# THE STATISTICAL THEORY OF TURBULENCE OF AN INCOMPRESSIBLE FLUID AT LARGE REYNOLDS NUMBERS

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We consider the statistical theory of the turbulent motion of a viscous incompressible fluid, using an analysis of the probability for a velocity distribution in the fluid at several points at the same time. We give the extra conditions which these functions satisfy. Assuming the independence of large and small scale motions for large Reynolds numbers we study the structure of the expansion of the distribution function in terms of the small parameter $R^{-1/4}$.

## 1. EQUATIONS FOR THE MANY-POINT VELOCITY DISTRIBUTION FUNCTIONS

THE complete statistical description of turbulence can be accomplished by giving the set of functions $F_n(\mathbf{v}_1, \mathbf{x}_1, \ldots, \mathbf{v}_n, \mathbf{x}_n; t)$ such that the probability that at time t the velocities of the fluid in the points $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$ lie in the ranges $d\mathbf{v}_1, \ldots, d\mathbf{v}_n$ is equal to $F_n d\mathbf{v}_1 \ldots d\mathbf{v}_n$.

One sees easily that in the case of laminar motion

$$F_n = \prod_{i=1}^{n} \delta(\mathbf{v}_i - \mathbf{v}(\mathbf{x}_i, t)),$$

where $\mathbf{v}(\mathbf{x}, t)$ is a solution of the Navier-Stokes equation. In the case of fully developed turbulence $F_n$ will be the superposition

$$F_n = \sum C[\mathbf{v}(\mathbf{x}, t)] \prod_{i=1}^{n} \delta(\mathbf{v}_i - \mathbf{v}(\mathbf{x}_i, t)),$$

where the sum is taken over all velocity fields which are realized with a probability $C[\mathbf{v}(\mathbf{x}, t)]$. This superposition will give a complicated picture for the distribution of n velocities which is far from laminar. A study of strong turbulence using directly the Navier-Stokes equation and its moments is therefore practically hopeless. Indeed, when $R/R_{cr} \sim 10^3$ it is necessary to give $10^7$ parameters (for instance, velocity Fourier components) in order to approximate the velocity field satisfactorily. It is, for instance, clear that it is difficult to achieve success when solving the problem of statistics for normal mechanical systems with a large number of degrees of freedom without introducing distribution functions and operating directly upon the equations of motion for many particles and the moments of these equations.

By virtue of the evident statistical nature of the problem it is apparent that only a description using distribution functions will enable us to introduce hypotheses directly using the fact that the Reynolds number, which is a measure of the number of degrees of freedom, is large.

Let us derive the equations for the functions $F_n$.

The Navier-Stokes equations to describe the motion of a viscous incompressible fluid have the form

$$\frac{\partial v^\alpha}{\partial t} + v^\beta \frac{\partial v^\alpha}{\partial x^\beta} + \frac{\partial p}{\partial x^\alpha} - \nu \Delta v^\alpha = 0, \quad \frac{\partial v^\alpha}{\partial x^\alpha} = 0. \tag{1.1}$$

We can eliminate the pressure from these equations in the usual way

$$p(\mathbf{x}_1, t) = -\frac{1}{4\pi} \int \frac{\partial v_2^\alpha \partial v_2^\beta}{\partial x_2^\beta \partial x_2^\alpha} \frac{d\mathbf{x}_2}{|\mathbf{x}_1 - \mathbf{x}_2|} + \psi(x_1, t), \tag{1.2}$$

the function $\psi(\mathbf{x}_1, t)$ is a harmonic function:

$$\Delta \psi(\mathbf{x}_1, t) = 0. \tag{1.3}$$

Although we shall assume in what follows that the velocity field is random we can show that the function $\psi$ is not a random quantity. We shall indicate by a bar over a quantity ensemble averaging and we introduce a new random function

$$\psi_1(\mathbf{x}_1, t) = \psi(\mathbf{x}_1, t) - \overline{\psi(\mathbf{x}_1, t)}$$

and its two-point second moment

$$f(\mathbf{x}_1, \mathbf{r}, t) = \overline{\psi_1(\mathbf{x}_1, t)\psi_1(\mathbf{x}_1 + \mathbf{r}, t)}. \tag{1.4}$$

When $\mathbf{x}_1$ is fixed the function f is a harmonic one. Moreover, it satisfies the following physically obvious requirement: it has no singularities and must tend to zero for large $\mathbf{r}$. A harmonic function with those properties vanishes. Taking the limit $\mathbf{r} \to 0$ in Eq. (1.4) we get

$$\overline{\psi_1^2(\mathbf{x}_1, t)} = 0,$$

which means that the random function $\psi$ is exactly the same as its average value:

$$\psi(\mathbf{x}_1, t) \equiv \overline{\psi(\mathbf{x}_1, t)}.$$

We can thus assume in our equations that $\psi$ is not a random quantity.

After integrating by parts in Eq. (1.2) the equations take the following form:

$$\frac{\partial v_1^\alpha}{\partial t} + v_1^\beta \frac{\partial v_1^\alpha}{\partial x_1^\beta} - \frac{1}{4\pi} \int v_2^\beta v_2^\gamma T^{\alpha\beta\gamma}(\mathbf{x}_1 - \mathbf{x}_2) d\mathbf{x}_2 + \frac{\partial \psi(\mathbf{x}, t)}{\partial x_1^\alpha} - \nu \Delta v_1^\alpha = 0, \tag{1.5}$$

where

$$T^{\alpha\beta\gamma}(\mathbf{x}_1 - \mathbf{x}_2) = \frac{\partial^3}{\partial x_1^\alpha \partial x_1^\beta \partial x_1^\gamma} \frac{1}{|\mathbf{x}_1 - \mathbf{x}_2|}.$$

We introduce a chain of equations which the functions $F_n$ satisfy. To do this we consider an arbitrary function $\varphi(\mathbf{v}(\mathbf{x}_1, t), \ldots, \mathbf{v}(\mathbf{x}_n, t))$ of the velocities in the points $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$ at time t. It is clear that

$$\frac{\partial\varphi}{\partial t}=\sum_{i=1}\frac{\partial\varphi}{\partial v_i^\alpha}\frac{\partial v_i^\alpha}{\partial t},$$

and, using Eq. (1.5) we can write

$$\frac{\partial\varphi}{\partial t}=\sum_{i=1}^{n}\frac{\partial\varphi}{\partial v_i^\alpha}\Big[-v_i^\beta\frac{\partial v_i^\alpha}{\partial x_i^\beta}-\frac{\partial\psi_i}{\partial x_i^\alpha}+\nu\Delta v_i^\alpha+\frac{1}{4\pi}\int v_{n+1}^{\beta}v_{n+1}^{\gamma}T^{\alpha\beta\gamma}(\mathbf{x}_1-\mathbf{x}_{n+1})\,d\mathbf{x}_{n+1}\Big]. \quad (1.6)$$

