THE GEOMETRIC OPTICS APPROXIMATION IN THE GENERAL CASE OF INHOMOGENEOUS AND NONSTATIONARY MEDIA WITH FREQUENCY AND SPATIAL DISPERSION

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An asymptotic solution of Maxwell's equations is obtained by the geometric optics method for the general case of a weakly nonstationary and weakly inhomogeneous medium with frequency and spatial dispersion and also weak absorption. Waves in an anisotropic medium as well as transverse and longitudinal waves in an isotropic medium are considered. It is shown that a term due to the nonstationarity of the medium appears in the energy conservation equation; terms also appear which are due to the presence of frequency dispersion in a nonstationary medium and to spatial dispersion in an inhomogeneous medium.

THE propagation of electromagnetic waves in inhomogeneous and nonstationary media with frequency and spatial dispersion is described by Maxwell's equations

$$\operatorname{rot}\mathbf{H}=\frac{1}{c}\frac{\partial\mathbf{D}}{\partial t},\quad \operatorname{rot}\mathbf{E}=-\frac{1}{c}\frac{\partial\mathbf{B}}{\partial t},\tag{1}$$

which must be solved together with the material equation

$$D_\alpha(\mathbf{r},t)=\int_{-\infty}^{t}dt'\int_{|\mathbf{r}-\mathbf{r}'|\leqslant c(t-t')}d\mathbf{r}'\,\varepsilon_{\alpha\beta}(t-t',t;\mathbf{r}-\mathbf{r}',\mathbf{r})E_\beta(\mathbf{r}',t')\tag{2}$$

[it is well known that the magnetic permeability tensor can be assumed to be equal to $\delta_{\alpha\beta}$ so that in Eqs. (1) $\mathbf{H}=\mathbf{B}$]. The inhomogeneity and nonstationarity of the medium manifests itself in that the dielectric permittivity tensor $\epsilon_{\alpha\beta}$ depends not only on the differences $t-t'$ and $\mathbf{r}-\mathbf{r}'$ but also on t and $\mathbf{r}$. The specific form of this tensor is determined by considering the microscopic processes in the medium.

Maxwell's equations can be solved by the methods of geometric optics in the case of weakly nonstationary and weakly inhomogeneous media under the condition that the wave is "almost plane" and "almost monochromatic." If T and L are quantities characterizing the nonstationarity and inhomogeneity of the medium (i.e., characteristic scales of the variation of $\epsilon_{\alpha\beta}$ over the variables t and $\mathbf{r}$), then for the method of geometric optics to be applicable it is necessary that the following inequalities be fulfilled:

$$T\gg\bar{\tau},\quad L\gg\bar{\lambda},\tag{3}$$

where $\bar{\tau}$ is some average period and $\bar{\lambda}$—the average spatial scale of the field. When conditions (3) are fulfilled the wave can be assumed to be plane and monochromatic over intervals $\Delta t\sim\bar{\tau}$ and distances $\Delta r\sim\bar{\lambda}$.

With the additional condition

$$T\gg\tau_0,\quad L\gg\lambda_0,\tag{4}$$

where τ_0 and λ_0 characterize the frequency and spatial dispersion (i.e., τ_0 and λ_0 determine the scale of variation of $\epsilon_{\alpha\beta}$ as a function of the variables $t-t'$ and $\mathbf{r}-\mathbf{r}'$) the wave can also be considered plane and monochromatic within the space-time region $\Delta t\sim\tau_0$ and $\Delta r\sim\lambda_0$ important for the integration in (2). It is important to note that conditions (3) and (4) make it possible to consider, just as in the scalar problem,[1] both weak ($\tau_0\ll\bar{\tau}$, $\lambda_0\ll\bar{\lambda}$) and strong ($\tau_0\gg\bar{\tau}$, $\lambda_0\gg\bar{\lambda}$) frequency and spatial dispersion.

