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SELF-FOCUSING OF THE ION ACOUSTIC WAVE IN A PLASMA

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An asymptotic method is used to analyze the propagation of longwave perturbations in a plasma in the presence of a periodic ion-acoustic wave (PW). It is found that a plasma in which a PW is excited is unstable against perturbations that propagate at a large angle with respect to the PW. The growth time for the PW instability is much smaller than the breaking time of the acoustic wave.

1. The propagation of a finite-amplitude ion-acoustic wave in a plasma can have a strong effect on the collective properties of the plasma as a whole. It is therefore of interest to analyze the small oscillations of an infinite uniform plasma in which there is excited a plane periodic ion-acoustic wave with arbitrary amplitude. The solution of the plasma equations in the form of a periodic wave whose profile remains fixed in time has been obtained by Sagdeev in[1]. The propagation of weak longwave perturbations in a plasma with a periodic ion-acoustic wave (PW) has been investigated in[2]. It is found that these perturbations propagate along the PW with the group velocity of the PW, which diminishes as the amplitude of the PW increases. Perturbations that propagate in the opposite direction are carried by a PW of finite amplitude in such a way that their velocity is also found to be directed along the velocity of the PW, and this velocity is an increasing uniform function of the amplitude of the PW.

A PW of modest amplitude is stable against longwave perturbations that propagate along the vector corresponding to the velocity of the PW but, as will be shown below, the PW is unstable against longwave perturbations that propagate at large angles with respect to the PW. In other words, because of the nonlinearity of the medium there arises a feedback effect on the PW which leads to the modification of the uniformity of the front associated with the PW. A similar phenomenon occurs in nonlinear optics. It is well known that a monochromatic plane wave that propagates in a nonlinear medium is broken up into a beam of filaments because of the feedback effect; at some distance from the source the filaments themselves will be modified.[3] The characteristic length for the decay of the PW due to the instability is given by $\lambda\sqrt{c/v_0}$ where c is the ion-acoustic velocity, λ is the wavelength of the PW and v_0 is the velocity of the PW. This decay length can be smaller than $\lambda c/v_0$, the "breaking" length for an acoustic wave with the same wavelength and amplitude.[4]

The growth rate found for the instability is larger than the growth rate found by Zakharov[5] for the resonance decay of a weak ion-acoustic wave into two ion-acoustic waves. In contrast with the present work, it is assumed in[5] that the wave spectrum is continuous but that the width of the wave packet is small. (The spectrum of the PW is obviously discrete.)

The investigation of the stability of the ion-acoustic PW reduces to an investigation of the stability of the nontrivial stationary solution of a system of nonlinear partial differential equations. In the present work, as in[2,6], the analysis is carried out as follows. We first find an expression for the profile of the PW in terms of the integrals of the motion of the PW, that is to say, the amplitude and phase of the PW, the energy of the ions, and the mean ion velocity in the rest system of the PW. We then carry out a transformation of the variables in the equations in such a way that the hydrodynamic equations for the velocity, density and electric field become equations for the integrals of the motion of the PW. It is evident from the equations that are obtained that the propagation of a weak longwave perturbation in a plasma with a PW corresponds to small oscillations of the integrals of the motion of the PW. We then use the van der Pol method of averaging to average the plasma equations in the new variables and linearize with respect to these small oscillations of the integrals of the PW. By this means we obtain the dispersion equation for the weak longwave perturbations in a plasma in which a PW is excited. The existence of complex roots in the dispersion equation then indicates an instability of the PW.

2. We start with the hydrodynamic equation for plasma with cold ions in which the time t and the coordinate x are replaced by the following variables:[2]

$$Dx_1 = x + u_1 t, \quad Dx_2 = x + u_2 t. \tag{1}$$

Here, D is the electron Debye radius; u_1 and u_2 are fixed velocities. The quantity v_x, the hydrodynamic ion velocity along x, the electron density n_e and the ion density n_i are replaced by the new variables f, q and N which are defined by the relations

$$v_x = c\sqrt{f} - u_1, \quad n_i = n_0 q/\sqrt{f}, \quad n_e = n_0 N e^{-f/2}, \quad c \equiv \sqrt{T/m_i}. \tag{2}$$

Here, T is the electron temperature, m_i is the ion mass, and n_0 is the mean particle density in the plasma. The meaning of the new variables can be understood as follows (cf. Sec. 3). If a plasma supports a periodic wave that propagates along x with a velocity u_1, then in the rest system of the PW $c\sqrt{f}$ is the hydrodynamic ion velocity, cq is the mean ion velocity, which is equal to the reciprocal of the average of $1/c\sqrt{f}$, and $T \ln N$ is the total ion energy. If there is no PW then q and N are constant and f is a periodic function of x_1.

