CLASSICAL THEORY OF NON-IDEAL COHERENT LIGHT BEAMS

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Approximate solutions of Maxwell's equations are obtained for the electric and magnetic field components of a polarized monochromatic light beam in free space. Its geometric and physical properties are discussed. It is found that the flux lines of the beam energy possess space curvature and torsion. It is demonstrated that the appearance of longitudinal field components leads to the possibility of ascribing a rest mass to each finite section of the beam. It is found that the narrower the energy channel of the beam the greater is the role which partial plane waves moving at a large angle to its axis play in formation of the beam.

INTRODUCTION

WHEN considering phenomena connected with the propagation of narrow quasimonochromatic light beams, it is customary to use the results of the theory developed for plane waves. A number of recent papers[1-4] consider beams with a finite aperture. In [5,6] are given solutions of the scalar wave equation in the diffusion approximation, containing information on the spatial distribution of only the amplitude and phase of the beam between the mirrors of the resonator. Solutions of this type describe well certain energy features of light beams, and also make it possible to develop a theory of their reflection and transmission through passive optical systems.[6,7] However, the scalar solution does not contain the entire information concerning the macro- and microstructure of the beam, concerning its ponderomotive action. It is therefore of interest to obtain a vector solution of the problem.

A common property of all real beams of electromagnetic waves is the dependence of the field amplitude and of the shape of the phase front on the coordinates, and also the nonvanishing of the transverse components of the wave vector. This always leads to the appearance of longitudinal components of the electromagnetic field, in contrast to the case of arbitrary superposition of plane waves propagating in one direction and forming an ideal light beam. For the considered non-ideal beam, the square of the length of the fourth-momentum does not vanish, making it possible to ascribe a rest mass to it, as is customary in the theory of hollow waveguides and resonators.

In free space, each mode of such a beam represents an autonomous formation propagating as a unit. A narrow light beam passing through a layer of matter or being refracted or reflected also behaves like a formation which on the whole does not obey the laws of geometrical optics.

ELECTROMAGNETIC FIELD OF LIGHT BEAM

The vector functions **E** and **H**, describing the electromagnetic field of a narrow monochromatic light beam propagating in the direction of the z axis in free space, will be sought in the form

$$\mathbf{E}(x, y, z, t) = \mathbf{E}_0(x, y, z) \exp[i(k_0 z - \omega t)],$$
$$\mathbf{H}(x, y, z, t) = \mathbf{H}_0(x, y, z) \exp[i(k_0 z - \omega t)]. \quad (1)$$

We assume that the beam is circularly polarized in the plane v = const[1)]

$$E_y = jE_x, \quad j^2 = -1, \quad j = \pm i, \quad (2)$$

and that the transverse components of the electric and magnetic vector in this plane are perpendicular, i.e.,

$$E_{0x} = H_{0y}, \qquad E_{0y} = -H_{0x}. \quad (3)$$

Conditions (2) and (3) can be satisfied simultaneously by assuming that the amplitudes E_{0z} and H_{0z} in the propagation direction change little over the extent of a wavelength

$$\lambda\!\!\!^{-} |\partial_z E_{0z}| \ll |E_{0z}|, \qquad \lambda\!\!\!^{-} |\partial_z H_{0z}| \ll |H_{0z}|, \quad (4)$$

where $\lambda\!\!\!^{-} = 1/k_0 = c/\omega$. We then obtain from Maxwell's equations a system of equations for the amplitudes $\mathbf{E}_0$ and $\mathbf{H}_0$:

$$(\partial_x^2 + \partial_y^2 + 2ik_0\partial_z)E_{0x} = 0, \quad (5a)$$
$$E_{0y} = jE_{0x}, \quad (5b)$$
$$E_{0z} = i\lambda\!\!\!^{-}(\partial_x + j\partial_y)E_{0x}, \quad (5c)$$
$$\mathbf{H} = -j\mathbf{E}, \quad (5d)$$

making it possible to determine uniquely all the components of the field intensity vectors of the light beam propagating in the direction of the z axis.

We note that when conditions (2)–(4) are satisfied we obtain directly for E_{0x} not the complete amplitude equation, but the Schrödinger-type Eq. (5a). It takes into account diffraction phenomena, for example the effect of transverse "diffusion" of the ray amplitudes over the front of the wave as the wave propagates. By specifying the function $E_{0x}(x, y, 0)$ in the plane $z = 0$ we determine the behavior of $E_{0x}(x, y, z)$ (and of the remaining components of the electromagnetic field) in all of space.