We shall seek a solution of Eqs. (1) in the form

$$\mathbf{E}=\tilde{\mathbf{E}}e^{i\varphi},\quad \mathbf{H}=\tilde{\mathbf{H}}e^{i\varphi},\quad \mathbf{D}=\tilde{\mathbf{D}}e^{i\varphi},\tag{5}$$

considering the amplitudes $\tilde{\mathbf{E}}$, $\tilde{\mathbf{H}}$, $\tilde{\mathbf{D}}$, as well as the quantities $\nabla\varphi$ and $\partial\varphi/\partial t$, to change slowly over intervals $\Delta t\sim\bar{\tau}$ and τ_0 and distances $\Delta r\sim\bar{\lambda}$ and λ_0. Making use of inequalities (4), we write for $E_\beta(\mathbf{r}',t')$ the following approximate expression obtained by expanding the amplitude and phase in powers of $t'-t$ and $\mathbf{r}'-\mathbf{r}$:

$$\begin{aligned}E_\beta(\mathbf{r}',t')\cong{}&\exp[i\varphi+i\omega(t-t')-i\mathbf{k}(\mathbf{r}-\mathbf{r}')]\\&\cdot\Big\{\tilde{E}_\beta+\frac{\partial\tilde{E}_\beta}{\partial x_j}(x_j'-x_j)+\frac{\partial\tilde{E}_\beta}{\partial t}(t'-t)\\&+\frac{i\tilde{E}_\beta}{2}\Big[(x_j'-x_j)(x_m'-x_m)\frac{\partial^2\varphi}{\partial x_j\partial x_m}\\&+2(t'-t)(x_j'-x_j)\frac{\partial^2\varphi}{\partial t\,\partial x_j}+(t'-t)^2\frac{\partial^2\varphi}{\partial t^2}\Big]+\ldots\Big\},\end{aligned}\tag{6}$$

where $\tilde{E}_\beta=\tilde{E}_\beta(\mathbf{r},t)$, $\varphi=\varphi(\mathbf{r},t)$, and where we have introduced the notation:

$$\mathbf{k}=\mathbf{k}(\mathbf{r},t)\equiv\nabla\varphi(\mathbf{r},t),\quad \omega=\omega(\mathbf{r},t)\equiv-\partial\varphi(\mathbf{r},t)/\partial t.\tag{7}$$

Defining the complex tensor $\epsilon_{\alpha\beta}(\omega,t;\mathbf{k},\mathbf{r})$ as

$$\varepsilon_{\alpha\beta}(\omega,t;\mathbf{k},\mathbf{r})=\int_{-\infty}^{t}dt'\int d\mathbf{r}'\,\varepsilon_{\alpha\beta}(t-t',t;\mathbf{r}-\mathbf{r}',\mathbf{r})e^{i\omega(t-t')-i\mathbf{k}(\mathbf{r}-\mathbf{r}')},$$

we obtain from (2) with the aid of (6)

$$\begin{aligned}\tilde{D}_\alpha={}&\varepsilon_{\alpha\beta}\tilde{E}_\beta-i\frac{\partial\tilde{E}_\beta}{\partial x_j}\frac{\partial\varepsilon_{\alpha\beta}}{\partial k_j}+i\frac{\partial\tilde{E}_\beta}{\partial t}\frac{\partial\varepsilon_{\alpha\beta}}{\partial\omega}\\&-\frac{i\tilde{E}_\beta}{2}\Big\{\frac{\partial^2\varphi}{\partial x_j\partial x_m}\frac{\partial^2\varepsilon_{\alpha\beta}}{\partial k_j\partial k_m}-2\frac{\partial^2\varphi}{\partial t\,\partial x_j}\frac{\partial^2\varepsilon_{\alpha\beta}}{\partial\omega\,\partial k_j}+\frac{\partial^2\varphi}{\partial t^2}\frac{\partial^2\varepsilon_{\alpha\beta}}{\partial\omega^2}\Big\}+\ldots\end{aligned}\tag{8}$$

By virtue of (3) and (4) the terms with the factor i are small compared with $\epsilon_{\alpha\beta}\tilde{E}_\beta$.

