A POLARIZATION OPERATOR FOR QUANTIZED FIELDS

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Submitted April 15, 1968

Zh. Eksp. Teor. Fiz. 55, 1390–1392 (October, 1968)

It is proposed to describe fields by means of representations of an extended Poincaré group which depends on 11 parameters and includes, besides Lorentz transformations and translations, the "duality rotation" operator introduced by Misner and Wheeler.[1]

A main feature of a classification of fields must be the concept of reducibility or irreducibility of the representations of a symmetry group which governs these fields. For example, in the case of the electromagnetic field the complex fields $\mathbf{F} = \mathbf{H} + i\mathbf{E}$, $\mathbf{F}^* = \mathbf{H} - i\mathbf{E}$ form the basis of irreducible three-rowed representations of the Lorentz group, whereas the usual fields $\mathbf{H}$, $\mathbf{E}$ form the basis of a six-rowed reducible representation. The resolution of "reducible" fields into irreducible ones is accomplished by means of "polarization operators," whose eigenspaces give the irreducible representations. The polarization operators arise naturally in an extension of the fundamental symmetry group of the fields. In this note we wish to illustrate this principle with the example of the electromagnetic field.

As has recently been common practice, we replace the Lorentz group with the group SL(2) of unimodular transformations of a complex two-dimensional space. We extend this group to the group SL(2) × U(1) of all transformations whose determinant is of absolute value unity. This group depends on seven parameters. Then there is adjoined to the infinitesimal generators of the group SL(2) the generator Π of multiplication by a phase factor $e^{i\beta}$. In a paper by Misner and Wheeler[1] this transformation is called "duality rotation," but its group significance is not developed.

The spin tensor representations of SL(2), which give the fields, are at the same time representations of the group SL(2) × U(1). The group SL(2) × U(1) can be supplemented with translations in the usual way, just as one goes from the Lorentz group to the Poincaré group; the resulting group depends on 11 parameters (ω_{ik}, β, a_k). In the fundamental representation (action of the group on spinors) the infinitesimal generator Π is a unit operator. In the representation of the Lie algebra, however, (unlike that of the Lie group) a unit operator can also be represented by a nonunit operator, since in the Lie algebra the commutators are preserved, but not products.

It is most convenient to give the basis of representations of the group SL(2) × U(1) in the spintensor form, in which spintensors of type (j, j′) have 2j undotted and 2j′ dotted indices and the index σ which numbers the elements of the basis $\psi_\sigma(j, j')$ runs through $(2j+1)(2j'+1)$ values.

The transformations of the resulting group are described by the formulas

$$e^{{}^1\!/_2 i\omega_{ik}\hat{I}_{ik}}\Psi_\sigma^{(j,j')}(x)\,e^{-{}^1\!/_2 i\omega_{ik}\hat{I}_{ik}} = D_{\sigma\tau}^{(j,j')}[\Lambda^{-1}]\Psi_\tau^{(j,j')}(\Lambda x), \qquad (1)$$

$$e^{i\beta\hat{\Pi}}\Psi_\sigma^{(j,j')}(x)\,e^{-i\beta\hat{\Pi}} = e^{2i(j-j')}\Psi_\sigma^{(j,j')}(x), \qquad (2)$$

$$e^{ia_k\hat{P}_k}\Psi_\sigma(x)\,e^{-ia_k\hat{P}_k} = \Psi_\sigma^{(j,j')}(x+a), \qquad (3)$$

and in the infinitesimal form the formulas are

$$[\hat{I}_{ik}, \Psi_\sigma^{(j,j')}(x)] = \left\{\frac{x_i}{i}\frac{\partial}{\partial x^k} - \frac{x_k}{i}\frac{\partial}{\partial x^i} + S_{ik}^{(j,j')}\right\}\Psi_\sigma^{(j,j')}(x), \qquad (1')$$

$$[\Pi, \Psi_\sigma^{(j,j')}(x)] = 2(j-j')\Psi_\sigma^{(j,j')}(x), \qquad (2')$$

$$[P_k, \Psi_\sigma^{(j,j')}(x)] = \frac{1}{i}\frac{\partial}{\partial x^k}\Psi_\sigma^{(j,j')}(x). \qquad (3')$$

