nu_\perp k_\perp^2 - \frac{\omega_*^2}{D_e k_z^2}\right) \pm \left[9\left(9\nu_\perp k_\perp^2 - \frac{\omega_*^2}{D_e k_z^2}\right)^2 + 12\left(\frac{\omega^*}{D_e^* k_z^2} - \nu_\perp k_\perp^2\right)\left(9\nu_\perp k_\perp^2 - \frac{\omega_*^2}{D_e k_z^2}\right)\right]^{1/2}\right\}. \tag{24}$$

We now assume that the viscosity lies in the range defined by

$$1 > \nu_\perp k_\perp^2 D_e k_z^2/\omega_*^2 \geqslant 1/4. \tag{25}$$

Near the upper bound of the inequality in (25), that is to say, near the threshold for excitation of the first harmonic, from Eqs. (22)–(24) we have

$$k_z^2 v_1^2 = \frac{48 b^2 \omega^{*2}}{D_e k_z^2}\left(\frac{\omega_*}{D_e k_z^2} - \nu_\perp k_\perp^2\right), \tag{26}$$

$$k_z^2 v_2^2 = 4b^2\left(\frac{\omega_*^2}{D_e k_z^2} - \nu_\perp k_\perp^2\right)^2, \tag{27}$$

where we have taken the minus sign in front of the radical in Eq. (24). It is evident that the nonlinearity leads to a saturation of the amplitude of the stationary wave; in the present case this wave is essentially harmonic since the quantities v_2 and v_3 are higher order quantities. Furthermore, it is evident that as the friction is reduced ($D_e \to \infty$), in which case the growth rate for small perturbations is reduced, the amplitude of the stationary drift wave also tends to vanish: because of the strong viscous damping the amplitude cannot increase significantly.

It is evident from Eq. (23) for the amplitude v_1 that this amplitude vanishes at the lower bound of the inequality in (25), that is to say, it is a function with a maximum with respect to the transverse viscosity which, in turn, is determined by the magnetic field. It is easy to show that the amplitude of the second harmonic approaches a finite value at the lower limit of (25). This means that regardless of the linear excitation, the amplitude of the first harmonic is found to be very close to zero and this result is evidently due to the strong nonlinear distortion which implies a rapid transfer of energy from the first harmonic to the second.

The description given above is found to be in agreement with the experimental data of Hendel et al[1] who, in particular, have investigated the dependence of the relative amplitude of the drift waves on magnetic field for various modes. However, it should be noted that these experiments are carried out in an apparatus with end plates so that any quantitative comparison requires that the effect of these end plates be taken into account.

It is evident that taking account of nonlinear effects and effects that are associated with the excitation of the instability leads to a plasma model in which one can explain the stationary drift waves that have been observed in a number of experimental devices.

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