LOCALIZATION OF ELECTROMAGNETIC OSCILLATIONS IN A PLASMA

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It is shown that in a two-dimensional system the equations of nonlinear electrodynamics allow solutions which describe a wave that travels in a circle. Under these conditions the pressure associated with the wave field in the plasma leads to the formation of a self-sustaining circular waveguide. The possibility of localized magnetic oscillations in a toroidal configuration in three-dimensional space is also discussed.

1. A number of investigators have studied waves that propagate in rectilinear fashion in a self-sustaining waveguide (self-focusing).[1-4] In the present work we consider the two-dimensional case and show that the equations of nonlinear electrodynamics allow solutions that describe a wave that propagates in a circle. Under these conditions the pressure associated with the wave field in a plasma leads to the formation of a circular waveguide. We also discuss the possibility of localized electromagnetic oscillations in a three-dimensional space.

2. The analysis will be carried out for the case of a fully ionized plasma under the assumption that the frequency of the oscillations of the electromagnetic field is much greater than the collision frequency ($\omega \gg \nu$). When the energy associated with the oscillations of an electron in the high-frequency field becomes comparable with the kinetic energy of the electron the nonlinear properties of the plasma become important. The spatial distribution of electron density can then be given by the following relation:[5,6]

$$N(\mathbf{r}) = N \exp\left(-E^2(\mathbf{r}) / \mathscr{E}^2\right), \tag{1}$$

where N is the electron density in the region in which there is no field, E(r) is the amplitude at the electric field and $\mathscr{E}^2 = 8m\kappa T\omega^2/e^2$. We note that the ion density is distributed in space in the same way as the electron density by virtue of the Coulomb interaction between the electrons and ions.

The relation in (1) applies when the amplitude of the field does not vary significantly over the distance corresponding to the electron excursion in one period of the oscillation:

$$\Delta r \gg v_e / \omega, \quad \Delta r \gg eE / m\omega^2, \tag{2}$$

where $v_e = \sqrt{2\kappa T/m}$ is the electron thermal velocity and Δr is the characteristic distance over which there is a significant variation in the field amplitude.

The dielectric constant can be written in the form

$$\varepsilon = 1 - \alpha \exp\left(-\frac{E^2(\mathbf{r})}{\mathscr{E}^2}\right), \quad \alpha = \frac{4\pi N e^2}{m\omega^2}. \tag{3}$$

The electric field of a monochromatic wave is then described by the equation

$$\operatorname{rot}\operatorname{rot}\mathbf{E} = \frac{\omega^2}{c^2}\left[1 - \alpha\exp\left(-\frac{E^2}{\mathscr{E}^2}\right)\right]\mathbf{E}. \tag{4}$$

The solution of this equation will be written in a cylindrical coordinate system (r, φ, z) using a form in which all quantities are proportional to $\exp(in\varphi - i\omega t)$ under the assumption that the electric field is directed along the z axis. Equation (4) then becomes

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{dE}{dr}\right) - \frac{n^2}{r^2}E = -\frac{\omega^2}{c^2}\left[1 - \alpha\exp\left(-\frac{E^2}{\mathscr{E}^2}\right)\right]E. \tag{5}$$

Equation (5) describes the dependence of the amplitude of the electric field on the distance from the z axis. We shall be interested in solutions of this equation which describe a field that diminishes when $r \to \infty$ and $r \to 0$. We note that the necessary condition on the behavior of the field at infinity (the localization condition) is the inequality $\epsilon(r \to \infty) < 0$.

3. Before considering the spatial distribution of the field and plasma density we wish to find the conditions for which the plasma pressure is equilibrated by the field forces. We denote by r_0 the point at which the field E(r) reaches its maximum value. The analysis will be limited to cases in which the effective width of the self-sustaining waveguide, the distance Δr, in which the field differs from zero by a significant amount, is appreciably smaller than r_0. In this case the quantity r_0 characterizes the radius of curvature of the self-sustaining wave guide, which plays the role of a potential barrier, dividing the system into two regions: $r < r_0$ and $r > r_0$. Assuming that the potential barrier is large enough ($E^2(r = r_0) \gg \mathscr{E}^2$) we will also assume that these regions are generally characterized by different temperatures and unperturbed densities, these quantities being denoted by $T_1(\mathscr{E}_1^2)$ and $N_0(\alpha_0)$ when $r < r_0$ and by $T_2(\mathscr{E}_2^2)$ and $N_\infty(\alpha_\infty)$ when $r > r_0$. We shall also assume that the pressures in these regions are the same at the point r_0, i.e.,