In the new variables the ion equation of continuity $\partial n_i/\partial t + \nabla n_i v = 0$ is written in the following form:[2]

$$\frac{\partial q}{\partial x_1}+\frac{\partial}{\partial x_2}\left(q-\frac{uq}{\sqrt{f}}\right)+\nabla_\perp\frac{\mathbf{v}_\perp q}{\sqrt{f}}=0,$$

$$u\equiv\frac{u_1-u_2}{c},\quad \nabla_\perp\equiv D\left(0,\frac{\partial}{\partial y},\frac{\partial}{\partial z}\right),\quad D^2=\frac{T}{4\pi e^2 n_0}. \tag{3}$$

Here, $\mathbf{v}_\perp$ is the ion velocity perpendicular to the x axis divided by c, and satisfies the Euler equation

$$\sqrt{f}\frac{\partial \mathbf{v}_\perp}{\partial x_1}+(\sqrt{f}-u)\frac{\partial \mathbf{v}_\perp}{\partial x_2}+(\mathbf{v}_\perp\nabla_\perp)\mathbf{v}_\perp=-\frac{e}{T}\nabla_\perp\varphi, \tag{4}$$

where φ is the electric potential. Using Eq. (2), we can write the Euler equation

$$\frac{\partial v_x}{\partial t}+(\mathbf{v}\nabla)v_x=-\frac{e}{m_i}\frac{\partial\varphi}{\partial x}$$

in the following form:

$$\frac{\partial f}{\partial x_1}+\frac{\partial}{\partial x_2}(\sqrt{f}-u)^2+2(\mathbf{v}_\perp\nabla_\perp)\sqrt{f}=-\frac{2e}{T}\left(\frac{\partial}{\partial x_1}+\frac{\partial}{\partial x_2}\right)\varphi. \tag{5}$$

Making use of the fact that the electron thermal velocity is much greater than the ion-acoustic velocity and using Eq. (2) we have

$$n_e=n_0 e^{e\varphi/T},\quad \ln N=\frac{e\varphi}{T}+\frac{f}{2}. \tag{6}$$

3. If a periodic ion-acoustic wave (PW) propagates in the plasma along the x axis with a velocity u_1, the quantity $\mathbf{v}_\perp$, and all of the derivatives except for derivatives with respect to x_1 are found to vanish in (3)–(5). Then, following[2] and using the Poisson equation $\partial^2\varphi/\partial x^2 = 4\pi e(n_e - n_i)$ and (3)–(6) we obtain the equation for a nonlinear oscillator in which the role of the time is played by the variable x_1:

$$f''=2(q_0/\sqrt{f}-N_0e^{-f/2}). \tag{7}$$

Here, the primes denote derivatives with respect to x_1 while q_0 and N_0 are constants indicating the values of q and N from (2) in the periodic wave, where $q_0 = u_1/c$.

Integrating (7) we have

$$f'^2=8(W+q_0\sqrt{f}+N_0e^{-f/2}), \tag{8}$$

where W is a constant of integration which specifies the amplitude of the periodic wave. With the values of q_0, N_0 and W for some bounded region. Equations (7) and (8) determine f as a periodic function of x_1 that we will denote by $\Phi(W, x_1)$. Using Eqs. (7) and (8) we find

$$\Phi'\frac{\partial\Phi'}{\partial W}-\Phi''\frac{\partial\Phi}{\partial W}=4. \tag{9}$$

Making use of the condition that the electron and ion density in the unperturbed PW must be n_0, we obtain

$$\langle q_0/\sqrt{\Phi}\rangle=N_0\langle e^{-\Phi/2}\rangle=1, \tag{10}$$

where the angle brackets denote averages over x_1.