We average Eq. (1.6). We have

$$\overline{\frac{\partial\varphi}{\partial t}}=\int\frac{\partial F_n}{\partial t}\varphi(\mathbf{v}_1,\ldots,\mathbf{v}_n)\,d\mathbf{v}_1\ldots d\mathbf{v}_n,$$

$$\overline{\frac{\partial\varphi}{\partial v_i^\alpha}v_i^\beta\frac{\partial v_i^\alpha}{\partial x_i^\beta}}=\int\frac{\partial F_n}{\partial x_i^\beta}v_i^\beta\varphi(\mathbf{v}_1,\ldots,\mathbf{v}_n)\,d\mathbf{v}_1\ldots d\mathbf{v}_n.$$

The other terms in Eq. (1.6) transform analogously. Using the fact that $\varphi$ is an arbitrary function, we get the following set of equations for the functions $F_n$:

$$\frac{\partial F_n}{\partial t}+\sum_{i=1}^{n}\Big[v_i^\alpha\frac{\partial F_n}{\partial x_i^\alpha}+\frac{\partial F_n}{\partial v_i^\alpha}\frac{\partial\psi_i}{\partial x_i^\alpha}+\frac{1}{4\pi}\frac{\partial}{\partial v_i^\alpha}\int F_{n+1}v_{n+1}^{\beta}v_{n+1}^{\gamma}T^{\alpha\beta\gamma}(\mathbf{x}_i-\mathbf{x}_{n+1})\,d\mathbf{v}_{n+1}\,d\mathbf{x}_{n+1}+\nu\frac{\partial}{\partial v_i^\alpha}\int\delta(\mathbf{x}_i-\mathbf{x}_{n+1})\,d\mathbf{x}_{n+1}\,\Delta_{n+1}F_{n+1}v_{n+1}^{\alpha}\,d\mathbf{v}_{n+1}\Big]=0. \quad (1.7)$$

For the case of homogeneous turbulence without additional conditions Eqs. (1.7) were derived in[1].

We give the conditions which the functions $F_n$ must satisfy.

1. Normalization condition:

$$\int F_{n+1}\,d\mathbf{v}_{n+1}=F_n,\qquad\int F_n\,d\mathbf{v}_1\ldots d\mathbf{v}_n=1.$$

2. Continuity condition:

$$\lim_{\mathbf{x}_i\to\mathbf{x}_k}F_{n+1}=F_n\delta(\mathbf{v}_i-\mathbf{v}_k).$$

3. Symmetry condition:

$$F_n(\ldots,\mathbf{v}_i,\mathbf{x}_i,\ldots,\mathbf{v}_k,\mathbf{x}_k,\ldots;t)=F_n(\ldots,\mathbf{v}_k,\mathbf{x}_k,\ldots,\mathbf{v}_i,\mathbf{x}_i,\ldots;t).$$

4. Incompressibility condition:

$$\int\frac{\partial F_n}{\partial x_i^\alpha}v_i^\alpha\,dv_i=0.$$

5. Consistency condition:

$$\frac{\partial F_n}{\partial x_i^\alpha}=-\frac{\partial}{\partial v_i^\beta}\int\frac{\partial F_{n+1}}{\partial x_{n+1}^{\alpha}}v_{n+1}^{\beta}\delta(\mathbf{x}_{n+1}-\mathbf{x}_i)\,d\mathbf{v}_{n+1}\,d\mathbf{x}_{n+1}.$$

Only when all conditions are satisfied can we assume that the functions $F_n$ are functions with independent variables.

## 2. INTRODUCTION OF RELATIVE COORDINATES AND VELOCITIES

To formulate the basic assumptions we change to another set of coordinates:

$$\mathbf{X}=\mathbf{x}_1,\quad\boldsymbol{\xi}_2=\mathbf{x}_2-\mathbf{x}_1,\ldots,\boldsymbol{\xi}_n=\mathbf{x}_n-\mathbf{x}_1$$

and of random velocities

$$\mathbf{V}=\mathbf{v}_1,\quad\boldsymbol{\omega}_2=\mathbf{v}_2-\mathbf{v}_1,\ldots,\boldsymbol{\omega}_n=\mathbf{v}_n-\mathbf{v}_1.$$

If we assume that $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$ lie in some volume, $\mathbf{V}$ characterizes the motion of that volume of the fluid as a whole while the $\omega_i$ describe in some approximation the relative motion. Changing to the new variables means that we consider instead of the functions $F_n(\mathbf{v}_1, \mathbf{x}_1, \ldots, \mathbf{v}_n, \mathbf{x}_n; t)$ the new functions

$$\tilde F_n(\mathbf{V},\mathbf{X};\boldsymbol{\omega}_2,\boldsymbol{\xi}_2,\ldots,\boldsymbol{\omega}_n,\boldsymbol{\xi}_n;t)=\int F_n(\mathbf{v}_1,\mathbf{x}_1,\ldots,\mathbf{v}_n,\mathbf{x}_n;t)\cdot\delta(\mathbf{V}-\mathbf{v}_1)\delta(\mathbf{X}-\mathbf{x}_1)\prod_{i=2}^{n}\delta(\boldsymbol{\omega}_i-\mathbf{v}_i+\mathbf{v}_1)\delta(\boldsymbol{\xi}_i-\mathbf{x}_i+\mathbf{x}_1)\,d\mathbf{v}_1\,d\mathbf{x}_1\ldots d\mathbf{v}_n\,d\mathbf{x}_n.$$

In the new variables the equations for the functions $F_n$ look as follows:

$$\frac{\partial\tilde F_n}{\partial t}+V^\alpha\frac{\partial\tilde F_n}{\partial X^\alpha}+\frac{\partial\tilde F_n}{\partial V^\alpha}\frac{\partial\psi(\mathbf{X})}{\partial X^\alpha}+\sum_{i=2}^{n}\Big[\omega_i^\alpha\frac{\partial\tilde F_n}{\partial\xi_i^\alpha}+\frac{\partial\tilde F_n}{\partial\omega_i^\alpha}\Big(\frac{\partial\psi(\mathbf{X}+\boldsymbol{\xi}_i)}{\partial X^\alpha}-\frac{\partial\psi(\mathbf{X})}{\partial X^\alpha}\Big)$$
$$+\frac{1}{4\pi}\frac{\partial}{\partial\omega_i^\alpha}\int\frac{\partial^2\tilde F_{n+1}}{\partial\xi_{n+1}^{\beta}\partial\xi_{n+1}^{\gamma}}(T^\alpha(\boldsymbol{\xi}_{n+1}-\boldsymbol{\xi}_i)-T^\alpha(\boldsymbol{\xi}_{n+1}))\,\omega_{n+1}^{\beta}\omega_{n+1}^{\gamma}\,d\boldsymbol{\omega}_{n+1}\,d\boldsymbol{\xi}_{n+1}$$
$$+\nu\frac{\partial}{\partial\omega_i^\alpha}\int d\boldsymbol{\xi}_{n+1}\,d\boldsymbol{\omega}_{n+1}(\delta(\boldsymbol{\xi}_{n+1}-\boldsymbol{\xi}_i)-\delta(\boldsymbol{\xi}_{n+1}))\,\Delta_{n+1}\tilde F_{n+1}\,\omega_{n+1}^{\alpha}\Big]$$
$$+\frac{1}{4\pi}\frac{\partial}{\partial V^\alpha}\int\frac{\partial^2\tilde F_{n+1}}{\partial\xi_{n+1}^{\beta}\partial\xi_{n+1}^{\gamma}}T^\alpha(\boldsymbol{\xi}_{n+1})\,\omega_{n+1}^{\beta}\omega_{n+1}^{\gamma}\,d\boldsymbol{\omega}_{n+1}\,d\boldsymbol{\xi}_{n+1}$$
$$+\nu\frac{\partial}{\partial V^\alpha}\int d\boldsymbol{\xi}_{n+1}\,d\boldsymbol{\omega}_{n+1}\,\delta(\boldsymbol{\xi}_{n+1})\,\Delta_{n+1}\tilde F_{n+1}\,\omega_{n+1}=0. \quad (2.1)$$

It is necessary also to rewrite in the new coordinates all additional conditions which the velocity distribution functions satisfy.

