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ANALYTIC PROPERTIES OF THE EFFECTIVE DIELECTRIC CONSTANT OF RANDOMLY INHOMOGENEOUS MEDIA

Yu. A. RYZHOV

Radiophysics Institute of the Gor'kiĭ State University

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The general properties of the effective dielectric constant of randomly inhomogeneous media are investigated. The Kramers-Kronig theorem is considered as applied to an inhomogeneous medium; the corresponding relations between the imaginary and real parts of $\epsilon_{ij}^{\mathrm{eff}}(\omega, \mathbf{k})$ are an expression of the causality principle in the scattering medium. It is shown that the correlation functions of the thermal electromagnetic field in a randomly inhomogeneous medium are expressed in terms of the effective dielectric constant. The influence of the spatial dispersion connected with the inhomogeneities of the medium on the radiation from sources placed in such a medium is analyzed. In particular, allowance for the spatial dispersion is necessary in calculating the scattering of the quasi-static fields from a source by the near-zone inhomogeneities.

1. INTRODUCTION

THE average field in a randomly inhomogeneous medium is described with the aid of the effective dielectric constant. For a statistically homogeneous medium, the latter was calculated in a number of papers. In some of them[1-3], the problem was solved by perturbation theory. The results of these papers are valid at small relative fluctuations of the dielectric constant $\epsilon(\mathbf{r})$ of the medium. Considerations going beyond the perturbation methods are contained in[4-10]. In[11,12], a study is made of the effective dielectric constant of inhomogeneous anisotropic media. In the present paper we analyze the general analytic properties of $\epsilon_{ij}^{\mathrm{eff}}(\omega, \mathbf{k})$ and its connection with certain averaged energy quantities (quadratic in the field) in a randomly inhomogeneous medium.

2. EFFECTIVE DIELECTRIC CONSTANT OF A RANDOMLY INHOMOGENEOUS MEDIUM

Maxwell's equations for the random field at a frequency ω in a medium with a dielectric constant $\epsilon(\omega, \mathbf{r})$ are of the form

$$[\operatorname{rot}\operatorname{rot}\mathbf{E}(\omega,\mathbf{r})]_i = k_0^2\varepsilon(\omega,\mathbf{r})E_i(\omega,\mathbf{r}) + k_0^2K_i(\omega,\mathbf{r}). \tag{1}$$

Here $k_0 = \omega/c$, and $K_i(\omega, \mathbf{r})$ is the induction connected with the specified extraneous currents $\mathbf{j}_{\mathrm{ext}}(\omega, \mathbf{r})$ by the formula

$$\mathbf{j}_{\mathrm{ext}}(\omega,\mathbf{r}) = -(i\omega/4\pi)\mathbf{K}(\omega,\mathbf{r}). \tag{2}$$

We write (1) in the form

$$K_i(\omega,\mathbf{r}) = \hat{L}_{ij}(\omega,\mathbf{r})E_j(\omega,\mathbf{r}). \tag{3}$$

Let us average (3) over the ensemble of realizations of the random function $\epsilon(\omega, \mathbf{r})$. We obtain an equation for the average field $\langle E_i(\omega, \mathbf{r}')\rangle$:

$$K_i(\omega,\mathbf{r}) = \hat{L}_{ij}^{\mathrm{eff}}(\omega,\mathbf{r},\mathbf{r}')\langle E_j(\omega,\mathbf{r}')\rangle. \tag{4}$$

In a statistically homogeneous medium, the operator $\hat{L}_{ij}^{\mathrm{eff}}(\omega, \mathbf{r}, \mathbf{r}')$ is of the form of an operator of a homogeneous medium with a certain effective dielectric constant $\hat{\epsilon}_{ij}^{\mathrm{eff}}$, defined by the relation

$$\langle D_i(\omega,\mathbf{r})\rangle = \int \varepsilon_{ij}(\omega,\mathbf{r}-\mathbf{r}')\langle E_j(\omega,\mathbf{r}')\rangle d\mathbf{r}'. \tag{5}$$