Let us introduce weak absorption in the medium. To this end we separate from $\epsilon_{\alpha\beta}(\omega,t;\mathbf{k},\mathbf{r})$ the antihermitian part $\nu_{\alpha\beta}=(\epsilon_{\alpha\beta}-\epsilon^*_{\beta\alpha})/2$ which we shall assume to be small compared with the hermitian part $(\epsilon_{\alpha\beta} + \epsilon^*_{\beta\alpha})/2$ for which we retain the notation $\epsilon_{\alpha\beta}$.

Substituting (5) and (8) in (1), we obtain in the zeroth approximation the system of equations

$$i[\mathbf{k}\tilde{\mathbf{H}}^0]_\alpha + i\frac{\omega}{c}\varepsilon_{\alpha\beta}\tilde{E}_\beta{}^0 = 0, \qquad i[\mathbf{k}\tilde{\mathbf{E}}^0]_\alpha - i\frac{\omega}{c}\tilde{H}_\alpha{}^0 = 0, \tag{9*}$$

whereas the amplitude of the first approximation satisfies the system

$$i[\mathbf{k}\tilde{\mathbf{H}}^1]_\alpha + i\frac{\omega}{c}\varepsilon_{\alpha\beta}\tilde{E}_\beta{}^1 = -\operatorname{rot}_\alpha \tilde{\mathbf{H}}^0 + \frac{1}{c}\frac{\partial}{\partial t}(\varepsilon_{\alpha\beta}\tilde{E}_\beta{}^0)$$
$$-\frac{\omega}{c}\frac{\partial \tilde{E}_\beta{}^0}{\partial x_j}\frac{\partial \varepsilon_{\alpha\beta}}{\partial k_j} + \frac{\omega}{c}\frac{\partial \tilde{E}_\beta{}^0}{\partial t}\frac{\partial \varepsilon_{\alpha\beta}}{\partial \omega} - i\frac{\omega \nu_{\alpha\beta}}{c}\tilde{E}_\beta{}^0$$
$$-\frac{\omega \tilde{E}_\beta{}^0}{2c}\left\{\frac{\partial^2 \varphi}{\partial x_j \partial x_m}\frac{\partial^2 \varepsilon_{\alpha\beta}}{\partial k_j \partial k_m} - 2\frac{\partial^2 \varphi}{\partial t\, \partial x_j}\frac{\partial^2 \varepsilon_{\alpha\beta}}{\partial \omega\, \partial k_j} + \frac{\partial^2 \varphi}{\partial t^2}\frac{\partial^2 \varepsilon_{\alpha\beta}}{\partial \omega^2}\right\} \equiv X_\alpha,$$
$$i[\mathbf{k}\tilde{\mathbf{E}}^1]_\alpha - i\frac{\omega}{c}\tilde{H}_\alpha{}^1 = -\operatorname{rot}_\alpha \tilde{\mathbf{E}}^0 - \frac{1}{c}\frac{\partial \tilde{H}_\alpha{}^0}{\partial t} \equiv Y_\alpha. \tag{10}$$

According to (9) the coupling between the amplitudes $\tilde{\mathbf{E}}^0$ and $\tilde{\mathbf{H}}^0$ turns out to be the same as in a homogeneous stationary medium, although now the quantities **k** and ω are functions of **r** and t [see (7)]. Setting the determinant of the system (9) equal to zero, we obtain for the phase φ an eikonal equation (or, what is the same,—a "local" dispersion equation) in the form

$$\det \|D_{\alpha\beta}\| \equiv \det \left\| k^2\delta_{\alpha\beta} - k_\alpha k_\beta - \frac{\omega^2}{c^2}\varepsilon_{\alpha\beta}\right\| = 0. \tag{11}$$

The solution of this equation can be sought by the method of characteristics (see, for example,[2-5]).