1. Continuity condition:

$$\lim_{\boldsymbol{\xi}_i\to\boldsymbol{\xi}_k}\tilde F_{n+1}=\tilde F_n\delta(\boldsymbol{\omega}_i-\boldsymbol{\omega}_k),\quad\lim_{\boldsymbol{\xi}_k\to 0}\tilde F_{n+1}=\tilde F_n\delta(\boldsymbol{\omega}_k).$$

2. Normalization condition:

$$\int\tilde F_n\,d\mathbf{V}\,d\boldsymbol{\omega}_2\ldots d\boldsymbol{\omega}_n=1,$$

$$\int\tilde F_{n+1}\,d\boldsymbol{\omega}_{n+1}=\tilde F_n,\qquad\int\tilde F_n\,d\boldsymbol{\omega}\ldots d\boldsymbol{\omega}_n=F_1(\mathbf{V},\mathbf{X},t).$$

3. The consistency condition, which reflects the fact that in the Euler variables the variables $\mathbf{x}_i$ are not random variables but are the discrete indices of the random variables $\mathbf{v}_i$, has now the form

$$\frac{\partial\tilde F_n}{\partial\xi_{n+1}^{\alpha}}=-\frac{\partial}{\partial\omega_i^\beta}\int\frac{\partial\tilde F_{n+1}}{\partial\xi_{n+1}^{\alpha}}\delta(\boldsymbol{\xi}_{n+1}-\boldsymbol{\xi}_i)\,\omega_{n+1}^{\beta}\,d\boldsymbol{\omega}_{n+1}\,d\boldsymbol{\xi}_{n+1},$$

$$\frac{\partial\tilde F_n}{\partial X^\alpha}=-\frac{\partial}{\partial V^\beta}\int\frac{\partial\tilde F_{n+1}}{\partial\xi_{n+1}^{\alpha}}\delta(\boldsymbol{\xi}_{n+1})\,\omega_{n+1}^{\beta}\,d\boldsymbol{\omega}_{n+1}\,d\boldsymbol{\xi}_{n+1}$$
$$-\sum_{i=2}\frac{\partial}{\partial\omega_i^\beta}\int\frac{\partial\tilde F_{n+1}}{\partial\xi_{n+1}^{\alpha}}(\delta(\boldsymbol{\xi}_{n+1})-\delta(\boldsymbol{\xi}_{n+1}-\boldsymbol{\xi}_i))\,\omega_{n+1}^{\beta}\,d\boldsymbol{\omega}_{n+1}\,d\boldsymbol{\xi}_{n+1}. \quad (2.2)$$

4. The symmetry condition is written in the form of two relations as was the consistency condition:

a) symmetry with respect to permutation of any velocities and coordinates except $\mathbf{v}_1$ and $\mathbf{x}_1$ means that the function is not changed when the pairs $\omega_i$, $\xi_i$ and $\omega_k$, $\xi_k$ are interchanged:

$$P_{ik}{}^{a}\tilde F_n=\tilde F_n;$$

b) symmetry when $\mathbf{v}_1\mathbf{x}_1$ and $\mathbf{v}_k$, $\mathbf{x}_k$ are interchanged is now expressed by the equation

$$P_k{}^{b}\tilde F_n=\tilde F_n(\mathbf{V}+\boldsymbol{\omega}_k,\mathbf{X}+\boldsymbol{\xi}_k,\boldsymbol{\omega}_2-\boldsymbol{\omega}_k,\boldsymbol{\xi}_2-\boldsymbol{\xi}_k,\ldots,-\boldsymbol{\omega}_k,-\boldsymbol{\xi}_k,\ldots,\boldsymbol{\omega}_n-\boldsymbol{\omega}_k,\boldsymbol{\xi}_n-\boldsymbol{\xi}_k;t)=\tilde F_n(\mathbf{V},\mathbf{X},\boldsymbol{\omega}_2,\boldsymbol{\xi}_2,\ldots,\boldsymbol{\omega}_n,\boldsymbol{\xi}_n).$$

5) The incompressibility condition is written in the

form

$$\int \frac{\partial F_n}{\partial \xi_i^\alpha} \omega_i^\alpha d\omega_i = 0.$$

We shall in the following denote the relative velocity distribution functions $\widetilde{F}_n$ by $F_n$.

## 3. BASIC ASSUMPTIONS. ESTIMATE OF THE ORDER OF MAGNITUDE OF THE DIFFERENT TERMS IN THE EQUATIONS FOR THE DISTRIBUTION FUNCTIONS. ZEROTH APPROXIMATION

We have already mentioned that we shall try to use in solving the chain of equations the fact that the Reynolds number is large. We assume without proof that in that case our hydrodynamic motion has a statistical character and possesses necessarily (due to $R \gg 1$) a large number of degrees of freedom. Moreover, it is very clear that such a motion will be characterized by local velocity gradients which are much larger than the average gradients. If $v_\lambda$ is a characteristic difference in velocities over a distance $\lambda$, this means that $v_\lambda/\lambda \gg U/L$, where U is a characteristic change in velocity over the dimensions L given by the problem. In terms of the distribution functions this means, for instance, that

$$V^\alpha \frac{\partial F_n}{\partial X^\alpha} \ll \omega_i^\alpha \frac{\partial F_n}{\partial \xi_i^\alpha},$$

if $|\xi_i| \ll L$.