$$N_0 T_1 \exp\left(-\frac{E_{max}^2}{\mathscr{E}_1^2}\right) = N_\infty T_2 \exp\left(-\frac{E_{max}^2}{\mathscr{E}_2^2}\right). \tag{6}$$

Under these conditions there is no macroscopic plasma transfer through the potential barrier.

Multiplying the left and right sides of Eq. (5) by the quantity dE/dr and integrating with respect to r from 0 to infinity, we have

$$N_\infty\kappa T_2 - N_0\kappa T_1 = \frac{1}{16\pi}\frac{c^2}{\omega^2}\int_0^\infty \frac{1}{r}\left[\frac{n^2}{r^2}E^2 - \left(\frac{dE}{dr}\right)^2\right]dr. \tag{7}$$

The expression on the right side of Eq. (7) characterizes the pressure and field forces which equilibrate the pressure differential of the plasma between the regions $r \ll r_0$ and $r \gg r_0$.

It is interesting to note that Eq. (7) can also be obtained by starting with the momentum conservation relation

$$\frac{\partial}{\partial x_\beta}(P_{\alpha\beta} + T_{\alpha\beta}) = 0,$$

where $P_{\alpha\beta}$ is the pressure tensor in the medium and $T_{\alpha\beta}$ is the field force tensor as averaged over time. In this case, however, it is also necessary to take account of the Coulomb forces that arise by virtue of charge separation.

4. We shall first investigate the case in which the plasma density is small in the inner region $\alpha_0 \ll 1$. Under these conditions, Eq. (5) indicates that in the region $r < r_0$ the field can be described by Bessel functions of order n. Consequently, the amplitude of the electric field reaches a maximum value at values of r corresponding to the first maximum in the Bessel function. When $n \gg 1$ we have

$$r_0 \approx \frac{c}{\omega}(n + 0{,}8\, n^{1/3}). \tag{8}$$

The electric field in the region $r > r_0$ is also characterized by Bessel functions up to values of r for which the inequality

$$\alpha_\infty \exp\left(-\frac{E^2}{\mathscr{E}_2^2}\right) \ll 1 - \left(\frac{nc}{r\omega}\right)^2.$$

is violated. Hence, the effective width of the waveguide can be estimated from the Bessel functions. When $n \gg 1$ we find

$$\Delta r \approx (c/\omega)\, n^{1/3}. \tag{9}$$

Now, using Eq. (7) and making use of Eqs. (8) and (9) we can find the maximum field amplitude. This is found to be

$$E_{max} \approx \alpha_\infty^{1/2}\, n^{1/3} \mathscr{E}_2. \tag{10}$$

In Fig. 1 we show the amplitude of the electric field as a function of coordinates in the case for which $n = 1000$ and $\alpha_\infty = 4$. When $r < r_0$ this functional dependence is proportional to $J_n(\omega r/c)$. In the region $r > r_0$ the solution of Eq. (5) can be found by means of a computer. It is found that $E_{max} \approx 12.5\ \mathscr{E}_2$ for the conditions considered here. Other solutions of Eq. (5), which are shown by dashed lines in Fig. 1, describe nonlocalized standing electromagnetic waves.

Finally, we find the conditions for which it is possible to neglect the effect of particles that penetrate the potential barrier. Per unit time there will penetrate into the inner region a number of particles of order

$$2\pi r_0 N_\infty v_i \exp(-E_{max}^2/\mathscr{E}_2^2),$$

where v_i is the ion thermal velocity. If we neglect the leakage of particles from the region $r < r_0$ the plasma density in this region is given by

$$N_0 + t\frac{v_i}{r_0} N_\infty \exp\left(-\frac{E_{max}}{\mathscr{E}_2^2}\right).$$

Our analysis holds so long as the condition $\alpha \ll 1$ is satisfied in the region $r < r_0$. We find

$$t \ll \frac{r_0}{v_i \alpha_\infty} \exp\left(\frac{E_{max}}{\mathscr{E}_2^2}\right).$$

If this condition is violated, the change in plasma density due to particle leakage through the potential barrier can lead to an important modification of the spatial distribution of the wave field.

