NUCLEAR-SPIN RELAXATION IN SMALL-SIZE SUPERCONDUCTORS

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We consider the rate of nuclear-spin relaxation for superconductors having dimensions $l \ll \xi_0$, where ξ_0 is the correlation parameter. We show that the deviation from the expression for a bulky superconductor occurs only when the spin-orbit interaction of the electrons with the lattice inhomogeneities is taken into account; in spite of the weakness of this interaction, it can strongly change the relaxation rate in the case of sufficiently small l, so that the ratio of the relaxation rate for the superconductor and of the normal metal goes over into the corresponding expression for the ultrasonic-wave absorption.

ONE of the first experiments confirming the coherence of the electronic states in superconductors, as predicted by the theory of Bardeen, Cooper, and Schrieffer, was the measurement by Hebel and Slichter[1] of the nuclear-spin relaxation. Using the concept of quasiparticle states, they have shown that the theory leads to a qualitative agreement with experiment. On the other hand, the quantitative discrepancies were subsequently attributed to the real anisotropy of metals.

It would be of interest, however, to see how the result is affected by the smallness of the samples used in the experiments, since the damping of the excitations becomes larger than the excitation energy when the dimensions are smaller than $\xi_0 \sim v/T_c \sim 10^{-4}$ cm, and the notion of quasiparticle particle states becomes meaningless. As will be shown below the result changes only when account is taken of the spin-orbit interaction with the inhomogeneities of the lattice. However, in spite of the weakness of this interaction, allowance for it can greatly alter the relaxation time in the case of sufficiently small samples.

We shall calculate the probability of nuclear-spin transition as a result of interaction with the electronic system, described by the Hamiltonian

$$H = \frac{8\pi}{3}\gamma_e\gamma_n \mathbf{I}(\psi^+(0)\hat{\sigma}\psi(0)).$$

For simplicity we consider here one nuclear spin situated at the origin of the coordinate system; γ_n, $\mathbf{I}$, γ_e, $\hat{\sigma}$—respectively the gyromagnetic coefficient and the spin operators of the nucleus and of the electron. The total nuclear-spin transition probabilities obtained by summing the square of the modulus of the matrix of the Hamiltonian of the interaction between the initial (a) and final (b) states of the nucleus and the electronic system over all the final states and averaged in the sense of Gibbs over all the initial states of the electronic system, with allowance of the conservation of the energy in the transition.

We have

$$w = \pi g_{\alpha\beta}\sum_{ab}\exp\{(\Omega + \mu N_a - E_a)/T\}\langle a|(\psi^+(0)\hat{\sigma}^{(\alpha)}\psi(0))|b\rangle$$
$$\times\langle b|\psi^+(0)\hat{\sigma}^{(\beta)}\psi(0))|a\rangle\,\delta(E_a - E_b + \chi_0),$$
$$g_{\alpha\beta} = 2\Big(\frac{8\pi}{3}\gamma_n\gamma_e\Big)^2\langle n|I^{(\alpha)}|m\rangle\langle m|I^{(\beta)}|n\rangle, \tag{1}$$

χ_0—change of nuclear-spin energy in the transition.

It can be shown (see, for example,[2], p. 205) that to calculate the probability of the transition (1) it is sufficient to know the two-particle Matsubara Green's function for pairwise-coinciding "temporal" arguments. This connection is given by

$$w = \frac{\operatorname{Im}\mathscr{K}^R(\chi_0)}{e^{-\chi_0/T} - 1}, \tag{2}$$

where $\mathscr{K}^R$—analytic continuation of the function

$$K(\chi) = g_{\alpha\beta}\sigma^{(\alpha)}_{\mu\lambda}\frac{T}{(2\pi)^6}\sum_{\omega_+}\int d\mathbf{p}_+ d\mathbf{p}_- K^{(1,\beta)}_{\lambda,\mu}(\mathbf{p}_+, \mathbf{p}_-; \omega_+, \omega_+ - \chi) \tag{3}$$

from the discrete point $i\chi = i\cdot 2\pi nT$ from the upper half plane to the real axis; $\mathbf{K}^{(1)}_{\lambda\mu}(\mathbf{p}_+, \mathbf{p}_-; \omega_+, \omega_-)$—Fourier component of the function

$$\mathbf{K}^{(1)}_{\lambda\mu}(x - y, y - z) = \langle T\{\psi_\lambda(x)(\psi^+(y)\hat{\sigma}\psi(y))\psi_\mu(z)\}\rangle.$$