The polarization of the wave is also obtained from (9). In the case of anisotropic media when there is no polarization degeneracy and the matrix $\|D_{\alpha\beta}\|$ is a matrix of the second rank it is convenient to describe the polarization of the field with the aid of a unit (in the general case complex) polarization vector **f** which characterizes the direction of the electric and magnetic field vectors:

$$\mathbf{E} = \Phi\mathbf{f}, \quad \mathbf{H} = \Phi\mathbf{h}, \quad \mathbf{h} = \frac{c}{\omega}[\mathbf{kf}] \tag{12}$$

(here and below we omit the zero and the tilde in the amplitudes of the zeroth approximation). Eliminating the vector **H** from (9), one can convince oneself that the polarization vector **f** satisfies the system of equations

$$D_{\alpha\beta}f_\beta = 0, \tag{13}$$

where the components of the tensor $D_{\alpha\beta}$ are given by expression (11).

The system (13) and the normalization condition $\mathbf{f}\cdot\mathbf{f}^* = 1$ determine the vector **f** with an accuracy up to an arbitrary phase factor $\exp(i\alpha)$. The arbitrariness in the choice of α can be removed by fixing, for example, the direction of the real part of **f** (in practice it is most convenient to follow the local symmetry axes of the medium). As in[5,6], in which no account was taken of nonstationarity and spatial dispersion, one can show that fixing the direction of the real part of **f** does not affect the values of **E** and **H**.

Thus in an anisotropic medium only the complex amplitude factor $\Phi = |\Phi|e^{i\delta}$ remains undetermined in the zeroth geometric-optics approximation. The absolute value and the argument of Φ can be obtained from the consistency condition of the equations of the first approximation, i.e., from the orthogonality condition of the six-vector (**X**, **Y**) with the six-vector ($\mathbf{E}^*$, $-\mathbf{H}^*$) which satisfies the system of homogeneous equations transposed with respect to (9):

$$\mathbf{XE}^* - \mathbf{YH}^* = 0. \tag{14}$$

Let us define the density of electromagnetic energy W averaged over the period of the oscillations and an averaged Poynting vector **S** in analogy with the way in which this is done for a homogeneous stationary medium[7,8]:

$$W = \frac{1}{16\pi}\left[\frac{\partial(\omega\varepsilon_{\alpha\beta})}{\partial\omega}E_\alpha{}^*E_\beta + \mathbf{HH}^*\right] = \frac{|\Phi|^2}{16\pi}\left[\frac{\partial(\omega\varepsilon_{\alpha\beta})}{\partial\omega}f_\alpha{}^*f_\beta + \mathbf{hh}^*\right], \tag{15}$$

$$\mathbf{S} = \frac{c}{16\pi}([\mathbf{E}^*\mathbf{H}] + [\mathbf{EH}^*]) - \frac{\omega}{16\pi}\frac{\partial\varepsilon_{\alpha\beta}}{\partial\mathbf{k}}E_\alpha{}^*E_\beta$$
$$= \frac{|\Phi|^2}{16\pi}\left[c([\mathbf{f}^*\mathbf{h}] + [\mathbf{fh}^*]) - \omega\frac{\partial\varepsilon_{\alpha\beta}}{\partial\mathbf{k}}f_\alpha{}^*f_\beta\right], \tag{16}$$

but we shall keep in mind that **k** and ω are functions of **r** and t.