We see thus that the largest terms in the equations in the chain will be of order $(\omega/\xi)F_n$. The other terms will be small with extra factors $U\xi/L\omega$. The term containing the harmonic function $\psi$ and the derivative with respect to $\omega_i$ will, for instance, be of second order:

$$\frac{\partial F_n}{\partial \omega_i^\alpha}\left(\frac{\partial \psi(X+\xi_i)}{\partial X^\alpha} - \frac{\partial \psi(X)}{\partial X^\alpha}\right) \approx F_n \frac{\omega}{\xi}\left(\frac{U\xi}{L\omega}\right)^2,$$

since $\psi \sim U^2$. Dropping all small terms, we get the following equations

$$\frac{\partial F_n}{\partial t} + \hat{L}_{0n} F_{n+1} = 0, \tag{3.1}$$

where

$$\hat{L}_{0n} F_{n+1} = \sum_{i=2}^{n} \Big\{ \omega_i^\alpha \frac{\partial F_n}{\partial \xi_i^\alpha} + \frac{1}{4\pi} \frac{\partial}{\partial \omega_i^\alpha} \int \frac{\partial^2 F_{n+1}}{\partial \xi_i^\beta \partial \xi_i^\gamma} [T^\alpha(\xi_{n+1} - \xi_i) - T^\alpha(\xi_{n+1})]\, \omega_{n+1}^\beta \omega_{n+1}^\gamma \, d\omega_{n+1} d\xi_{n+1}$$
$$+ \nu \frac{\partial}{\partial \omega_i^\alpha} \int d\xi_{n+1} d\omega_{n+1} [\delta(\xi_{n+1} - \xi_i) - \delta(\xi_{n+1})] \Delta_{n+1} F_{n+1} \omega_{n+1}^\alpha \Big\}. \tag{3.2}$$

We have purposefully not given very exact estimates for the remaining and the dropped terms in the chain of equations for the distribution functions, which are more complicated than we have just written down. At the moment it is only important for us that the terms dropped are much smaller than those retained. It is thus clear that the evolution of the distribution function $F_n$ is described by equations which contain only operators of the relative velocities and coordinates. The velocity **V** in a given spot and the coordinate **X** "of the spot" enter in this equation merely as parameters and the equations themselves are the same as the equations for the distribution functions $\varphi_n = \int F_n dV$ in the uniform case. We note that this is true only if we neglect the terms dropped and is a consequence from the fact that the Reynolds number is large which makes the dropped terms small.

If we now assume (as a consequence, not proved by us, of the fact that the number of degrees of freedom is large) that the equation written down by us for the relative velocities distribution functions contains at small distances a relaxation to some stationary distribution which is independent of the initial distribution, this stationary state will be independent of the origin of the turbulence, i.e., the geometry, the method of excitation, and other "macroscopic" properties of the problem considered. These stationary functions must thus be universal, uniform and isotropic ones. This assumption about the relaxation was just the one we had in mind when we said that our hydrodynamic motion was statistical in character.

It is useful to repeat our discussion from a somewhat different point of view based directly upon experimental data. Experiments show that for large Reynolds numbers turbulent motion occurs in incompressible fluids which 1) has large velocity gradients at small distances; 2) depends only on the "macroscopic" conditions of the problem, i.e., the conditions which are imposed in the usual laminar case and which are independent of the character and form of the initial fluctuations.

Starting from these very general experimental facts and dealing with the chain of equations for the distribution functions we can conclude that apart from small quantities we have for the distribution functions $F_n = F_1(\mathbf{V}, \mathbf{X}, t)\varphi_n^{(0)}(\omega_2, \xi_2, \ldots)$ where the functions $\varphi_n^{(0)}$ are independent of the problem and are universal, uniform, and stationary functions. Kolmogorov[2] formulated in 1941 this character of the functions $\varphi_n^{(0)}$ as a self-consistent hypothesis.

We shall consider stationary problems, and in nonstationary problems we shall assume that

$$\frac{\partial F_n}{\partial t} \sim \frac{U}{L} F_n.$$

We get thus for the zeroth approximation $F_n^{(0)} = F_1\varphi_n^{(0)}$ the zeroth approximation stationary equations (3.1). These equations do not contain the time t and the coordinate **X** explicitly, as we have already noted.

The zeroth approximation functions depend on the time t and the coordinate **X** only through the variables $F_1(\mathbf{V}, \mathbf{X}, t)$ and $\epsilon(\mathbf{X}, t)$. To track the slow dependence of the distribution functions on **X** and t we shall look for the general solution in the form of a series of approximations

$$F_n = F_n^{(0)} + F_n^{(1)} + \ldots,$$

where

$$F_n^{(k)} = F_n^{(k)}(\mathbf{V}, \omega_2, \xi_2, \ldots, \omega_n, \xi_n \,|\, F_1(\mathbf{V}, \mathbf{X}, t), \varepsilon(\mathbf{X}, t))$$

depend on **X** and t only through the functional dependence on $F_1$ and $\epsilon$. As scales for the velocities and the coordinates we can introduce the Kolmogorov scales $v_0 = (\nu\epsilon)^{1/4}$ and $\lambda_0 = (\nu^3/\epsilon)^{1/4}$ after which the equations become dimensionless with $\nu = 1$. The required solution will be a symmetric, isotropic, now already necessarily, and universal solution. This solution, i.e., the function $\varphi_n^{(0)}$ must satisfy all additional conditions, apart from first order corrections.

To estimate the order of magnitude we shall use the Kolmogorov relations $\omega \sim v_0\xi/\lambda_0$ when $\xi \ll \lambda_0$ and $\omega \sim (\epsilon\xi)^{1/3}$ when $\xi \gg \lambda_0$. The basis for the first relation is the continuity condition. As to the second relation (in the so-called inertial range) it follows from the zeroth approximation solution for $|\xi_i| \gg \lambda_0$ and will be justified in a following paper. At the moment we shall use them without proof.

Before we turn to a study of the first approximation equations, we verify that the zeroth approximation functions satisfy in the necessary accuracy the additional conditions 1—5 and make clear to what conditions on the functions $\varphi_n^{(0)}$ these lead.

From the continuity condition it follows that

$$\lim_{\xi_i \to \xi_k} \varphi_{n+1}^{(0)} = \varphi_n^{(0)} \delta(\omega_i - \omega_k), \quad \lim_{\xi_i \to 0} \varphi_{n+1}^{(0)} = \varphi_n^{(0)} \delta(\omega_i).$$

The normalization condition changes to the following conditions:

$$\int \varphi_{n+1}^{(0)} d\omega_{n+1} = \varphi_n^{(0)}, \quad \int \varphi_n^{(0)} d\omega_2 \ldots d\omega_n = 1.$$

The consistency conditions take the form

$$\frac{\partial \varphi_n^{(0)}}{\partial \xi_i^\alpha} = -\frac{\partial}{\partial \omega_i^\beta} \int \frac{\partial \varphi_{n+1}^{(0)}}{\partial \xi_{n+1}^\alpha} \delta(\xi_{n+1} - \xi_i)\, \omega_{n+1}^\beta\, d\omega_{n+1}\, d\xi_{n+1},$$

$$\sum_{i=2}^n \frac{\partial \varphi_n^{(0)}}{\partial \xi_i^\alpha} + \frac{\partial}{\partial \omega_i^\beta} \int \frac{\partial \varphi_{n+1}^{(0)}}{\partial \xi_{n+1}^\alpha} \delta(\xi_{n+1})\, \omega_{n+1}^\beta\, d\omega_{n+1}\, d\xi_{n+1} = 0.$$

We have dropped small terms in the second consistency condition. We note that for n = 1 the consistency condition in zeroth approximation will look as follows:

$$\int \frac{\partial \varphi_2^{(0)}}{\partial \xi_2^\alpha} \omega_2^\beta\, \delta(\xi_2)\, d\omega_2\, d\xi_2 = 0.$$

This condition is trivial for an isotropic function.