In the absence of lattice inhomogeneities, $\mathbf{K}^{(1)}$ can be readily determined by expanding the T-product in accordance with the weak theorem for superconductors:

$$\mathbf{K}^{(1)}_{\lambda\mu} = \sigma_{\lambda\mu}\{\mathfrak{G}^{(0)}_{\omega+}(\mathbf{p}_+)\mathfrak{G}^{(0)}_{\omega-}(\mathbf{p}_-) + \mathscr{F}^{(0)}_{\omega+}(\mathbf{p}_+)\mathscr{F}^{(0)}_{\omega-}(\mathbf{p}_-)\}.$$

Allowance for the lattice inhomogeneities leads to the need for solving the integral equations obtained by using the technique of averaging over the inhomogeneity positions, developed by Abrikosov and Gor'kov[2]. These equations were derived in[3] and have the following form

$$\hat{\mathbf{L}}^{(1)}(p_+, p_-) = \frac{n}{(2\pi)^3}\int d\mathbf{p}\hat{f}(\mathbf{p}, \mathbf{p}')\,\{[\mathfrak{G}(p_+')\mathfrak{G}(p_-') + \mathscr{F}(p_+')\mathscr{F}(p_-')]$$
$$\times[\hat{\sigma} + \hat{\mathbf{L}}^{(1)}(p_+', p_-')] + [\mathfrak{G}(p_+')\mathscr{F}(p_-')$$
$$+ \mathscr{F}(p_+')\mathfrak{G}(p_-')]\hat{\mathbf{L}}^{(2)}(p_+', p_-')\}\hat{f}(\mathbf{p}', \mathbf{p}),$$

$$\hat{\mathbf{L}}^{(2)}(p_+, p_-) = \frac{n}{(2\pi)^3}\int d\mathbf{p}\hat{f}^*(\mathbf{p}, \mathbf{p}')\,\{[\mathscr{F}(p_+')\mathfrak{G}(-p_-') - \mathfrak{G}(p_+')\mathscr{F}(p_-')]$$
$$\times[\hat{\sigma} + \hat{\mathbf{L}}^{(1)}(p_+', p_-')] + [\mathscr{F}(p_+')\mathscr{F}(p_-')$$
$$- \mathfrak{G}(p_+')\mathfrak{G}(p_-')]\hat{\mathbf{L}}^{(2)}(p_+', p_-')\}f^*(\mathbf{p}', \mathbf{p}).$$

$$p_+ = p + \frac{k}{2}, \quad p_- = p - \frac{k}{2}, \quad p_+' = p' + \frac{k}{2}, \quad p_-' = p - \frac{k}{2}.$$

Here $\hat{\mathbf{L}}^{(i)}(\mathbf{p}_+, \mathbf{p}_-)$ are defined by the relations

$$\hat{\mathbf{L}}^{(i)}(p_+, p_-) = \frac{n}{(2\pi)^3}\int\hat{f}(\mathbf{p}, \mathbf{p}')\hat{\mathbf{K}}^{(i)}(p_+', p_-')\hat{f}(\mathbf{p}', \mathbf{p})\,d\mathbf{p}',$$

where $\hat{\mathbf{K}}^{(2)}(p_+, p_-)$ are the Fourier components of the function

$$\mathbf{K}^{(2)}_{\lambda\mu}(x-y, y-z) = \langle T\{\psi_\lambda(x)\{\psi^+(y)\hat{\sigma}\psi(y)\}(\psi(z)\hat{I})_\mu\}\rangle,$$

$$\hat{I} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

Here $\mathfrak{G}$ and $\mathcal{F}$ are the Green's function of the superconducting alloy, and s(p, p′) is the amplitude of scattering by the lattice inhomogeneities with allowance for the spin-orbit interaction:

$$f_{\alpha\beta}(\mathbf{p}, \mathbf{p}') = a(\mathbf{p}, \mathbf{p}')\delta_{\alpha\beta} + ib(\mathbf{p}, \mathbf{p}')([\mathbf{pp}']\sigma_{\alpha\beta})\frac{1}{p_0^2}.$$

The quantity of interest, $\hat{\mathbf{K}}^{(1)}(p_+, p_-)$, is connected with $\mathbf{L}^{(1)}$ and $\mathbf{L}^{(2)}$ as follows:

$$\begin{aligned}\mathbf{K}^{(1)}(p_+, p_-) = [\mathfrak{G}(p_+)\mathfrak{G}(p_-) + \mathcal{F}(p_+)\mathcal{F}(p_-)][\hat{\sigma} + \hat{\mathbf{L}}^{(1)}(p_+, p_-)] \\ + [\mathfrak{G}(p_+)\mathcal{F}(p_-) + \mathcal{F}(p_+)\mathfrak{G}(p_-)]\hat{\mathbf{L}}^{(2)}(p_+, p_-).\end{aligned} \tag{5}$$