The following are the representations to be considered for the electromagnetic field.[2]

I. The vector-potential representation: (j, j′) = (½, ½);

$$\begin{pmatrix} a_{1\dot{1}} & a_{1\dot{2}} \\ a_{2\dot{1}} & a_{2\dot{2}} \end{pmatrix} = \begin{pmatrix} A_0 + A_3 & A_1 - iA_2 \\ A_1 + iA_2 & A_0 - A_3 \end{pmatrix}. \qquad (4)$$

In this case Π commutes with A_k.

II. The field-strengths representation: (j, j′) = (1, 0), (j, j′) = (0, 1);

$$\begin{pmatrix} f_{11}, & f_{12} \\ f_{21}, & f_{22} \end{pmatrix} = \begin{pmatrix} F_1 + iF_2, & F_3 \\ F_3, & -F_1 + iF_2 \end{pmatrix},$$

$$\begin{pmatrix} f_{\dot{1}\dot{1}} & f_{\dot{1}\dot{2}} \\ f_{\dot{2}\dot{1}} & f_{\dot{2}\dot{2}} \end{pmatrix} = \begin{pmatrix} \overset{*}{F}_1 - i\overset{*}{F}_2 & \overset{*}{F}_3 \\ \overset{*}{F}_3 & -\overset{*}{F}_1 - i\overset{*}{F}_2 \end{pmatrix}. \qquad (5)$$

In this case it follows from (2′) that

$$[\Pi, f_{\alpha\beta}] = 2f_{\alpha\beta}, \qquad [\Pi, f_{\dot{\alpha}\dot{\beta}}] = -2f_{\dot{\alpha}\dot{\beta}}, \qquad (6)$$

III. The energy-momentum representation: (j, j′) = (1, 1);

$$T_{\lambda\mu\dot{\sigma}\dot{\tau}} = f_{\lambda\mu}f_{\dot{\sigma}\dot{\tau}}. \qquad (7)$$

in this case Π commutes with $T_{\lambda\mu\dot{\sigma}\dot{\tau}}$. This means that the components of the energy-momentum tensor remain unchanged under transformations of the subgroup U(1) (cf.[1], page 236).

The operator Π must not be confused with the well known charge operator Q (cf.[3], page 36), which is not connected with the symmetry group SL(2) × U(1) and plays the part of a "superselection operator" in the theory. The formulas that hold for the operator Q are

$$[Q, \Psi_\sigma(x)] = \Psi_\sigma(x), \qquad [Q, \bar{\Psi}_\sigma(x)] = -\bar{\Psi}_\sigma(x),$$

from which it is seen that the operator Q, like the operators

$$\frac{1}{i}\frac{\partial}{\partial x^k}, \quad \left(\frac{x_i}{i}\frac{\partial}{\partial x^k} - \frac{x_k}{i}\frac{\partial}{\partial x^i}\right)$$

acts in the same way in all representations, whereas the operator [like the spin operator S_{ik}, Eq. (1)] acts differently on the bases of representations of different types (j, j′).

In relation to the representations of the group $SL(2) \times U(1)$ the operator Π plays the role of a Casimir operator [cf. (2′)]; in the case of the electromagnetic field it distinguishes between the right-handed polarization (1, 0) and the left-handed polarization (0, 1). The field states corresponding to right-handed and left-handed polarizations are pure states with regard to Π. These states, however, are not pure states with regard to the real components of the field strengths **H** and **E**.

[1] C. Misner and J. A. Wheeler, Ann. of Phys. 2, 525 (1957).

[2] O. Laporte and G. E. Uhlenbeck, Phys. Rev. 37, 1380 (1931).

[3] W. Pauli, in: Niels Bohr and the Development of Physics, New York, McGraw-Hill, 1955, pages 30 ff.

Translated by W. H. Furry
156