The symmetry conditions a) will be simple:

$$P_{ik}{}^a \varphi_n^{(0)} = \varphi_n^{(0)}.$$

We obtain the symmetry condition b) for the $\varphi_n^{(0)}$ in zeroth approximation by integrating the complete symmetry condition over **V**:

$$P_k{}^b \varphi_n^{(0)} = \varphi_n^{(0)}.$$

Using this equation the complete symmetry condition for the zeroth approximation function is reduced to the equation

$$F_1(\mathbf{V}, \mathbf{X}, t) = F_1(\mathbf{V} + \omega_k, \mathbf{X} + \xi_k, t).$$

One sees easily that in the approximation required by us (apart from first order terms) this equation is valid, as

$$F_1(\mathbf{V} + \omega_k, \mathbf{X} + \xi_k, t) = F_1(\mathbf{V}, \mathbf{X}, t) + O(R^{-1/4}).$$

The incompressibility condition is in zeroth approximation obtained at once from the total conditions:

$$\int \frac{\partial \varphi_n^{(0)}}{\partial \xi_i^\alpha} \omega_i^\alpha\, d\omega_i = 0.$$

Considering the additional conditions in zeroth approximation, we note a very important fact: they all contain solely the functions $\varphi_n$, are independent of $F_1$, and are universal ones.

## 4. FIRST APPROXIMATION

Before we turn to a consideration of the first approximation we note that the general solution of the zeroth approximation equations can be looked for in the form

$$F_1(\mathbf{V}, \mathbf{X}, t)\, \varphi_n(\mathbf{V}, \omega_2, \xi_2, \ldots, \omega_n, \xi_n | F_1, \varepsilon).$$

Our approximations are valid when $|\xi_i| \ll L$ but then the $\varphi_n$ are completely independent of **V** in zeroth approximation. It is thus natural to assume that at small distances this function depends weakly on **V** and that we can expand it in a series and look for the solution of the zeroth approximation equations $\Phi_n$ in the form

$$\Phi_n = F_1(\mathbf{V}, \mathbf{X}, t) \left[ \varphi_n^{(0)} + \frac{u^\alpha}{\bar{u}^2} v_0 \varphi_{n\alpha}^{(1)} + \ldots \right],$$

since the dependence of $\varphi_n$ on **V** is slow compared with the dependence of $F_1(\mathbf{V}, \mathbf{X}, t)$ on this variable. Here $u^\alpha = V^\alpha - \overline{V^\alpha}$.

We now consider the first approximation. First of all we write down the set of equations for the first approximation functions:

$$\hat{L}_{0n} F_{n+1}^{(1)} + \frac{1}{4\pi} \frac{\partial F_1}{\partial V^\alpha} \int \frac{\partial^2 \varphi_{n+1}^{(0)}}{\partial \xi_{n+1}^\beta \partial \xi_{n+1}^\gamma} T^\alpha(\xi_{n+1})\, \omega_{n+1}^\beta\, \omega_{n+1}^\gamma\, d\omega_{n+1}\, d\xi_{n+1}$$
$$+ \nu \frac{\partial F_1}{\partial V^\alpha} \int d\xi_{n+1}\, d\omega_{n+1}\, \delta(\xi_{n+1})\, \Delta_{n+1} \varphi_{n+1}^{(0)}\, \omega_{n+1}^\alpha = 0. \quad (4.1)$$

We shall look for the solution of the inhomogeneous Eq. (4.1) in the form

$$F_n^{(1)} = \frac{\partial F_1}{\partial V^\alpha} \chi_{\alpha n}^{(1)}.$$

We get the following equations for the functions $\chi_{\alpha n}^{(1)}$:

$$\frac{1}{4\pi} \int \frac{\partial^2 \varphi_{n+1}^{(0)}}{\partial \xi_{n+1}^\beta \partial \xi_{n+1}^\gamma} T^\alpha(\xi_{n+1})\, \omega_{n+1}^\beta\, \omega_{n+1}^\gamma\, d\omega_{n+1}\, d\xi_{n+1}$$
$$+ \nu \int d\omega_{n+1}\, d\xi_{n+1}\, \delta(\xi_{n+1})\, \Delta_{n+1} \varphi_{n+1}^{(0)}\, \omega_{n+1}^\alpha + \hat{L}_{0n} \chi_{\alpha\, n+1}^{(1)} = 0.$$

We shall not write out all additional conditions in view of their unwieldiness. One can verify that as in the zeroth approximation all additional conditions reduce, after second order terms have been dropped, to relations which contain only the functions $\chi_{n\alpha}^{(1)}$ and $\varphi_{n\alpha}^{(1)}$ and which are independent of the variables **V**, **X**, t. The functions $\chi_{n\alpha}^{(1)}$ and $\varphi_{n\alpha}^{(1)}$ are thus universal, uniform, and isotropic vector functions depending on $\omega_2, \xi_2, \ldots, \omega_n, \xi_n$. However, the symmetry conditions b) enable us, as we shall see below, by using the tensor properties of the first approximation corrections, to determine it in explicit form for n = 2. The symmetry conditions reduce in the necessary accuracy for the first approximation to the relations

$$P_k{}^b \varphi_{n\alpha}^{(1)} = \varphi_{n\alpha}^{(1)}, \quad P_k{}^b \chi_{n\alpha}^{(1)} - \chi_{n\alpha}^{(1)} + \omega_k^\alpha \varphi_n^{(0)} = 0. \quad (4.2)$$

Solving these equations one finds easily that

$$\chi_{\alpha n}^{(1)} = -\frac{1}{(n-2)!} \varphi_n^{(0)} \sum_{i=2} \omega_i^\alpha + \sum_k P_k{}^b \chi_{\alpha n}^{(1)} \frac{1}{(n-1)!},$$

$$\varphi_{n\alpha}^{(1)} = \sum_k P_k{}^b \varphi_{n\alpha}^{(1)} \frac{1}{(n-1)!}.$$

Bearing in mind that the functions $\varphi_{n\alpha}^{(1)}$ and $\chi_{n\alpha}^{(1)}$ are isotropic vectors it follows from these relations for n = 2 that $\chi_{2\alpha}^{(1)} = \frac{1}{2}\omega_2^\alpha \varphi_2^{(0)}$ and $\varphi_{2\alpha}^{(1)} = 0$. We find thus that

$$F_2^{(1)} = \frac{1}{2}\omega_2{}^\alpha \varphi_2^{(0)} \frac{\partial F_1}{\partial V^\alpha}. \tag{4.3}$$

One can check that this function satisfies identically all additional conditions (it is impossible to check the consistency condition directly since the functions with n = 3 enter into it).