We are interested in the solution of (4) for the case $l \ll \xi_0$, $l_{so} \gg \xi_0$, where l_{so}—mean free path due only to the spin-orbit interaction. Representing $\hat{\mathbf{L}}^{(i)}$ in the form

$$\hat{\mathbf{L}}^{(i)} = L_\sigma^{(i)}\hat{\sigma} + L_p^{(i)}(\mathbf{p}\hat{\sigma})\mathbf{p}\frac{1}{p_0^2}$$

and substituting in (4), we obtain, under our assumptions concerning l and l_{so}, a system of linear equations for $\mathbf{L}_\sigma^{(i)}$ and $\mathbf{L}_p^{(i)}$; their solution is of the form

$$L_\sigma^{(i)} = L_{\sigma,0}^{(i)} + L_{\sigma,1}^{(i)},$$

$$L_{\sigma,0}^{(1)} = \frac{1}{2\tau_0(\sqrt{\omega_+^2 + \Delta^2} + \sqrt{\omega_-^2 + \Delta^2})}\left(1 + \frac{\Delta^2 - \omega_+\omega_-}{\sqrt{\omega_+^2 + \Delta^2}\sqrt{\omega_-^2 + \Delta^2}}\right)$$

$$L_{\sigma,0}^{(2)} = \frac{i(\omega_+ + \omega_-)}{2\tau_0(\sqrt{\omega_+^2 + \Delta^2} + \sqrt{\omega_-^2 + \Delta^2})\sqrt{\omega_+^2 + \Delta^2}\sqrt{\omega_-^2 + \Delta^2}},$$

$$L_{\sigma,1}^{(i)} = -\frac{4}{3\tau_1}L_{\sigma,0}^{(i)}(\sqrt{\omega_+^2 + \Delta^2} + \sqrt{\omega_-^2 + \Delta^2})^{-1},$$

$$L_p^{(i)} = -\frac{\tau_0'}{6\tau_1}L_{\sigma,0}^{(i)}, \qquad \tau_1^{-1} = \tau^{-1} - \tau_0'^{-1},$$

$$\tau_0^{-1} = \frac{nmp_0}{(2\pi)^2}\int |a|^2 d\Omega, \quad \tau_0'^{-1} = \frac{nmp_0}{2(2\pi)^2}\int |a|^2 \sin^2\theta\, d\Omega.$$

The obtained expressions for $\mathbf{L}^{(i)}$ must now be substituted in (5), after which, with the aid of (3) and a subsequent analytic continuation using (2), we obtain the quantity of interest to us.

Let us consider first the contribution made to (3) by the term

$$[\mathfrak{G}(p_+)\mathfrak{G}(p_-) + \mathcal{F}(p_+)\mathcal{F}(p_-)]\hat{\sigma}.$$

It is obvious that this expression does not depend on effects that are characteristic of alloys, for following integration over the momenta it contains Green's functions with coinciding spatial arguments, and therefore[2] they are equal to the corresponding functions of the pure superconductor. For this reason, expression (6) yields, after performance of the indicated program, the transition probability in the pure superconductor, equal to[1]

$$\begin{aligned} w^{(0)} = \frac{g_{\alpha\alpha}(p_0m)^2}{\pi^3}\int_\Delta^\infty \frac{d\omega[\omega(\omega+\chi_0) + \Delta^2]\varphi(\omega)[1 - \varphi(\omega+\chi_0)]}{\sqrt{\omega^2 - \Delta^2}\sqrt{(\omega+\chi_0)^2 - \Delta^2}}, \\ \varphi(\omega) = (e^{\omega/T} + 1)^{-1}. \end{aligned} \tag{7}$$

Let us calculate now the contribution made to the transition probability by the quantities $\mathbf{L}_{\sigma,1}^{(i)}$. Integrating over the momenta, we obtain for the corresponding contribution to (3)

$$K_{\sigma,1}(\chi) = \frac{g_{\alpha\alpha}(p_0m)^2T}{3\tau_0\tau_1\pi^2}\sum_{\omega_+} R(\omega_+, \chi),$$

where

$$R(\omega_+, \chi) = -\frac{\Delta^2 - \omega_+(\omega_+ - \chi) + \sqrt{\omega_+^2 + \Delta^2}\sqrt{(\omega_+ - \chi)^2 + \Delta^2}}{(\sqrt{(\omega_+ - \chi)^2 + \Delta^2} + \sqrt{\omega_+^2 + \Delta^2})^2\sqrt{\omega_+^2 + \Delta^2}\sqrt{(\omega_+ - \chi)^2 + \Delta^2}}.$$

After algebraic transformation, this equation can be rewritten in the form

$$R(\omega_+, \chi) = \chi^{-2}\left[-1 + \frac{\Delta^2 + \omega_+(\omega_+ - \chi)}{\sqrt{\omega_+^2 + \Delta^2}\sqrt{(\omega_+ - \chi)^2 + \Delta^2}}\right]$$