Substituting in (14) expressions for **X** and **Y** from (10) and setting the real and imaginary part of (14) separately equal to zero, we obtain two equations

$$\frac{\partial W}{\partial t} + \operatorname{div}\mathbf{S} = \left(2i\omega\nu_{\alpha\beta} - \frac{\partial\varepsilon_{\alpha\beta}}{\partial t} + \omega\frac{\partial^2\varepsilon_{\alpha\beta}}{\partial\omega\,\partial t} - \omega\frac{\partial^2\varepsilon_{\alpha\beta}}{\partial\mathbf{r}\,\partial\mathbf{k}}\right)\frac{E_\alpha{}^*E_\beta}{16\pi}, \tag{17}$$

$$W\frac{\partial\delta}{\partial t} + \mathbf{S}\nabla\delta + \frac{|\Phi|^2}{16\pi}N = 0, \tag{18}$$

where

$$N = \operatorname{Im}\left\{c\mathbf{f}\operatorname{rot}\mathbf{h}^* + c\mathbf{h}^*\operatorname{rot}\mathbf{f} + h_\alpha{}^*\frac{\partial h_\alpha}{\partial t} + f_\alpha{}^*\frac{\partial f_\beta}{\partial t}\frac{\partial(\omega\varepsilon_{\alpha\beta})}{\partial\omega} - \omega f_\alpha{}^*\frac{\partial f_\beta}{\partial x_j}\frac{\partial\varepsilon_{\alpha\beta}}{\partial k_j}\right\}. \tag{19}$$

The Poynting vector **S** in the energy conservation equation (17) is equal to the product of the density of the electromagnetic energy W and the group velocity vector $\mathbf{u} = d\omega/d\mathbf{k}$. One can verify this most readily by multiplying the first of equations (9) by $\mathbf{E}^*$ and the second—by $\mathbf{H}^*$, adding the resulting expressions and differentiating the result with respect to **k**. This was done approximately in[7,8]. Rytov obtained the equality $\mathbf{S} = \mathbf{u}W$ considerably earlier as one of the results of considering field equations in their four-dimensional form.[9]

The term $2i\omega\nu_{\alpha\beta}$ in the right-hand side of (17) describes weak absorption. The three following terms

$$-\frac{\partial\varepsilon_{\alpha\beta}}{\partial t} + \omega\frac{\partial^2\varepsilon_{\alpha\beta}}{\partial\omega\partial t} - \omega\frac{\partial^2\varepsilon_{\alpha\beta}}{\partial x_j\partial k_j} \tag{20}$$

determine the change in the energy as a result of the nonstationarity and inhomogeneity of the medium. Unlike the first term in (20), the second and third terms are related with the presence of frequency dispersion in a nonstationary medium and of spatial dispersion in an inhomogeneous medium. They can be considered as additions to the hermitian part of the dielectric permittivity tensor

$$i(\nu_{\alpha\beta})_{\text{eff}} = i\nu_{\alpha\beta} + \frac{1}{2}\left(\frac{\partial^2\varepsilon_{\alpha\beta}}{\partial\omega\,\partial t} - \frac{\partial^2\varepsilon_{\alpha\beta}}{\partial x_j\,\partial k_j}\right). \tag{21}$$

*$[\mathbf{kH}^0]_\alpha = [\mathbf{k}\times\mathbf{H}^0]_\alpha$.

They thus make a certain contribution to the enhancement (or weakening) of the field in a nonstationary and inhomogeneous dispersive medium and must be taken into account, for example, in discussing the problem of the stability of a weakly inhomogeneous plasma.[10,11]

The term $\frac{1}{2}\partial^2\epsilon/\partial\omega\partial t$ has been derived by Pitaevskiĭ[12] for the case of an isotropic dielectric, and the term with $\partial^2\epsilon/\partial x_j\partial k_j$ has been noted by Petviashvili for a weakly turbulent plasma,[13,14] as well as in[1] where the problem of the propagation of a monochromatic wave in a medium with spatial dispersion was considered. As communicated to the author by Stepanov, analogous terms were obtained by the latter in conjunction with Ostrovskiĭ in a study of waves in a nonstationary isotropic plasma.

The last two terms in (20) are symmetric with respect to the space and time variables. On the other hand, there is no such symmetry in the first term, because we have assumed that the medium as a whole is at rest. Going over to a relativistically invariant form of notation should apparently establish the four-dimensional symmetry of both Eq. (17) and (18).