From the fact that the operators $P_i^b$ commute with the chain operators $L_{on}$ it follows that $\omega_k^\alpha \varphi_n^{(0)} \partial F_1/\partial V^\alpha$ must be a solution of the inhomogeneous problem with an inhomogeneous term which is antisymmetrized with respect to the interchange $P_k^b$. It turns out that this is indeed the case if we take into account that $\varphi_n^{(0)}$ satisfies the zeroth approximation equations. Indeed,

$$\hat{L}_{on}\,\omega_k{}^\alpha \frac{\partial F_1}{\partial V^\alpha}\varphi_{n+1}^{(0)} = \omega_k{}^\alpha \frac{\partial F_1}{\partial V^\alpha}\hat{L}_{0n}\,\varphi_{n+1}^{(0)}$$

$$+\frac{1}{4\pi}\frac{\partial F_1}{\partial V^\alpha}\int \frac{\partial^2 \varphi_{n+1}^{(0)}}{\partial \xi_{n+1}^\beta \partial \xi_{n+1}^\gamma}(T^\alpha(\xi_{n+1}-\xi_k) - T^\alpha(\xi_{n+1}))\,\omega_{n+1}^\beta\,\omega_{n+1}^\gamma\,d\omega_{n+1}\,d\xi_{n+1}$$

$$+\nu\frac{\partial F_1}{\partial V^\alpha}\int d\omega_{n+1}\,d\xi_{n+1}[\delta(\xi_{n+1}-\xi_k) - \delta(\xi_{n+1})]\,\Delta_{n+1}\,\varphi_{n+1}^{(0)}\,\omega_{n+1}^\alpha.$$

The first term in this equation is obtained if we do not differentiate $\omega_k^\alpha$, and vanishes. The second term is the same, as can easily be seen by a change in the integration variable, as the inhomogeneous part of the first approximation equations antisymmetrized with respect to the interchange $P_k^b$.

## 5. DERIVATION OF THE EQUATION FOR THE FIRST DISTRIBUTION FUNCTION

In this section we shall derive an approximate equation for the first distribution function $F_1(\mathbf{V}, \mathbf{X}, t)$. This is an important problem since by using this function one can construct the average velocity field, evaluate all single-point velocity moments and solve such problems as the calculation of the resistive force in turbulent flow and other non-uniform problems.

We shall see in what follows that the set of equations for the two functions $F_1(\mathbf{V}, \mathbf{X}, t)$ and $\epsilon(\mathbf{X}, t)$ is a closed one. In particular, a well-defined damping law for strong uniform turbulence will follow from this set of equations. When deriving the equation we need not know in detail the relative velocities distribution functions; in our equations only a finite number of averages of these enter. We need therefore only some properties and the general nature, for instance, of the functions $\varphi_n^{(0)}$.

The form of the zeroth approximation distribution functions at once suggests that such an equation exists at all. Indeed, the first equation of the chain has the form

$$\frac{\partial F_1}{\partial t} + V^\alpha \frac{\partial F_1}{\partial X^\alpha} + \frac{1}{4\pi}\frac{\partial}{\partial V^\alpha}\int \frac{\partial^2 F_2}{\partial \xi_2{}^\beta \partial \xi_2{}^\gamma} T^\alpha(\xi_2)\,\omega_2{}^\beta\,\omega_2{}^\gamma\,d\omega_2\,d\xi_2$$
$$+\nu\frac{\partial}{\partial V^\alpha}\int d\omega_2\,d\xi_2\,\delta(\xi_2)\Delta_2 F_2\,\omega^\alpha{}_2 + \frac{\partial F_1}{\partial V^\alpha}\frac{\partial \psi}{\partial x^\alpha} = 0. \tag{5.1}$$

If we substitute into this equation $F_2 = F_1\varphi_2^{(0)}(\omega_2, \xi_2)$ we get apparently an equation containing only $F_1$ and some averages. One sees, however, easily that these averages vanish:

$$\int \delta(\xi_2)\,d\xi_2\,d\omega_2\,\Delta_2\varphi_2^{(0)}\,\omega_2{}^\alpha = \int \frac{\partial^2 \varphi_2^{(0)}}{\partial \xi_2{}^\beta \partial \xi_2{}^\gamma} T^\alpha \omega_2{}^\beta \omega_2{}^\gamma\,d\omega_2\,d\xi_2 = 0,$$

since the function $\varphi_2^{(0)}$ is isotropic. We shall see in what follows that the contribution to the "kinetic" equation comes only from the first approximation corrections to the distribution function.

Let us consider the contribution made by the first approximation corrections. First of all we consider the "viscous" term. In first approximation we have

$$\nu\int d\omega_2\,d\xi_2\,\delta(\xi_2)\,\Delta_2\frac{\partial F_2^{(1)}}{\partial V^\alpha}\omega_2{}^\alpha$$
$$= \nu\frac{\partial^2 F_1}{\partial V^\alpha \partial V^\beta}\frac{1}{2}\int d\omega_2\,d\xi_2\,\delta(\xi_2)\,\Delta_2\varphi_2^{(0)}\,\omega_2{}^\alpha\omega_2{}^\beta = \frac{1}{3}\,\epsilon^{(0)}\,\Delta_V F_1,$$

$\epsilon^{(0)}$ is the zeroth order dissipated energy. For the second term which contains the pair function we get the following expression:

$$\frac{1}{4\pi}\frac{\partial^2 F_1}{\partial V^\alpha \partial V^\delta}\int \frac{\partial^2 \varphi_2^{(0)}}{\partial \xi_2{}^\beta \partial \xi_2{}^\gamma} T^\alpha(\xi_2)\,\omega_2{}^\beta\,\omega_2{}^\gamma\,\omega_2{}^\delta\,d\omega_2\,d\xi_2, \tag{5.2}$$

which turns out to be equal to zero by virtue of the incompressibility condition.

In the preceding considerations we have always implicitly used the assumption that the non-local terms in the chain which contain an integration over all distances converge fast and that the integration in fact takes place over a region with linear dimensions of the order $\lambda \ll L$. We can verify this directly for (5.2). A detailed proof of the correctness of the construction of a series of approximations for the correlation functions goes beyond the framework of the present paper. However, the non-local term in the first equation of the chain gives, as we check now, a contribution of order $(U/L)F_1$; to calculate it we must briefly dwell upon the analysis of the structure of non-local terms.

One sees easily that the integral term can be written in the form

$$\frac{\partial}{\partial V^\alpha}\int F_2(\mathbf{V}, \mathbf{X}, b_2, \mathbf{x}_2)\,b_2 T^\alpha(\mathbf{X}-\mathbf{x}_2)\,d\mathbf{x}_2\,db_2, \quad b_2 = \frac{\partial v_2{}^\alpha}{\partial x_2{}^\beta}\frac{\partial v_2{}^\beta}{\partial x_2{}^\alpha}, \tag{5.3}$$

where $F_2(\mathbf{V}, \mathbf{X}, b_2, \mathbf{x}_2)$ is the joint probability for the quantities $b_2$ and $\mathbf{V}$. The quantity $b_2$ is essentially determined by the slow scale. While the reduction of the correlation of two velocities in different points occurs at distances of the order of the main scale L, the reduction of the correlation between the velocity and its derivative in another point will happen fast.