Let us write down the solution of Eqs. (3) and (4):

$$E_i(\omega,\mathbf{r}) = \hat{L}_{ij}^{-1}(\omega,\mathbf{r},\mathbf{r}')K_j(\omega,\mathbf{r}'),$$
$$\langle E_i(\omega,\mathbf{r})\rangle = (\hat{L}_{ij}^{\mathrm{eff}})^{-1}K_j(\omega,\mathbf{r}') = \int (\hat{L}_{ij}^{\mathrm{eff}}(\omega,\mathbf{r}-\mathbf{r}'))^{-1}K_j(\omega,\mathbf{r}')d\mathbf{r}'. \tag{6}$$

Averaging the first equation of (6) and comparing the obtained relation with the second equation of (6), we get

$$(\hat{L}_{ij}^{\mathrm{eff}})^{-1} = \langle \hat{L}_{ij}^{-1}\rangle. \tag{7}$$

If the medium is statistically homogeneous and isotropic, then the tensor of the effective dielectric constant is expressed in terms of the tensor of the effective polarizability $\hat{\xi}_{ij}^{\mathrm{eff}}$ [7,8]. The random polarizability of the medium ξ is connected with the dielectric constant by the formula

$$\xi = 3(\varepsilon - \varepsilon_0)/(\varepsilon + 2\varepsilon_0), \tag{8}$$

where $\epsilon_0(\omega)$ is obtained from the equation $\langle\xi\rangle = 0$. For the tensor $\epsilon_{ij}^{\mathrm{eff}}(\omega, \mathbf{k})$ we have in this case the relations

$$\varepsilon_{ij}^{\mathrm{eff}}(\omega,\mathbf{k}) = \int \varepsilon_{ij}^{\mathrm{eff}}(\mathbf{r})e^{-i\mathbf{k}\mathbf{r}}d\mathbf{r},$$
$$\varepsilon_{ij}^{\mathrm{eff}}(\omega,\mathbf{k}) = \left(\delta_{ij} - \frac{k_ik_j}{k^2}\right)\varepsilon_{\mathrm{eff}}^{tr}(\omega,k) + \frac{k_ik_j}{k^2}\varepsilon_{\mathrm{eff}}^{l}(\omega,k).$$

Analogous formulas can be written for $\xi_{ij}^{\mathrm{eff}}(\omega, \mathbf{k})$. The connection between the tensors $\epsilon_{ij}^{\mathrm{eff}}(\omega, \mathbf{k})$ and $\xi_{ij}^{\mathrm{eff}}(\omega, \mathbf{k})$ is given by the equations

$$\varepsilon_{\mathrm{eff}}^{l}(\omega,k) = \varepsilon_0(\omega)\frac{1 + {}^2/_3\xi_{\mathrm{eff}}^{l}(\omega,k)}{1 - {}^1/_3\xi_{\mathrm{eff}}^{l}(\omega,k)},$$
$$\varepsilon_{\mathrm{eff}}^{tr}(\omega,k) = \varepsilon_0(\omega)\frac{1 + {}^2/_3\xi_{\mathrm{eff}}^{tr}(\omega,k)}{1 - {}^1/_3\xi_{\mathrm{eff}}^{tr}(\omega,k)}. \tag{10}$$

The tensor $\xi_{ij}^{\mathrm{eff}}(\omega, \mathbf{k})$ is expressed directly in terms of the mass operator $Q_{ij}(\mathbf{r}_1, \mathbf{r}_2)$, which in the case of a normal distribution of the quantity ξ is given by

$$Q_{ij}(\mathbf{r}_1,\mathbf{r}_2)=k_0^4\varepsilon_0^2B_\xi(\mathbf{r}_1,\mathbf{r}_2)G_{ij}'(\mathbf{r}_1,\mathbf{r}_2)$$
$$+k_0^8\varepsilon_0^4\int\int G_{il}'(\mathbf{r}_1,\mathbf{r}_3)G_{lm}'(\mathbf{r}_3,\mathbf{r}_4)G_{mj}'(\mathbf{r}_4,\mathbf{r}_2)B_\xi(\mathbf{r}_1,\mathbf{r}_4)B_\xi(\mathbf{r}_3,\mathbf{r}_2)\,d\mathbf{r}_3\,d\mathbf{r}_4+\ldots, \tag{11}$$