Equation (18) obtained by setting the imaginary terms in the consistency condition (14) equal to zero determines the change of phase δ. The law governing the variation of δ could already have been obtained from the work of Rytov[9] in which the imaginary terms were not analyzed, although it was noted that they concern the structure of the wave field to a greater degree than the law of conservation of energy. In the absence of spatial dispersion and in a stationary medium Eq. (18) is equivalent to the results obtained by Lewis[5] in a more abstract form (a medium with $n > 3$ spatial dimensions), and with the additional condition ω = const (monochromatic wave) it goes over into Eq. (13) of[6]. All the properties of δ established in[5] and[6] extend also to the more general case considered here.

For an isotropic medium the dispersion equation (11) takes on the form

$$\det\|D_{\alpha\beta}\| = -\frac{\omega^2}{c^2}\left(k^2-\frac{\omega^2}{c^2}\varepsilon_\perp\right)^2\varepsilon_\parallel = 0,$$

where $\epsilon_\perp$ and $\epsilon_\parallel$ are the transverse and longitudinal dielectric permittivities. The propagation of transverse waves for which

$$\mathbf{kE} = \mathbf{kH} = 0, \quad c^2k^2 = \omega^2\varepsilon_\perp,$$

is by characteristic polarization degeneracy (the matrix $\|D_{\alpha\beta}\|$ is a first-rank matrix, not a second-rank matrix as is the case for an anisotropic medium).

In this instance, as in the work of Rytov,[15,16] one must seek the fields $\mathbf{E}$ and $\mathbf{H}$ in the following form:

$$\mathbf{E} = \Phi_1\mathbf{n} + \Phi_2\mathbf{b}, \qquad \mathbf{H} = \sqrt{\varepsilon_\perp}(\Phi_1\mathbf{b} - \Phi_2\mathbf{n}), \tag{22}$$

where $\mathbf{n}$ and $\mathbf{b}$ are unit vectors of the normal and binormal to the ray. From the consistency conditions of the first-approximation equations

$$\mathbf{Xn} - \sqrt{\varepsilon_\perp}\mathbf{bY} = 0, \qquad \mathbf{Xb} + \sqrt{\varepsilon_\perp}\mathbf{nY} = 0 \tag{23}$$

we obtain then the law of conservation of energy (17) in which one must replace $\epsilon_{\alpha\beta}$ by $\epsilon_\perp\delta_{\alpha\beta}$ and W and $\mathbf{S}$ denote the following quantities:

$$W = \frac{1}{16\pi}\left[\frac{\partial(\omega\varepsilon_\perp)}{\partial\omega}\mathbf{EE}^* + \mathbf{HH}^*\right] = \frac{1}{16\pi}\frac{\partial(\omega^2\varepsilon_\perp)}{\omega\,\partial\omega}(|\Phi_1|^2+|\Phi_2|^2),$$

$$\mathbf{S} = \frac{1}{16\pi}\left[c([\mathbf{E}^*\mathbf{H}]+[\mathbf{EH}^*]) - \omega\frac{\partial\varepsilon_\perp}{\partial\mathbf{k}}\mathbf{EE}^*\right]$$

$$= \frac{1}{16\pi}\left(2c\sqrt{\varepsilon_\perp}\mathbf{t} - \omega\frac{\partial\varepsilon_\perp}{\partial\mathbf{k}}\right)(|\Phi_1|^2+|\Phi_2|^2),$$

where $\mathbf{t} = \mathbf{k}/k$ is a unit vector tangent to the ray.

In addition, we obtain with the aid of (22) and (23) Rytov's law of the rotation of the field vectors[15,16]

$$d\theta/d\sigma = 1/T, \tag{24}$$

which was initially obtained for monochromatic waves in a stationary inhomogeneous medium without spatial dispersion[1)]. In (24) $\theta = \tan^{-1}(\Phi_2/\Phi_1)$ is the angle between the vector $\mathbf{E}$ and the vector of the principal normal to the ray $\mathbf{n}$, T is the radius of torsion, and $d\sigma = u\,dt$ is an element of length of the ray. Thus (24) is valid in the general case of modulated waves in an inhomogeneous and nonstationary medium with frequency and spatial dispersion.