The function $F_2(\mathbf{V}, \mathbf{X}, b_2, \mathbf{x}_2)$ will split into the product $F_1(\mathbf{V}, \mathbf{X}, t)\cdot F(b_2, \mathbf{x}_2)$ at distances $\sim\lambda_0$. If we write (5.3) in the form

$$\frac{\partial}{\partial V^\alpha}\int [F_2(\mathbf{V}, \mathbf{X}, b_2, \mathbf{x}_2) - F_1(\mathbf{V}, \mathbf{X})F(b_2, \mathbf{x}_2)]\,T^\alpha(\mathbf{X}-\mathbf{x}_2)\,b_2\,db_2\,d\mathbf{x}_2$$
$$+\frac{\partial F_1}{\partial V^\alpha}\int \overline{b_2(\mathbf{x}_2)}\,T^\alpha(\mathbf{X}-\mathbf{x}_2)\,d\mathbf{x}_2,$$

the first of the integrals written down will thus be fast converging and will as far as order of magnitude is concerned be the same as the complete integral (of order $(U/L)R^{1/4}$). The second integral can completely be expressed in terms of the function $F_1(\mathbf{V}, \mathbf{X}, t)$ and will be of order U/L. We can similarly consider non-local terms for $n > 2$. In fact, in the first integral we can substitute the zeroth and first approximation functions which we have found. As a result of the substitution it was elucidated that the contribution from the first integral vanishes in the zeroth and first approximations. Up to terms of order $(U/L)R^{-1/4}$, we need therefore retain in the equation for $F_1$ only the second integral.

Finally, in first approximation the function $F_1$ satisfies the following equation

$$\hat{D}F_1 + {}^1\!/_3\varepsilon^{(0)}\Delta_V F_1 = 0, \tag{5.4}$$

where

$$\hat{D} = \frac{\partial}{\partial t} + V^\alpha \frac{\partial}{\partial X^\alpha} + \frac{\partial \bar{p}}{\partial X^\alpha}\frac{\partial}{\partial V^\alpha}.$$

$$\Delta \bar{p} = \frac{\partial^2}{\partial X^\alpha \partial X^\beta} \int V^\alpha V^\beta F_1(\mathbf{V}, \mathbf{X}, t)\, d\mathbf{V}. \tag{5.5}$$

Equation (5.4) contains the unknown function $\epsilon^{(0)}(\mathbf{X}, t)$. The derivation of the equation for $\epsilon^{(0)}(\mathbf{X}, t)$ requires higher approximations as does the derivation of the consistency condition for $F_1$.

## 6. SECOND APPROXIMATION

The set of equations for the second approximation functions takes the following form:

$$\hat{L}_{0n} F^{(2)}_{n+1} + \hat{D}F_1\varphi^{(0)}_n + \nu \frac{\partial}{\partial V^\alpha} \int d\xi_{n+1}\, d\omega_{n+1}\, \delta(\xi_{n+1})\, \Delta_{n+1} F^{(1)}_{n+1}\, \omega^{(\alpha)}_{n+1}$$

$$+ \frac{1}{4\pi}\frac{\partial}{\partial V^\alpha} \int \frac{\partial^2 F^{(1)}_{n+1}}{\partial \xi^\beta_{n+1} \partial \xi^\nu_{n+1}} T^\alpha(\xi_{n+1})\, \omega^\beta_{n+1}\, \omega^\nu_{n+1}\, d\omega_{n+1}\, d\xi_{n+1} = 0. \tag{6.1}$$

In accordance with the general procedure the second approximation correction $F^{(2)}_n$ has the form

$$F^{(2)}_n = \hat{D}F_1\chi^{(2)}_n + \frac{\partial^2 F_1}{\partial V^\alpha \partial V^\beta}\bar{\chi}^{(2)}_{n\alpha\beta} + F_1 \frac{u^\alpha u^\beta}{\bar{u}^2}\frac{v_0^2}{\bar{u}^2}\varphi^{(2)}_{n\alpha\beta}, \tag{6.2}$$

where $\chi^{(2)}_n$ and $\bar{\chi}^{(2)}_{n\alpha\beta}$ depend only on the relative velocities and coordinates and satisfy each their own set of equations obtained from (6.1), while $u^\alpha = V^\alpha - \overline{V^\alpha}$.

Of the additional conditions we note the symmetry condition

$$P_k{}^b (F^{(0)}_n + F^{(1)}_n + F^{(2)}_n) = F^{(0)}_n + F^{(1)}_n + F^{(2)}_n\,,$$

whence, in the appropriate degree of accuracy

$$P_k{}^b \chi^{(2)}_n = \chi^{(2)}_n\,, \qquad P_k{}^b \varphi^{(2)}_{n\alpha\beta} = \varphi^{(2)}_{n\alpha\beta},$$

$$\chi^{(2)}_{n\alpha\beta} = \omega_k{}^\alpha \omega_k{}^\beta \varphi^{(0)}_n + {}^1\!/_2(\omega_k{}^\alpha P_k{}^b \chi^{(1)}_{n\beta} + \omega_k{}^\beta P_k{}^b \chi^{(1)}_{n\alpha}) + P_k{}^b \bar{\chi}^{(2)}_{n\alpha\beta}.$$

One sees easily that the second approximation does not contribute to the "kinetic" equation since the first part of the correction $\chi^{(2)}_2$ is an isotropic scalar function while $\bar{\chi}^{(2)}_{2\alpha\beta}$ is an isotropic tensor function.

The second approximation gives, however, the first non-vanishing contribution to the right-hand side of the consistency condition (2.2) for n = 1:

$$\frac{\partial F_1}{\partial X^\alpha} = -\frac{\partial}{\partial V^\beta} \int \frac{\partial F_2}{\partial \xi_2{}^\alpha}\, \delta(\xi_2)\, \omega_2{}^\beta\, d\omega_2\, d\xi_2,$$

into which we must substitute (6.2) which gives

$$\frac{\partial F_1}{\partial X^\alpha} = k_1 \varepsilon \frac{\partial}{\partial V^\alpha} \Delta_V F_1 + \frac{\varepsilon}{\bar{u}^2}\left[k_2 \frac{\partial}{\partial V^\alpha} F_1 u^2 + k_3 \frac{\partial}{\partial V^\beta} F_1 u^\alpha u^\beta\right],$$

where the universal constants $k_1$, $k_2$, $k_3$ are determined by integrals of $\bar{\chi}^{(2)}_{2\alpha\beta}$ and $\varphi^{(2)}_{2\alpha\beta}$, respectively, and are connected through the relation $3k_2 + k_3 = 0$.