5. Now we consider the case in which the plasma pressure in the region $r \ll r_0$ is approximately the same as the plasma pressure in the region $r \gg r_0$:

$$N_0 \varkappa T_1 \approx N_\infty \varkappa T_2 \equiv P.$$

This is the case when

$$r_0 - cn/\omega \gg \Delta r.$$

Under these conditions the following equation holds in the waveguide region, that is to say, near the point r_0:

$$\frac{d^2E}{dr^2} - \frac{n^2}{r_0^2}E = -\frac{\omega^2}{c^2}\left[1 - \alpha \exp\left(-\frac{E^2}{\mathscr{E}_2^2}\right)\right]E. \tag{11}$$

We note that a similar equation has been treated in [3] in which rectilinear propagation of a wave was considered in the two-dimensional case. Equation (11) yields the following dependence of field amplitude on coordinates:

$$r - r_0 = \pm\frac{c}{\omega}\int_E^{E_{max}} dE \left\{\left[\left(\frac{cn}{\omega r_0}\right)^2 - 1\right]E^2 + 32\pi P\left[1 - \exp\left(-\frac{E^2}{\mathscr{E}_2^2}\right)\right]\right\}^{-1/2}. \tag{12}$$

Thus we have

$$E_{max}^2 \approx 32\pi P[1 - (nc/r_0\omega)^2]^{-1}, \tag{13}$$

$$\Delta r \approx (c/\omega)[1 - (nc/r_0\omega)^2]^{-1/2}. \tag{14}$$

Using the relations in (6) and (7) we can determine the connection between the values of the density at $r = 0$ and far from the system. We have

$$N_0 - N_\infty \approx 2N_\infty \frac{c}{r_0\omega}\left[1 - 2\left(\frac{nc}{r_0\omega}\right)^2\right]\left[1 - \left(\frac{nc}{r_0\omega}\right)^2\right]^{-3/2}.$$

It follows from this equation that the plasma density is larger in the inner region of the system than in the region $r \gg r_0$ if

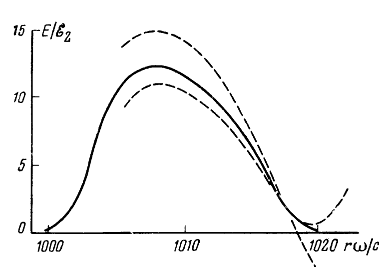

FIG. 1

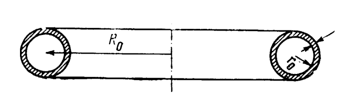

FIG. 2

$$n < 2^{-1/2} r_0 \omega / c. \qquad (15)$$

We note that these results apply for a time

$$t \ll \frac{2r_0}{v_i}\frac{E_{max}^2}{\mathscr{E}^2}\exp\frac{E_{max}^2}{\mathscr{E}^2}. \qquad (16)$$

6. Localized electromagnetic oscillations in a three-dimensional space[1)] can be represented in the form of a wave that propagates in a self-sustaining wave guide of toroidal shape, the propagation occurring along the minor circle of the torus (cf. Fig. 2, where the wave-guide region is shown by cross-hatching). If the major radius of the torus is significantly greater than the minor radius $R_0 \gg r_0$ the description of this wave can be given in terms of the solution obtained above for the two-dimensional case.

We note that this system is subject to the field forces which try to reduce the major radius of the torus. The results obtained here are valid for times that are appreciably smaller than the characteristic time in which R_0 is reduced. We find

$$t \ll \left\{\frac{16\pi R_0^2 r_0 N_0 M}{\Delta r E_{max}^2[2-(nc/r_0\omega)^2-(c/\omega\Delta r)^2]}\right\}^{1/2}.$$

It is probable that these field forces can be balanced by the centrifugal forces in a torus that rotates around its principle axis.