Sharply-directional radiation formed in lasers is characterized by a Gaussian distribution of the field intensity over the beam cross section. A scalar function possessing this property and satisfying Eq. (5a), was obtained for the case of a confocal resonator in [6].

1)The upper and lower sign respectively denote throughout right-hand and left-hand circular polarization.

Using the results obtained there, in accordance with (5), we get an expression for the components of the electric and magnetic fields of a circularly polarized light beam:

$$E_x = \sum_{m,n=0}^{\infty} E_x^{mn} = \sum_{m,n=0}^{\infty} a_{mn}\Psi_m\left(\frac{x}{\sigma}\right)\Psi_n\left(\frac{y}{\sigma}\right)e^{i\Phi_{mn}};\quad E_y = jE_x,$$

$$E_z = \sum_{m,n=0}^{\infty} E_z^{mn} = \sum_{m,n=0}^{\infty} a_{mn}\left\{-\frac{\lambda r}{\sigma_0\sigma}\Psi_m\left(\frac{x}{\sigma}\right)\Psi_n\left(\frac{y}{\sigma}\right)e^{i\alpha}\right.$$

$$\left.+\frac{i\lambda}{\sigma}\sqrt{2m}\,\Psi_{m-1}\left(\frac{x}{\sigma}\right)\Psi_n\left(\frac{y}{\sigma}\right) \mp \frac{\lambda}{\sigma}\sqrt{2n}\,\Psi_m\left(\frac{x}{\sigma}\right)\Psi_{n-1}\left(\frac{y}{\sigma}\right)\right\}e^{i\Phi_{mn}};$$

$$\mathbf{H} = -j\mathbf{E}. \tag{6}$$

Here

$$\Phi_{mn} = -\omega t + k_0 z + r^2/2\lambda R - (m+n+1)\operatorname{arctg}(\lambda z/\sigma_0^2),$$
$$\alpha = \pm\operatorname{arctg}(y/x) + \operatorname{arctg}(\sigma_0^2/\lambda z),$$

a_{mn}—constant dimensional coefficient which determines the contribution of different modes to the field of the beam,

$$\sigma(z) = \sigma_0\{1 + (\lambda z/\sigma_0^2)^2\}^{1/2} \tag{7}$$

—running effective radius of the light beam, $R(z) = \sigma_0^2\sigma^2/\lambda^2 z$—radius of curvature of the phase front near the z axis, $r = (x^2 + y^2)^{1/2}$ and Ψ_m—normalized Hermite functions. The vector potential of the beam field is $\mathbf{A} = \pm\lambda\mathbf{H}$.

The aggregate of all the possible spatial modes $(\mathbf{E}^{mn}, \mathbf{H}^{mn})$ (6) forms a non-ideal monochromatic light beam. Each mode represents an independent solution of the system (5). An elliptically (or linearly) polarized beam can be obtained by superposition of circularly polarized beams of opposite polarization.

The dependence of the phases of the longitudinal components E_z^{mn} and H_z^{mn} on the coordinates does not coincide with the dependence for the transverse components. This circumstance distinguishes significantly the solutions (6) from the corresponding solutions of the quasi-optical approximation, where the phase functions of all the field components are assumed to be equally dependent on the coordinates. The latter causes the vectors **E** and **H** to lie in a plane that is tangent to the equal-phase surface, something that does not occur in our case. In this sense, the electromagnetic field described by (6) is not transverse. The obtained expressions (6) approximate well the field of a light beam in all of space under the condition $\sigma_0 > \lambda$.

The real components of the electric and magnetic fields of the fundamental (0, 0) mode of a circularly polarized beam are given by

$$\begin{aligned} E_x &= A(r,z)\cos\Phi, & H_x &= \pm A(r,z)\sin\Phi,\\ E_y &= \mp A(r,z)\sin\Phi, & H_y &= A(r,z)\cos\Phi,\\ E_z &= A(r,z)B(r,z)\cos(\Phi+\alpha), & H_z &= \pm A(r,z)B(r,z)\sin(\Phi+\alpha), \end{aligned} \tag{8}$$

where

$$A(r,z) = a_{00}\Psi_0\left(\frac{x}{\sigma}\right)\Psi_0\left(\frac{y}{\sigma}\right) = E_0\left(\frac{\sigma_0}{\sigma}\right)\exp\left(\frac{-r^2}{2\sigma^2}\right);$$
$$B(r,z) = -\lambda r/\sigma_0\sigma;\qquad \Phi = \Phi_{00},$$

and $E_0 = a_{00}\sqrt{\pi\sigma_0^2}$—amplitude of $|E_{0x}(0, 0, 0)|$ in the narrowest part of the beam.