4. When weak longwave (adiabatic) perturbations are superimposed on the periodic wave the plasma is described by the general equations (3)–(6) and Poisson's equation, in which the derivatives $\nabla_\perp$ and $\partial/\partial x_2$ play the role of small parameters in the perturbation.[2,7] The quantity f is replaced by two variables a and α which are defined by

$$f(\rho)=\Phi(W,x_1)+a(\rho)\partial\Phi/\partial W+\alpha(\rho)\Phi'\equiv\Phi+f_1, \tag{11}$$

where ρ denotes the ensemble of variables x_1, x_2, y and z. The term $a\partial\Phi/\partial W$ contains as a small parameter the ratio of the perturbation amplitude a to the amplitude of the periodic wave; the quantity α is the small perturbation of the phase of the PW.

In order to make the transformation unique and in order to obtain the first-order equation in $\partial/\partial x_1$ involving the parameters a and α, we impose the following condition on the new functions:[7]

$$\frac{\partial f}{\partial x_1}+\frac{\partial(\sqrt{f}-u)^2}{\partial x_2}=\Phi'+a\frac{\partial\Phi'}{\partial W}+\alpha\Phi''. \tag{12}$$

We now write the quantities q and N in the form $q = q_0 + q_1$ and $N = N_0 + N_1$ and linearize all equations with respect to q_1, N_1, a, α and $\mathbf{v}_\perp$. In other words, we are considering infinitesimal perturbations on a stationary background (PW) very much in the way one considers infinitesimal perturbations on the background of a uniform stationary plasma.

Substituting the electric field from the linearized equation (5) in Poisson's equation and taking account of Eq. (12) we have

$$\frac{1}{2}\left(\frac{\partial}{\partial x_1}+\frac{\partial}{\partial x_2}\right)\left(\Phi'+a\frac{\partial\Phi'}{\partial W}+a\Phi''\right)-\Delta_\perp\left\{\ln N-\frac{1}{2}(\Phi+f_1)\right\}$$
$$=\frac{q_0+q_1}{(\Phi+f_1)^{1/2}}-(N_0+N_1)\exp\left\{-\frac{1}{2}(\Phi+f_1)\right\}. \tag{13}$$

Here, the symbol $\Delta_\perp\equiv D^2(\partial^2/\partial_y^2+\partial^2/\partial z^2)$ represents the expression for the electric potential that follows from Eq. (6). From Eq. (7) which determines Φ we see that the principle parts are reduced in Eq. (13). Linearizing Eq. (13) we have

$$(a'+\dot{a})\frac{\partial\Phi'}{\partial W}+(\alpha'+\dot{\alpha})\Phi''-2\frac{\Delta_\perp N_1}{N_0}+\Delta_\perp f_1=2\left(\frac{q_1}{\sqrt{\Phi}}-N_1e^{-\Phi/2}\right), \tag{14}$$

where the dots denotes derivatives with respect to x_2. Linearizing Eq. (12) we have

$$(a'+\dot{a})\frac{\partial\Phi}{\partial W}+(\alpha'+\dot{\alpha})\Phi'=\left(\dot{a}\frac{\partial\Phi}{\partial W}+\dot{\alpha}\Phi'\right)\frac{u}{\sqrt{\Phi}}. \tag{15}$$

Substituting the expression for the electric field (6) in (5) and linearizing we have

$$N_1'+\dot{N}_1=uN_0\dot{R},\quad R\equiv\sqrt{\Phi+f_1}. \tag{16}$$

We now write (3) and (4) in the form

$$q_1'+\dot{q}_1\left(1-\frac{u}{\sqrt{\Phi}}\right)=-\frac{uq_0}{2}\Phi^{-3/2}\left(\dot{a}\frac{\partial\Phi}{\partial W}+\dot{\alpha}\Phi'\right)-\frac{q_0}{\sqrt{\Phi}}\nabla_\perp\mathbf{v}_\perp, \tag{17}$$

$$\mathbf{v}_\perp'+\dot{\mathbf{v}}_\perp\left(1-\frac{u}{\sqrt{\Phi}}\right)=-\frac{\nabla_\perp N_1}{N_0\sqrt{\Phi}}+\frac{\nabla_\perp f_1}{2\sqrt{\Phi}}. \tag{18}$$