where $B_\xi(\mathbf{r}_1, \mathbf{r}_2)$ is the correlation function of the process $\xi(\mathbf{r})$, and $G'_{ij}(\mathbf{r}_1, \mathbf{r}_2)$ is the regularized value of the Green's function for a homogeneous medium with dielectric constant $\epsilon = \epsilon_0(\omega)$:

$$G_{ij}'(\mathbf{r})=G_{ij}^0(\mathbf{r})-\frac{1}{3k_0^2\varepsilon_0}\delta(\mathbf{r}),$$
$$G_{ij}^0(\mathbf{r})=-\left(\delta_{ij}+\frac{1}{k_0^2\varepsilon_0}\frac{\partial^2}{\partial x_i\,\partial x_j}\right)\frac{\exp(ik_0\sqrt{\varepsilon_0}r)}{4\pi r},$$
$$G_{ij}'(\mathbf{r})=-\mathrm{P}\left[\delta_{ij}+\frac{1}{k_0^2\varepsilon_0}\frac{\partial^2}{\partial x_i\,\partial x_j}\right]\frac{\exp(ik_0\sqrt{\varepsilon_0}r)}{4\pi r}. \tag{12}$$

The connection between $\xi_{ij}^{\text{eff}}(\omega, \mathbf{k})$ and $Q_{ij}(\mathbf{r})$ is

$$\xi_{ij}^{\text{eff}}(\omega,\mathbf{k})=-\frac{1}{k_0^2\varepsilon_0}\int Q_{ij}(\mathbf{r})e^{-i\mathbf{kr}}\,d\mathbf{r}. \tag{13}$$

Formulas (9), (10), and (13), define the effective dielectric constant of a randomly inhomogeneous isotropic medium.

3. THE KRAMERS-KRONIG RELATIONS FOR THE EFFECTIVE DIELECTRIC CONSTANT. THERMAL FLUCTUATIONS IN A RANDOMLY INHOMOGENEOUS MEDIUM

The Kramers-Kronig relation for $\epsilon_{ij}^{\text{eff}}(\omega, \mathbf{k})$ follows directly from the corresponding formulas for $\epsilon(\omega, \mathbf{r})$, which are best written in the form[13]

$$\varepsilon(\omega,\mathbf{r})-1=\frac{1}{\pi i}\mathrm{P}\int_{-\infty}^{+\infty}d\omega'\frac{\varepsilon(\omega',\mathbf{r})-1}{\omega'-\omega}. \tag{14}$$

We multiply the left and right sides of (14) by the vector $\mathbf{E}(\epsilon, \mathbf{r})$ and average over the ensemble of the realizations of the inhomogeneities of the medium. From the obtained relation we get[1)]

$$\varepsilon_{ij}^{\text{eff}}(\omega,\mathbf{r}-\mathbf{r}')-\delta_{ij}\delta(\mathbf{r}-\mathbf{r}')$$
$$=\frac{1}{\pi i}\mathrm{P}\int_{-\infty}^{+\infty}d\omega'\cdot\frac{\varepsilon_{ij}^{\text{eff}}(\omega',\mathbf{r}-\mathbf{r}')-\delta_{ij}\delta(\mathbf{r}-\mathbf{r}')}{\omega'-\omega} \tag{15}$$

and the corresponding corrolaries, which express the causality principle in a randomly inhomogeneous medium:

$$\varepsilon_{ij}^{\prime\,\text{eff}}(\omega,\mathbf{r}-\mathbf{r}')-\delta_{ij}\delta(\mathbf{r}-\mathbf{r}')=\frac{1}{\pi}\mathrm{P}\int_{-\infty}^{+\infty}d\omega'\frac{\varepsilon_{ij}^{\prime\prime\,\text{eff}}(\omega',\mathbf{r}-\mathbf{r}')}{\omega'-\omega},$$
$$\varepsilon_{ij}^{\prime\prime\,\text{eff}}(\omega,\mathbf{r}-\mathbf{r}')=-\frac{1}{\pi}\mathrm{P}\int d\omega'\frac{\varepsilon_{ij}^{\prime\,\text{eff}}(\omega',\mathbf{r}-\mathbf{r}')-\delta_{ij}\delta(\mathbf{r}-\mathbf{r}')}{\omega'-\omega} \tag{16}$$

Formulas (16) are meaningful also in a physically transparent scattering medium, where there is no true absorption. The imaginary part of $\epsilon_{ij}^{\text{eff}}(\omega, \mathbf{k})$ describes the process of transformation of energy of the regular component of the electric field into the random component (scattered field). The connection between $\mathrm{Im}\,\epsilon_{\text{eff}}^{l}(\omega, \mathbf{k})$ and $\epsilon_{\text{eff}}^{tr}(\omega, \mathbf{k})$ and the scattering process becomes manifest most clearly if the Poynting theorem is used.

From the field equations we obtain directly for the energy flux averaged over the period

$$-\mathbf{E}\mathbf{j}_{\text{ext}}=\operatorname{div}\mathbf{S},\quad \mathbf{S}=\frac{c}{4\pi}[\mathbf{EH}]. \tag{17*}$$

On the other hand, for the equations of the average field, with the same specified current $\mathbf{j}_{\text{ext}}$, the conservation law takes the form

$$-\langle\mathbf{E}\rangle\mathbf{j}_{\text{ext}}=\frac{c}{4\pi}\operatorname{div}[\langle\mathbf{E}\rangle\langle\mathbf{H}\rangle]+Q_0,$$
$$Q_0=\frac{i\omega}{4\pi}(\langle\mathbf{E}_\omega\rangle\langle\mathbf{D}_\omega^*\rangle-\langle\mathbf{E}_\omega^*\rangle\langle\mathbf{D}_\omega\rangle),$$
$$\mathbf{E}(r,t)=\mathbf{E}_\omega e^{-i\omega t}+\mathbf{E}_\omega^*e^{i\omega t}, \tag{18}$$

where Q_0 is the density of the "effective heat" released into the space surrounding the radiator. If we average (17) over the inhomogeneities and subtract Eq. (18) from it, then the scattered-field flux averaged over the time and over the ensemble $\epsilon(\mathbf{r})$ (we have omitted the symbol for averaging over the period of the high-frequency)

$$\langle\mathbf{s}\rangle=\frac{c}{4\pi}[\mathbf{eh}]=\langle\mathbf{S}\rangle-\mathbf{S}_0,$$

($\mathbf{S}_0$—flux of average field) is given by

$$\operatorname{div}\langle\mathbf{s}\rangle=Q_0. \tag{19}$$

The source of the average flux of random radiation is the effective heat released in the medium as a result of the absorption energy of the average field. The total average flux of the scattered field through the closed surface is $\langle s\rangle_f = \oint Q_0\,d\mathbf{r}$. Moving the closed surface into the region where the average field can be regarded as absorbed ($\mathbf{S}_0 = 0$), we obtain for the total radiation flux from a point dipole with current $\mathbf{j}_{\text{ext}} = \mathbf{n}\delta(\mathbf{r})$ the expression