The damping time for these localized oscillations can be estimated from the relation $\tau \approx W/Q$ where W is the total field energy and Q is the energy evolved in the plasma per unit time by virtue of the dissipation of the field energy.[7] Estimates indicate a rather large value of τ since the damping is important only in the weak field region where $E \lesssim \mathscr{E}$. For example, with $T \approx 10^7$ degrees, $N \approx 10^{10}$ cm^{-3} and $r_0 \approx 10$ cm we find that τ is of the order of a second. However, this estimate should be viewed with some criticism since the stability of the system has not been studied. Questions of stability require additional analysis such as in the cases studied in[1-4].

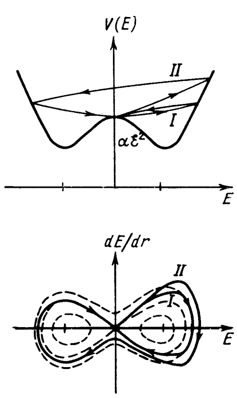

FIG. 3

The author is indebted to V. P. Silin for critical remarks and to B. I. Zaslavskiĭ for discussion.

Note added in proof (June 12, 1968). In conclusion we show that Eq. (5) has more than one solution corresponding to a localized field.

In the qualitative investigation of the solutions of Eq. (5) we have used a method which was applied in[8-10]; in particular, by E and dE/dr we are to understand the coordinate and velocity of some ficticious particle which we will treat as though this particle were moving in a potential well. In this case the quantity r plays the role of the time.

First we consider Eq. (5) without the terms $r^{-1}dE/dr$ and $(n^2/r^2)E$ in which case the analysis corresponds to that of a conservative system. The first integral of the motion

$$\frac{d^2E}{dr^2} = -\frac{\omega^2}{c^2}\left[1-\alpha\exp\left(-\frac{E^2}{\mathscr{E}^2}\right)\right]E \qquad (17)$$

is of the form

$$K = \frac{c^2}{\omega^2}\left(\frac{dE}{dr}\right)^2 + V(E), \quad V(E) = E^2 + \alpha\mathscr{E}^2\exp\left(-\frac{E^2}{\mathscr{E}^2}\right). \qquad (18)$$

The quantity K represents the total energy of the ficticious particle. In Fig. 3 we show schematically the functional dependence V(E) which characterizes the potential well; we also show the pattern of the integral curves in the phase plane corresponding to different values of K. When $K = \alpha\mathscr{E}^2$ Eq. (17) has a unique periodic solution.

Now, returning to Eq. (5) we write it in the form

$$\frac{\omega^2}{c^2}\frac{dK}{dr} = -\frac{1}{r}\left(\frac{dE}{dr}\right)^2 + \frac{n^2}{r^2}E\frac{dE}{dr}, \qquad (19)$$

where K as determined by (18) plays the role of the energy of a ficticious particle.

The total variation of K is given by

$$K_{r\to\infty} - K_{r=0} = -\frac{c^2}{\omega^2}\int_0^\infty\left[\frac{1}{r}\left(\frac{dE}{dr}\right)^2 - \frac{n^2}{r^2}E\frac{dE}{dr}\right]dr. \qquad (20)$$

The solutions of interest here must start and stop on the phase plane at the point E = 0, dE/dr = 0. Under these conditions the ficticious particle can execute some number of oscillations in the potential well.

Thus, in the cross section z = const. the pattern of the spatial distribution of the amplitude of the electric field is in the form of concentric rings. If the maximum values of the field amplitude are large ($|E_{max}| \gg E$) we can assume that the regions of space lying between the wave guide channels are fairly isolated and characterized by their own values of temperature and unperturbed plasma density. Under these conditions the spatial distribution in the field will be given by dependence similar to that shown in Fig. 4.

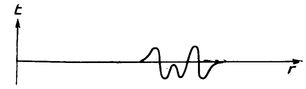

FIG. 4

[1)]The author has learned that localized fields in the shape of spherical configurations were discussed by L. V. Keldysh (report to the Session on General and Applied Physics, U.S.S.R. Acad. of Sciences, 1956) in an explanation of the nature of "ball lightning". Unfortunately, the results of this work were not published.

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