Solving (14) and (15) with respect to a' and α' and taking account of (9) we have

$$4a'+4\dot{a}=-\left(\dot{a}\frac{\partial\Phi}{\partial W}+\dot{\alpha}\Phi'\right)\frac{u}{\sqrt{\Phi}}\Phi''+2\frac{\Delta_\perp N_1}{N_0}\Phi'$$
$$-\left(\frac{\partial\Phi}{\partial W}\Delta_\perp a+\Phi'\Delta_\perp\alpha\right)\Phi'+2\Phi'\left(\frac{q_1}{\sqrt{\Phi}}-N_1e^{-\Phi/2}\right); \tag{19}$$

$$4\alpha'+4\dot{\alpha}=\frac{u}{\sqrt{\Phi}}\left(\frac{\partial\Phi}{\partial W}\dot{a}+\Phi'\dot{\alpha}\right)\frac{\partial\Phi'}{\partial W}-2\frac{\Delta_\perp N_1}{N_0}\frac{\partial\Phi}{\partial W}$$
$$+\left(\frac{\partial\Phi}{\partial W}\Delta_\perp a+\Phi'\Delta_\perp\alpha\right)\frac{\partial\Phi}{\partial W}-2\left(\frac{q_1}{\sqrt{\Phi}}-N_1e^{-\Phi/2}\right)\frac{\partial\Phi}{\partial W}. \tag{20}$$

Thus, we have obtained a system of equations (16)–(20) for oscillations of the parameters α, a, N_1, q_1 and $v_\perp$. It is evident from these equations that the derivative of each of these parameters with respect to x_1 is of the same order as the derivative with respect to x_2, y and z. The derivative operator with respect to x_1 is of the order of the "amplitude" of the dependence on x_1 divided by the length of the periodic wave while the derivative with respect to x_2 (or y, z) is the amplitude of the dependence on x_2 divided by the wavelength of the perturbation.[7] By hypothesis the length of the periodic wave is much smaller than the wavelength of the perturbation so that the "amplitude" of the dependence on x_2, y and z is much greater than the amplitude of the dependence on x_1. Hence, in all places in which there is no derivative of a parameter with respect to x_1 the dependence on this variable can be neglected.[2,7] (The last term in (19) does not contain the smallness parameter of the asymptotic expansion, which is equal to the derivatives with respect to x_2 or y, so that we must take account of the dependence of q_1 and N_1 on x_1 in this term. We note further that the periodic wave propagates along x with a velocity u_1 while in the linear approximation the perturbation propagates with a velocity v_2. Because of this difference in phase velocities there will be a rapidly oscillating dependence of the perturbation on x_1 and a "slowly varying" (in the sense of the definition in[7]) dependence on x_2 and y and z.)

In the light of these remarks we now can carry out an averaging of Eqs. (16)–(20) with respect to x_1. Taking account of the fact that the average of the derivatives of the parameters with respect to the variable x_1 must vanish, we have

$$\dot{N}_1 = uN_0 p_1 \dot{a}, \quad p_1 \equiv \frac{\partial}{\partial W}\langle\sqrt{\Phi}\rangle; \tag{21}$$

$$\frac{q_0 - u}{q_0}\dot{q}_1 = uq_0 p_2 \dot{a} - \nabla_\perp v_\perp, \quad p_2 \equiv \frac{\partial}{\partial W}\left\langle\frac{1}{\sqrt{\Phi}}\right\rangle; \tag{22}$$

$$\dot{v}_\perp = p_1 \nabla_\perp a; \tag{23}$$

$$4\dot{a} = -2u\left\langle\left(\frac{q_0}{\Phi} - \frac{N_0}{\sqrt{\Phi}}e^{-\Phi/2}\right)\frac{\partial\Phi}{\partial W}\right\rangle\dot{a} - \langle\Phi'^2\rangle\Delta_\perp a + 4\langle\sqrt{\Phi'}q_1\rangle - 2\langle N_1\Phi' e^{-\Phi/2}\rangle; \tag{24}$$

$$4\dot{\alpha} = -\frac{2}{N_0}\left\langle\frac{\partial\Phi}{\partial W}\right\rangle\Delta_\perp N_1 + \left\langle\left(\frac{\partial\Phi}{\partial W}\right)^2\right\rangle\Delta_\perp a - 4p_1 q_1 - 4\frac{\partial}{\partial W}\langle e^{-\Phi/2}\rangle N_1 + \frac{u}{2}\left\langle\frac{1}{\sqrt{\Phi}}\frac{\partial}{\partial W}\Phi'^2\right\rangle\dot{\alpha}. \tag{25}$$