To estimate the approximations made in the determination of the beam field components (6), we shall use the complete amplitude equation for E_{0x}

$$(\partial_x^2 + \partial_y^2 + 2ik_0\partial_z)E_{0x} = -\partial_z^2 E_{0x}.$$

Since the coordinate z enters in the expression for E_{0x} only in the combination $\lambda z/\sigma_0^2 = \zeta$, it follows that

$$\partial_z^2 E_{0x} = (\lambda/\sigma_0^2)^2\partial_\zeta^2 E_{0x}(r,\zeta),$$

where $\partial_\zeta^2 E_{0x}(r, \zeta)$ is a slowly varying function of ζ. Therefore the order of the correction δE_{0x} to the obtained solution E_{0x} does not exceed, at the most, $(\lambda/\sigma_0^2)^2$. In particular, it can be shown that for the fundamental mode

$$\delta E_{0x} = \frac{i\lambda z}{\sigma^2}\left(\frac{\lambda}{\sigma_0}\right)^2\exp\left[-2i\operatorname{arctg}\frac{\lambda z}{\sigma_0^2}\right]\left\{1 - \frac{r^2}{\sigma^2}\exp\left[-i\operatorname{arctg}\frac{\lambda z}{\sigma_0^2}\right]\right.$$
$$\left.+\frac{r^4}{8\sigma^4}\exp\left[-2i\operatorname{arctg}\frac{\lambda z}{\sigma_0^2}\right]\right\}E_{0x}.$$

The correction to E_{0z}, as can be readily verified from (5c), is proportional, at most, to $(\lambda/\sigma_0^2)^3$.

The components (8) constitute an exact solution of the approximate system (5). Since we are interested in solutions in the form of a circularly polarized wave, the corrections to the expressions in (8) enter in the form: $A \to A + \Delta A$, $B \to B + \Delta B$, $\Phi \to \Phi + \Delta\Phi$, where ΔA and $\Delta\Phi$ are of the order of λ^2, and $\Delta B \sim \lambda^3$. Henceforth, in calculating quantities that are quadratic in the field, we shall retain terms of order λ^2.

GEOMETRICAL PROPERTIES OF LIGHT BEAM

For the field of the fundamental mode (0, 0) of the beam, the components of the Poynting vector

$$S_x(r,z) = -\frac{c}{4\pi}A^2(r,z)B(r,z)\cos\alpha$$
$$S_y(r,z) = \mp\frac{c}{4\pi}A^2(r,z)B(r,z)\sin\alpha, \tag{9}$$
$$S_z(r,z) = \frac{c}{4\pi}(A(r,z)+\Delta A)^2 \approx \frac{c}{4\pi}A^2(r,z)$$

have non-zero transverse components, due to the presence of longitudinal components of the electric and magnetic fields. The energy streamlines of such a beam (see the figure)

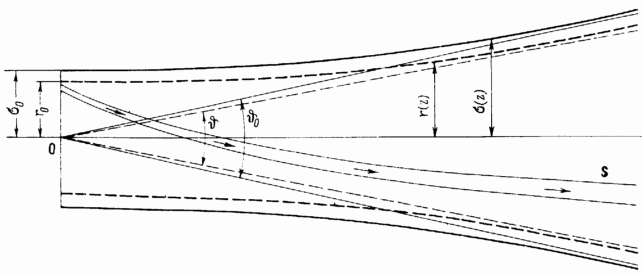
$$r = r_0\{1 + (\lambda z/\sigma_0^2)^2\}^{1/2}, \tag{10a}$$
$$\varphi = \varphi_0 \pm \operatorname{arctg}(\lambda z/\sigma_0^2) \tag{10b}$$

constitute a family of curves lying on the surfaces of single-cavity hyperbolids of revolution[2)] coaxial with the direction of beam propagation. These lines have a curvature $\kappa = \lambda^2 r_0/\sigma_0\sigma^3$ and a nonzero torsion $\tau = \pm\lambda/\sigma^2$. The phase difference between the transverse components of the field remained constant in this case, thus evidencing that in the case of circular polarization there is no phase shift during the propagation. In the case of linear polarization, this corresponds to conservation of the orientation of the plane of polarization in space.