$$\langle s\rangle_f=\langle S\rangle_f$$
$$=(2\pi)^3\frac{i\omega}{4\pi}\int d\mathbf{k}\,\{\varepsilon_{ij}^{*\,\text{eff}}(\omega,\mathbf{k})-\varepsilon_{ji}^{\text{eff}}(\omega,\mathbf{k})\}\langle E_i(\omega,\mathbf{k})\rangle\langle E_j^*(\omega,\mathbf{k})\rangle$$
$$=\int Q_0\,d\mathbf{r}=\frac{4}{3\pi\omega}\int_0^\infty k^2\,dk\left\{\frac{\mathrm{Im}\,\varepsilon_{\text{eff}}^{l}(\omega,k)}{|\varepsilon_{\text{eff}}^{l}(\omega,k)|^2}+2k_0^4\frac{\mathrm{Im}\,\varepsilon_{\text{eff}}^{tr}(\omega,k)}{|k^2-k_0^2\varepsilon_{\text{eff}}^{tr}(\omega,k)|^2}\right\},$$
$$E_i(\omega,\mathbf{k})=\frac{1}{(2\pi)^3}\int E_i(\omega,\mathbf{r})e^{-i\mathbf{kr}}\,d\mathbf{r}. \tag{20}$$

Formulas (18)—(20) describe from the energy point of view the process of scattering and the transition of the field to the random state.

Let us consider now the thermal fluctuations in a randomly inhomogeneous medium. The correlation function of the fluctuation field in the medium can be written in the form

$$(E_i(\mathbf{r})E_j(\mathbf{r}'))_\omega=i\hbar[L_{ji}^{-1*}(\omega,\mathbf{r}',\mathbf{r})-L_{ij}^{-1}(\omega,\mathbf{r},\mathbf{r}')]\operatorname{cth}\frac{\hbar\omega}{2\varkappa T}. \tag{21}$$

Averaging this expression and using (7), we get

$$\langle(E_i(\mathbf{r})E_j(\mathbf{r}'))_\omega\rangle$$
$$=i\hbar[(L_{ji}^{\text{eff}}(\omega,\mathbf{r}',\mathbf{r}))^{-1*}-(L_{ij}^{\text{eff}}(\omega,\mathbf{r},\mathbf{r}'))^{-1}]\operatorname{cth}\frac{\hbar\omega}{2\varkappa T}, \tag{22}$$

1)It should be noted that

$$\langle\varepsilon(\omega_1,\mathbf{r})E(\omega_2,\mathbf{r})\rangle=\varepsilon^{\text{eff}}(\omega_1,\mathbf{r},\mathbf{r}')\langle E(\omega_2,\mathbf{r}')\rangle.$$

*$[\mathbf{EH}] \equiv \mathbf{E}\times\mathbf{H}$.

or

$$\langle (E_i(\mathbf{r})E_j(\mathbf{r}'))_\omega\rangle = 2\hbar\Phi(\omega, \mathbf{r}-\mathbf{r}')\,\mathrm{cth}\frac{\hbar\omega}{2\varkappa T},$$

$$\Phi(\omega, \mathbf{R}) = \frac{1}{(2\pi)^3}\int d\mathbf{k}\, e^{i\mathbf{kR}}\Psi_{ij}(\omega, \mathbf{k}),$$

$$\Psi_{ij}(\omega, \mathbf{k}) = \frac{k_ik_j}{k^2}\frac{\operatorname{Im}\varepsilon^l_{\mathrm{eff}}(\omega, k)}{|\varepsilon^l_{\mathrm{eff}}(\omega, k)|^2}$$

$$+ k_0^4\left(\delta_{ij} - \frac{k_ik_j}{k^2}\right)\frac{\operatorname{Im}\varepsilon^{tr}_{\mathrm{eff}}(\omega, k)}{|k^2 - k_0^2\varepsilon^{tr}_{\mathrm{eff}}(\omega, k)|^2}. \quad (23)$$

Expressions (20) and (23) will be also considered as applied to a physically transparent inhomogeneous medium. In this case they express properties of the thermal radiation in the inhomogeneous medium, with allowance for scattering and diffraction of the field. If we neglect the spatial dispersion, then formulas (22) assume the form ($R = |\mathbf{r} - \mathbf{r}'|$)[13]

$$\langle (E(\mathbf{r})E(\mathbf{r}'))_\omega\rangle = 2\hbar\left\{\frac{\operatorname{Im}\varepsilon^{\mathrm{eff}}(\omega)}{|\varepsilon^{\mathrm{eff}}|^2}\delta(\mathbf{r}-\mathbf{r}')\right.$$

$$\left. - \frac{1}{4\pi}\frac{i\omega^2}{Rc^2}\left[\exp\left(-\frac{\omega}{c}\sqrt{-\varepsilon^{\mathrm{eff}}}R\right) - \exp\left(-\frac{\omega}{c}\sqrt{-\varepsilon^{*\mathrm{eff}}}R\right)\right]\right\}\mathrm{cth}\frac{\hbar\omega}{2\varkappa T}. \quad (24)$$