## 7. EQUATION FOR THE ENERGY DISSIPATION

When constructing the approximations we assumed that the zeroth approximation function $\varphi^{(0)}_n$ is a completely well-defined function satisfying the zeroth approximation stationary equation in which we have chosen as the scales for the velocities $\omega_i$ and the distances $\xi_i$ the quantities $v_0 = (\epsilon\nu)^{1/4}$ and $\lambda_0 = (\nu^3/\epsilon)^{1/4}$. Although the zeroth approximation equations become dimensionless for $\nu = 1$ when we introduce these scales, the solution of these equations will still be ambiguous. Indeed, if $\varphi^{(0)}_n(\omega_2, \xi_2, \ldots, \omega_n, \xi_n)$ is a solution, one checks easily that $\alpha^{1-n}\varphi^{(0)}_n(\alpha\omega_2, \alpha^{-1}\xi_2, \ldots, \alpha\omega_n, \alpha^{-1}\xi_n)$ will also be a solution satisfying all additional conditions for any $\alpha > 0$. We choose $\alpha$ in such a way that the dissipation $\epsilon^{(0)}$ evaluated using the zeroth approximation is exactly equal to the true dissipation

$$-\frac{\nu}{2}\int \delta(\xi_2)\, \Delta_2 F_2(\mathbf{V}, \mathbf{X}, t, \omega_2, \xi_2)\, \omega_2{}^2\, d\omega_2\, d\xi_2\, d\mathbf{V}.$$

Denoting by $\epsilon^{(k)}$ the contribution to the dissipation evaluated using the functions of the corresponding approximation we get for this choice of $\alpha$ that $\epsilon^{(1)} + \epsilon^{(2)} + \ldots = 0$. This is also an insufficient equation for $\epsilon$. Restricting ourselves to the second approximation (the first one does not contribute: $\epsilon^{(1)} = 0$) we have

$$\varepsilon^{(2)} = {}^1\!/_2\nu \int \delta(\xi_2)\, \Delta_2 F^{(2)}_2(\mathbf{V}, \mathbf{X}, t, \omega_2, \xi_2)\, \omega_2{}^2\, d\omega_2\, d\xi_2\, d\mathbf{V} = 0.$$

Substituting from (6.2) the expression $F^{(2)}_2$ we get the equation

$$k_4\left(\frac{\partial}{\partial t} + \bar{V}^\alpha \frac{\partial}{\partial X^\alpha}\right)\varepsilon^{1/2} + k_5 \frac{\varepsilon^{3/2}}{\bar{u}^2} = 0. \tag{7.1}$$

The equations (5.4), (6.3), and (7.1) form now a closed set of equations for $F_1(\mathbf{V}, \mathbf{X}, t)$ and $\epsilon(\mathbf{X}, t)$. The constants $k_4$ and $k_5$ are determined by the appropriate integrals of $\chi^{(2)}_2$ and $\varphi^{(2)}_{2\alpha\beta}$ and can be found experimentally.

As an example we consider the problem of the damping of uniform isotropic turbulence. In that case the set of equations takes the following form:

$$\frac{\partial F_1}{\partial t} + \frac{\varepsilon}{3}\Delta \mathbf{V} F_1 = 0, \tag{7.2}$$

$$\frac{\partial}{\partial V^\alpha}\Delta \mathbf{V} F_1 + \frac{k_2}{k_1}\frac{1}{\overline{V^2}}\left[\frac{\partial}{\partial V^\alpha} F_1 V^2 - 3\frac{\partial}{\partial V^\beta} F_1 V^\alpha V^\beta\right] = 0, \tag{7.3}$$

$$\frac{\partial \varepsilon}{\partial t} = -\frac{\varepsilon^2}{\overline{V^2}}\frac{2k_5}{k_4} \tag{7.4}$$

One can easily solve Eq. (7.3) in the isotropic case; the only solution which has no singularities corresponds to a Gaussian distribution for $F_1$:

$$F_1(\mathbf{V}, t) = \left[\frac{k_2}{2k_1}\frac{1}{2\pi\overline{V^2}}\right]^{3/2} \exp\left[-\left(\frac{k_2}{2k_1}\right)^{1/2}\frac{V^2}{\overline{V^2}}\right]. \tag{7.5}$$

Using this function to evaluate $\overline{V^2}$ one can find that $k_2/k_1 = 9/2$. Equations (7.2) and (7.4) lead to the fact that

$$\overline{V^2}(t)\, /\, \overline{V^2}(0) = (1 + t\, /\, t_0)^{(1-k_5/k_4)^{-1}},$$

which is the law for the damping of uniform isotropic turbulence. Experiment shows that with good accuracy $k_5/k_4 = 2$.

## CONCLUSION

The advantage of the formulation given here of the problem of turbulence using several-point distribution functions consists in that it enables us to separate in the problem a small parameter $R^{-1/4}$ directly connected with the fact that the Reynolds number is large. The immediate problem is the construction of approximate expressions for the evolution in time of the distribution function $F_1$ and the average dissipation velocity $\epsilon(\mathbf{X}, t)$ starting from the exact set of coupled equations. The derivation is similar to the derivation of the equations of hydrodynamics and finding the form of the kinetic coefficients from a chain of correlation functions for statistical systems with strong interaction[3] assuming small deviations from equilibrium. In our case the distributions of velocity differences at small distances (viscous, inertial range) are equilibrium ones and the role of the slowly changing hydrodynamic quantities is played by $F_1(V, \mathbf{X}, t)$ and $\epsilon(\mathbf{X}, t)$.

We succeeded in constructing the above-mentioned system without explicitly solving the equations in zeroth and subsequent approximations but studying the functional dependence upon $F_1$ and $\epsilon$ of the corrections in the expansion in $R^{-1/4}$. Then, as in the derivation of the hydrodynamic equations, we choose at once a solution in which the dependence on the variables $\mathbf{X}$, t is completely coupled to the dependence of the functions $F_1$ and $\epsilon$ on these variables. The proof of the correctness of the construction of such a solution is, apparently, just as complicated as the analogous proof for statistical systems with a strong interaction.

The set of equations obtained contains some universal constants and their evaluation can be realized by solving the equations in zeroth and subsequent approximations. Notwithstanding the similarity with the hydrodynamic approximation in statistical mechanics it is necessary to note the peculiar nature of the relaxation which is connected only with functions at small distances. The first attempt to use directly the existence of such a relaxation, in particular, the assumption about the stationarity at small distances is due to Kolmogorov[2] who in the equations for the third moment dropped the corresponding small terms.

It is also necessary to emphasize the very important statement that the distribution function $F_1$ must satisfy two equations: a ''kinetic'' equation and the so-called consistency equation. The latter reflects the fact that in the Euler picture of turbulence the coordinate $\mathbf{X}$ is not a random one and the above-mentioned dependence vanishes, for instance, when the distribution function is integrated over the velocities.

When we state the boundary problem the system must satisfy boundary conditions at solid surfaces or surfaces where one goes over into a non-turbulent fluid. It seems to us that in zeroth approximation the boundary condition consists in the vanishing of the velocity component along the outward normal and that it reflects a well-defined picture about the division boundary according to which fluid can only flow into but not out of the turbulent region. However, a regular derivation of such a boundary condition is the topic of a separate problem.

We note that for the construction of the theory we needed only very qualitative assumptions about the properties of the distribution functions at small distances which, apparently, are corroborated by experiment.

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