The expression for the time-averaged energy density of the beam in the approximation under consideration[3)]

[2)]The aperture angle ϑ of the asymptotic cones of these cylinders are determined by the relation $\tan(\vartheta/2) = \lambda r_0/\sigma_0$.

[3)]The correction ΔA in the first term can be neglected compared with A, since this affects the magnitude of the term insignificantly.



Schematic representation of a non-ideal coherent light beam (fundamental mode). $\sigma(z)$—running radius of the beam, defined by (7); S—Poynting vector (9); ϑ_0—beam aperture angle, ϑ and r(z)—asymptotic angle and radius of the cross section of the single-cavity hyperboloid of revolution (10a).

$$W(r,z)=\frac{(A+\Delta A)^2}{4\pi}+\frac{A^2B^2}{8\pi} \tag{11}$$

differs from the case of a plane wave having the corresponding amplitudes in the second term in the right side of the equation. The energy flux through the walls of a current tube (10) vanishes, and the energy propagates only along these tubes, with div S = 0. The rate of energy propagation v = S/W along the current tubes (10) is smaller than the propagation velocity c of a plane electromagnetic wave. In a sufficient vicinity to the z axis of the beam, the beam properties approach the properties of a plane wave. In this region, v = c and the well known relation S = cW is satisfied.

It is interesting to note that the variables x, y, and z enter in the expressions (6) for the field components in the form of the dimensionless combinations $\xi = x/\sigma_0$, $\eta = y/\sigma_0$, and $\zeta = z/l_0$, where $l_0 = \sigma_0^2/\lambda\!\!\!^{-}$, and therefore the parameters l_0 and σ_0 are the natural longitudinal and transverse scales of the beam.

Let us subdivide the entire region of propagation of the light beam into a near Fresnel zone with $z \lesssim l_0$, and a far Fraunhofer zone, where $z \gg l_0$. In the far zone, the energy streamlines can be regarded as straight and merging from the origin. In the near zone, these lines experience an invisible torsion.

The equal-phase surfaces

$$z/\lambda\!\!\!^{-} + r^2/2\lambda\!\!\!^{-}R - \operatorname{arc\,tg}(\lambda\!\!\!^{-}z/\sigma_0^2) = \text{const} \tag{12}$$

are surfaces of revolution, and their form changes from a plane as $z \to 0$ to an ellipsoid of revolution $(z-z_0)^2 + r^2/2 = z_0^2$ when $z \to \infty$. Here z_0 is a constant. The maximum deflection of the phase surface from a plane will occur at $z = l_0$. Then $R = l_0$, and Eq. (12) goes over (near the z axis) into the equation of a sphere, $r^2 + (z + l_0)^2 = R^2$. The latter denotes that the field components of the beam in question approximates radiation from a resonator with confocal mirrors.

The wave vector **k**, determined from the relation $\mathbf{k} = \nabla\Phi$,

$$k_\varphi = 0, \quad k_r = r/\lambda\!\!\!^{-}R, \quad k_z \approx 1/\lambda\!\!\!^{-}, \tag{13}$$

has a nonzero transverse component k_r, which determines the bending of the phase surface upon propagation of the beam. The phase trajectories of a narrow coherent beam have a curvature $\kappa = \lambda\!\!\!^{-}r^2/\sigma_0\sigma^3$ and constitutes the generatrices of hyperbolic cylinders (10a).

Expansion of the field (8) in a Fourier integral (in plane waves) yields

$$\mathbf{E}(\mathbf{k}) = E_0\frac{\sigma_0^2}{2\pi}\mathbf{G}(\mathbf{k})\,\delta(k_0 - k_z - k_r^2/2k_0)\exp(-\sigma_0^2k_r^2/2), \tag{14}$$

where

$$k_r = \sqrt{k_x^2 + k_y^2}, \qquad \varphi_k = \operatorname{arc\,tg}(k_y/k_x),$$
$$\mathbf{G}(\mathbf{k}) = \{1, j, -(k_r/k_0)\exp(j\varphi_k)\}.$$

As expected, the principal role in the formation of broad beams is played by plane waves with vectors **k** contained in a small narrow angle near the propagation direction. When $\sigma_0 \to \infty$, we obtain a plane wave with $k = k_0$. The narrower the energy channel of the beam, the more significant the contribution of the plane waves propagating at large angles to the z axis. When $\sigma_0 \to 0$, the beam degenerates into a standing wave.