For analytic continuation with respect to the variable χ [1)], it is convenient to change from summation to integration by means of the formula

$$2T\sum_{\omega_+} R(\omega_+, \chi) = \frac{i}{2\pi}\int_{C^+ + C^-} R(\omega, \chi)\,\mathrm{tg}\frac{\omega}{2T}\,d\omega. \tag{8}$$

(The contours C^+ and C^- are shown in the figure.) By making the substitution $\omega - \chi = -u$ we can verify that the integrals over the contours $C^{+(2)}$ and $C^{-(2)}$ are respectively equal to the integrals over the contours $C^{-(1)}$ and $C^{+(1)}$, i.e.,

$$T\sum_{\omega_+} R(\omega_+, \chi) = \frac{i}{2\pi}\int_{C^{+(1)} + C^{-(1)}} R(\omega, \chi)\,\mathrm{tg}\frac{\omega}{2T}\,d\omega. \tag{9}$$

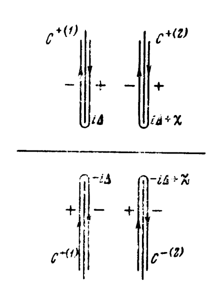
In (7) we can already go over to $\chi_0 = i\chi$ and, taking into account the signs of the imaginary parts of the roots, we obtain for the imaginary part of the analytic continuation

$$\begin{aligned}\mathrm{Im}\,\mathscr{K}_{\sigma,1}^R(\chi_0) = \frac{g_{\alpha\alpha}(p_0m)^2}{3\pi^2\tau_0\tau_1\chi_0^2}\int_\Delta^\infty \frac{[\omega(\omega+\chi_0) - \Delta^2]}{\sqrt{\omega^2 - \Delta^2}\sqrt{(\omega+\chi_0)^2 - \Delta^2}} \\ \times\left[\mathrm{th}\frac{\omega}{2T} - \mathrm{th}\frac{\omega+\chi_0}{2T}\right]d\omega.\end{aligned} \tag{10}$$

Substituting (10) in (2) we get

$$w^{(1)} = \frac{2g_{\alpha\alpha}(p_0m)^2}{3\pi^3\tau_0\tau_1\chi_0^2}\int_\Delta^\infty \frac{[\omega(\omega+\chi_0) - \Delta^2]\varphi(\omega)[1 - \varphi(\omega+\chi_0)]\,d\omega}{\sqrt{\omega^2 - \Delta^2}\sqrt{(\omega+\chi_0)^2 - \Delta^2}}. \tag{11}$$

Repeating a similar procedure, we can verify that $\mathbf{L}_{\sigma,0}^{(i)}$ and $\mathbf{L}_p^{(i)}$ make no contribution to the probability of the



Integration contours for (8). The choice of the signs of the imaginary parts of the roots $\sqrt{\Delta^2 - \omega^2}$ and $\sqrt{\Delta^2 - (\omega - i\chi)^2}$ on the cuts is indicated.

1) More details concerning the analytic continuation of similar quantities are given in [2], p. 412.

transition, since after the analytic continuation the imaginary part of $\varkappa^R_{\sigma,0}$ and $\varkappa^R_p$ turn out to be equal to zero. Thus, the total transition probability is equal to

$$w = w^{(0)} + w^{(1)}. \tag{12}$$

Going over to a consideration of the obtained results, we note, first that owing to the smallness of χ_0 (Zeeman energy) $w^{(0)}$ and $w^{(1)}$ actually diverge, the former like $\ln \chi_0$ and the latter like χ_0^{-2}. It is obvious that this divergence can be eliminated by taking into account the real anisotropy of the metals. However, since the metal becomes isotropic with decreasing mean free path, we can expect for sufficiently small samples of sufficiently heavy metals (in which the spin-orbit interaction is not vanishingly small) that the second term in (12) will prevail over the first, and we obtain for the ratio of the transition probabilities in the superconductor and in the normal metal

$$\frac{w^{(s)}}{w^{(n)}} = \frac{2}{e^{\Delta/T}+1}, \tag{13}$$

which agrees with the corresponding expression for the damping of the ultrasonic waves in the superconductors.

[1] L. C. Hebel and C. P. Slichter, Phys. Rev. **113**, 1504 (1959).

[2] A. A. Abrikosov, L. P. Gor'kov, and I. E. Dzyaloshinskiĭ, Metody kvantovoĭ teorii polya v statisticheskoĭ fizike (Quantum Field Theoretical Methods in Statistical Physics), Fizmatgiz, 1962 [Pergamon, 1965].

[3] A. A. Abrikosov and L. P. Gor'kov, Zh. Eksp. Teor. Fiz. **42**, 1088 (1962) [Sov. Phys.-JETP **15**, 752 (1962)].

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