For longitudinal waves in an isotropic medium (for which the eikonal equation is of the form $\epsilon_\parallel = 0$) $\mathbf{E}\times\mathbf{k} = 0$ and $\mathbf{H} = 0$. Therefore the electric field should be sought in the form $\mathbf{E} = \Phi\mathbf{t} = \Phi\mathbf{k}/k$. From the consistency condition of the first-approximation equations for longitudinal waves, $\mathbf{X}\cdot\mathbf{k} = 0$, we obtain the law of conservation of energy in the form (17) with $\epsilon_{\alpha\beta}$ replaced by $\epsilon_\parallel\delta_{\alpha\beta}$. Here one must bear in mind that for longitudinal waves

$$W = \frac{1}{16\pi}\frac{\partial(\omega\varepsilon_\parallel)}{\partial\omega}\mathbf{EE}^* = \frac{1}{16\pi}\frac{\partial(\omega\varepsilon_\parallel)}{\partial\mathbf{k}}|\Phi|^2,$$

$$\mathbf{S} = -\frac{\omega}{16\pi}\frac{\partial\varepsilon_\parallel}{\partial\mathbf{k}}\mathbf{EE}^* = -\frac{\omega}{16\pi}\frac{\partial\varepsilon_\parallel}{\partial\mathbf{k}}|\Phi|^2.$$

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[1] Yu. A. Kravtsov, Izv. vyssh. uch. zav., Radiofizika 2, No. 11 (1968).
[2] R. Courant, Partial Differential Equations, (Russ. transl.) Mir, 1964.
[3] S. Weinberg, Phys. Rev. 126, 1899 (1962).
[4] L. M. Kovrizhnykh, A. A. Rukhadze, and V. P. Silin, Zh. Eksp. Teor. Fiz. 44, 1953 (1963) [Sov. Phys.-JETP 27, 1314 (1963)].
[5] R. M. Lewis, Arch. Ratl. Mech. Anal. 20, 191 (1965).
[6] Yu. A. Zaĭtsev, Yu. A. Kravtsov, and Yu. Ya. Yashin, Izv. vyssh. uch. zav., Radiofizika 2 (1968), in press.
[7] T. H. Stix, The Theory of Plasma Waves, McGraw-Hill, 1962 (Russ. Transl., Atomizdat, 1965).
[8] V. D. Shafranov, Voprosy teorii plazmy (Problems of Plasma Theory) No. 3, Atomizdat, 1963, p. 3.
[9] S. M. Rytov, Zh. Eksp. Teor. Fiz. 17, 930 (1947).

[1)] The generalization of (24) for the case of modulated waves is given in [5].

[10] V. P. Silin, Zh. Eksp. Teor. Fiz. **44**, 1271 (1963) [Sov. Phys.-JETP **27**, 857 (1963)].
[11] A. B. Mikhaĭlovskiĭ, Voprosy teorii plazmy (Problems of Plasma Theory) No. 3, Atomizdat, 1963, p. 141.
[12] L. P. Pitaevskiĭ, Zh. Eksp. Teor. Fiz. **39**, 1450 (1960) [Sov. Phys.-JETP **12**, 1008 (1961)].
[13] V. I. Petviashvili, Zh. Eksp. Teor. Fiz. **44**, 1993 (1963) [Sov. Phys.-JETP **17**, 1340 (1963)].
[14] V. I. Petviashvili, Zh. Eksp. Teor. Fiz. **45**, 1467 (1963) [Sov. Phys.-JETP **18**, 1014 (1964)].
[15] S. M. Rytov, Doctoral dissertation, Trudy FIAN **2**, 3 (1940).
[16] S. M. Rytov, Dokl. Akad. Nauk SSSR **18**, 263 (1938).

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