PHYSICAL PROPERTIES OF LIGHT BEAMS

The energy U of a segment of a beam with thickness Δz, contained between two planes perpendicular to the z axis,

$$U = U_\perp + U_\parallel \approx \frac{E_0^2}{4\pi}\pi\sigma_0^2\,\Delta z + \frac{E_0^2}{8\pi}\left(\frac{\lambda\!\!\!^{-}}{\sigma_0}\right)^2\pi\sigma_0^2\,\Delta z \tag{15}$$

and the total energy flux

$$\Pi_z = \Pi = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} S_z\,dx\,dy = \frac{c}{4\pi}E_0^2\pi\sigma_0^2 \tag{16}$$

are constant quantities and are determined by its effective cross section $\pi\sigma_0^2$. A monochromatic pulse of finite length Δz will broaden upon propagation only in the transverse direction. Some 86% of the entire beam energy passes through the cross section σ, and approximately 99.97% passes through the section 2σ.

The momentum density S/c^2, as seen from (9), is conserved along the energy streamlines. The momentum of the segment of the beam is

$$P_x = P_y = 0, \quad P_z = \frac{U_\perp}{c} \approx \frac{E_0^2}{4\pi c}\pi\sigma_0^2\,\Delta z. \tag{17}$$

The square of the four-momentum of the light beam differs from zero

$$U^2 - c^2P^2 = m_0^2c^4 \neq 0, \tag{18}$$

making it possible to ascribe to the beam segments a rest mass equal to

$$m_0 = \frac{1}{c^2}\sqrt{2U_\perp U_\parallel + U_\parallel^2} \approx \frac{E_0^2}{4\pi c^2}\frac{\lambda\!\!\!^{-}}{\sigma_0}\pi\sigma_0^2\,\Delta z. \tag{19}$$

The fact that the rest mass of a light beam differs from zero[9] is of fundamental importance and is due, in final analysis, to the presence of the longitudinal field compounds. Each segment of such a beam can be regarded as a wave packet propagating along the z axis with velocity

$$v_z = v = \frac{c^2P_z}{U} = \frac{c}{1 + U_\parallel/U_\perp} \approx \frac{c}{1 + \lambda\!\!\!^{-2}/2\sigma_0^2},$$

which is lower than the velocity of light, and with self-mass determined by formula (18). Relation (18) has in our case a simple geometrical interpretation. The angle between the sides U and cP of the triangle is equal to half the beam-aperture angle ϑ.

The components of the time-averaged angular-momentum density are given by

$$M_x = \frac{A^2(r,z)}{4\pi c}[y \pm zB(r,z)\sin\alpha], \quad M_y = -\frac{A^2(r,z)}{4\pi c}[x + zB(r,z)\cos\alpha],$$

$$M_z = \mp\frac{r\sigma_0}{\sigma}\frac{A^2(r,z)}{4\pi c}B(r,z). \tag{20}$$

and therefore the components of the total momentum of the beam segments are

$$M_x = M_y = 0, \quad M_z = \pm\lambda\frac{E_0^2}{4\pi c}\pi\sigma_0^2\Delta z. \tag{21}$$

The intrinsic (spin) angular momentum of the fundamental mode of the beam

$$\mathbf{M}_s = \frac{1}{4\pi c}\int [\mathbf{EA}]\,d\mathbf{r}$$

coincides with the total angular momentum (21), and its orbital angular momentum is equal to zero.

For a circularly polarized light beam, the following well known relations are satisfied

$$U_\perp / M_z = \pm\omega, \quad U_\perp = cP, \tag{22}$$

thus evidencing that the approximations employed here are correct and noncontradictory.

In conclusion we note that the expressions for the energy density, total energy, energy flux, momentum, and rest mass of a linearly polarized beam coincide with the corresponding formulas (11), (15), (16), (17), and (19). The angular momentum and the intrinsic (spin) momentum of such a beam vanish, and its energy streamlines have no torsion.

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