Following the averaging process we integrate the last two term in Eq. (24) by parts and substitute the expressions for q_1' and N_1' from Eq. (16) and (17); in this way we find

$$\dot{a} = (\langle\sqrt{\Phi}\rangle - u)\dot{q}_1 + \langle e^{-\Phi/2}\rangle \dot{N}_1 - \frac{1}{4}\langle\Phi'^2\rangle\Delta_\perp\alpha + q_0\nabla_\perp v_\perp. \tag{26}$$

We have now obtained a system of linear equations with constant coefficients (21)–(26). Certain of these coefficients have been computed for the general case using an electronic computer.[2] For a low-amplitude PW, in which W is less than $-(1+q_0^2)$ these coefficients can be computed by expansion of Eqs. (7) and (8):

$$\langle\sqrt{\Phi}\rangle \approx q_0 - \frac{A}{2q_0}, \quad p_2 \approx \frac{3}{2q_0^3(1-q_0^2)},$$

$$\langle\Phi'^2\rangle \approx 4(1-q_0^2)A, \quad \left\langle\frac{1}{\sqrt{\Phi}}\frac{\partial}{\partial W}\Phi'^2\right\rangle \approx \frac{4}{q_0}, \tag{27}$$

$$\left\langle\left(\frac{\partial\Phi}{\partial W}\right)^2\right\rangle \approx \frac{q_0^2}{(1-q_0^2)^2 A}, \quad A \approx \left\langle\frac{v_x^2}{c^2}\right\rangle \approx \frac{W + q_0^2 + 1}{1 - q_0^2}.$$

We now write the dependence of the averaged parameters on the coordinate in the form $\exp(iDk_\parallel x_2 + ik_\perp r_\perp)$. For a low-amplitude PW, taking account of Eq. (27), from Eqs. (21) and (26) we then obtain the following dispersion equation for u:

$$k_\parallel^2(u - q_0 + q_0^3) = \frac{D^2 q_0^4 k_\perp^4}{2(u-2q_0)} - \frac{3uk_\parallel^2 + q_0 k_\perp^2}{4q_0(u-q_0)}A + \frac{k_\perp^2 A}{2(u-q_0)(u-2q_0)}\left[\frac{3u(2q_0-u)}{q_0} + \frac{k_\perp^2}{k_\parallel^2}q_0\right]. \tag{28}$$

This equation is a cubic equation in $u_2 = c(q_0 - u)$, the phase velocity of the perturbation along x; it then follows that small oscillations with negative energy are possible in a plasma that supports an ion-acoustic PW.[8]

We now investigate the dispersion equation (28) for an infinitesimal amplitude of the PW; in this case we multiply by $(u - q_0)(u - 2q_0)$ and write A = 0. Then the two roots of (28) assume the form

$$u_2 = \tfrac{1}{2}c\{q_0^3 - q_0 \pm [(q_0^3 + q_0)^2 + 2q_0^4 D^2 k_\perp^4 k_\parallel^{-2}]^{1/2}\}.$$

It is evident that in the propagation of a perturbation along the PW, i.e., $k_\perp = 0$, the phase velocity of the perturbation is $u_2 = cq_0^3 = u_1^3 c^{-2}$, which coincides with the group velocity of a weak ion-acoustic wave characterized by a phase velocity u_1.

The phase velocity of perturbations that propagate against the PW is $-u_1$. For propagation at a right angle with respect to the PW, i.e., $k_\parallel \to 0$ the phase velocity of the perturbation along x approaches $\pm\infty$ and the phase velocity itself $k_\parallel u_2 (k_\parallel^2 + k_\perp^2)^{-1/2}$ approaches $ck_\perp D/\sqrt{2}$ and is much smaller than the phase velocity of the PW since we take $k_\perp D \ll 1$ by hypothesis. The third root $u = q_0$ yields $u_2 = 0$. In the linear case this root corresponds to the phase velocity of the standing wave. In the nonlinear case there are no solutions of the plasma equations in the form of a standing wave, but the PW can interact with weak long-wave oscillations and can convert them into standing waves.

Thus, the phase velocity of longwave perturbations in a plasma containing a PW does not coincide with the ion-acoustic velocity even if the PW exhibits a small amplitude provided, obviously, that the amplitude of perturbation is much smaller than the amplitude of the PW.

The small quantity A makes a finite contribution in (28) when the propagation is essentially transverse to the PW; in this case $(k_\perp/k_\parallel)^2 \gg 1$. Under these conditions, neglecting small terms we can write Eq. (28) in the form

$$2(u - q_0 + q_0^3)(u - q_0)(u - 2q_0) = D^2 q_0^4 k_\perp^4 k_\parallel^{-2}(u - q_0) + q_0(k_\perp/k_\parallel)^4 A. \tag{29}$$

When $(k_\perp/k_\parallel)^4 A > (1/4)q_0^6(2 + q_0^2)$ the first term on the right side can be neglected and Eq. (29) exhibits complex roots:

$$\frac{u_2}{c} \approx \frac{1}{2}q_0^3 \pm i\left[\left(\frac{k_\perp}{k_\parallel}\right)^4 \frac{A}{2+q_0^2} - \frac{q_0^6}{4}\right]^{1/2}, \tag{30}$$

which indicates the instability of the PW with respect to excitation of these perturbations.

The quantity $k_{\parallel}^{-1}$ is limited from above by the length of the interval over which the PW exists, the effective length of the PW. When $k_{\parallel} \to 0$ the imaginary part of the Eq. (30) approaches infinity. This means that the effective length of the PW must be bounded.

Let us now assume that an infinite plane radiates a periodic ion-acoustic wave into a semi-infinite plasma and that because of the nonlinear effects the neighborhood of the region in the front of the PW experiences an interaction that modifies the PW. Taking $q_0 = 1$ we can use Eq. (30) to estimate the distance L from the radiating plane to the point at which an appreciable distortion of the PW occurs. The maximum growth rate obtains for the largest value of $k_{\perp}/k_{\parallel}$. We take $k_{\parallel} \sim 1/L$ and $k_{\perp} \sim 1/3\lambda$, where λ is the wavelength of the PW. Then, using Eq. (30) we find that the growth time for the perturbation is $9\lambda^2/cL\sqrt{A}$.

Multiplying this quantity by $c/2$, the velocity of propagation of the perturbation, we find that the distance in which the perturbation grows significantly is of order $L \sim 9\lambda^2/2L\sqrt{A}$, whence $L \sim \lambda A^{-1/4}$, which can be verified experimentally.

The results obtained here apply only for a low-amplitude PW, in which case the expansion in (27) is valid. It is evident from Eq. (30) that the larger the amplitude of the PW the smaller the value of the minimum angle at which the propagation of unstable perturbations is possible. However, at reasonably high amplitudes for the PW we find that Eq. (27) is no longer satisfied and in this case one must resort to numerical calculations in order to obtain the coefficients in the dispersion equation. These calculations have been carried out in[2] for the case in which the longwave perturbation propagates along the PW ($k_{\perp} = 0$). It is found that under these conditions an instability appears for an amplitude of the ion variation in the PW which is approximately equal to half the mean ion density in the plasma.

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[1] R. Z. Sagdeev, Reviews of Plasma Physics, Consultants Bureau, New York, 1966, Vol. 4.

[2] V. I, Petviashvili, Zh. Tekh. Fiz. **38**, 205 (1968) [Sov. Phys.-Tech. Phys. **13**, 149 (1968)].

[3] G. A. Askar'yan, Zh. Eksp. Teor. Fiz. **42**, 1567 (1962) [Sov. Phys. JETP **15**, 1088 (1962)]; A. G. Litvak and V. I. Talanov, Izv. vyssh. uch zav. Radiofizika **9**, 900 (1965).

[4] L. D. Landau and E. M. Lifshitz, Fluid Mechanics, Addison-Wesley, Reading, Mass., 1969.

[5] V. E. Zakharov, Zh. Eksp. Teor. Fiz. **41**, 1107 (1960) [Sov. Phys. JETP **14**, 791 (1962)].

[6] V. I. Petviashvili, Zh. Tekh. Fiz. [Sov. Phys.-Tech. Phys.] (in press).

[7] N. N. Bogolyubov and Yu. A. Mitropolskii, Asymptotic Methods in the Theory of Nonlinear Oscillations, Gordon and Breach, New York, 1961.

[8] B. B. Kadomtsev, A. M. Mikhaĭlovskiĭ and A. V. Timofeev, Zh. Eksp. Teor. Fiz. **47**, 2266 195 (1964) [Sov. Phys. JETP **20**, 1517 (1965)].

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