The second term from (24) in the transparent homogeneous medium determines the intensity of the thermal radiation[1]:

$$(E^2(\mathbf{r}))_\omega = \frac{\omega^3\hbar}{\pi c^3}n\,\mathrm{cth}\frac{\hbar\omega}{2\varkappa T}. \quad (25)$$

If the medium is inhomogeneous, but the scale of the inhomogeneities is large compared with the wavelength, then in this formula, in the geometrical-optics approximation, n can be regarded as a slow function of the coordinates. Averaging (25) over the inhomogeneities and calculating

$$\langle n(r)\rangle = \langle\sqrt{\langle\varepsilon\rangle + \Delta\varepsilon}\rangle = \sqrt{\langle\varepsilon\rangle} - {}^1\!/_8\langle\Delta\varepsilon^2\rangle/\langle\varepsilon\rangle^{3/2} + \ldots,$$

we obtain approximate formulas for the field intensity in a large-scale inhomogeneous medium. The same expression is obtained from (24) by recognizing that in the present case

$$\varepsilon^{\mathrm{eff}}(\omega) = \langle\varepsilon\rangle - {}^1\!/_4\langle\Delta\varepsilon^2\rangle/\langle\varepsilon\rangle + \ldots \text{[3]}.$$

4. UNIQUENESS OF THE SOLUTION OF THE EQUATION $\langle\xi\rangle = 0$, WHICH DETERMINES $\epsilon_0(\omega)$.

The value of the dielectric constant $\epsilon_0(\omega)$, which enters as a factor in formulas (10) for $\epsilon^l_{\mathrm{eff}}(\omega, \mathbf{k})$ and $\epsilon^{tr}_{\mathrm{eff}}(\omega, \mathbf{k})$, is determined from the equation $\langle\xi\rangle = 0$. Depending on the character of the distribution function $W(x)$ of the dielectric constant $\epsilon(\omega, \mathbf{r})$, the following cases can occur: if $\epsilon(\omega, \mathbf{r})$ assumes values of only one sign (say $W(x) = 0$ when $x \le 0$), then the equation $\langle\xi\rangle = 0$ has only one real solution (positive value of ϵ_0). If ϵ can vanish, then the equation $\langle\xi\rangle = 0$ has two complex conjugate roots, from which we choose the one determining the damping of the average field. The equation $\langle\xi\rangle = 0$ can be written in the form

$$F(z) = 0,$$

where

$$F(z) = 1 + \frac{3}{2}z\int_{-\infty}^{+\infty}\frac{W(\xi)}{\xi - z}d\xi, \quad W(\xi) \geqslant 0, \quad -\infty < \xi < +\infty,$$

$$\int_{-\infty}^{+\infty}W(\xi)d\xi = 1, \quad z = -2\varepsilon_0. \quad (26)$$

A detailed analysis of (26) is given in the Appendix. If $\epsilon(\mathbf{r})$ can vanish, then ϵ_0 is complex, and this circumstance can be explained by means of the following physical considerations. In such a medium there can exist longitudinal field oscillations, determined by the equation $\epsilon^l_{\mathrm{eff}}(\omega, \mathbf{k}) = 0$. It breaks up into two equations, of which one, $\xi^l_{\mathrm{eff}}(\omega, \mathbf{k}) = -{}^3\!/_2$, describes a longitudinal wave that attenuates over the correlation radius of the random inhomogeneity[8], and the second, $\epsilon_0(\omega) = 0$, makes it possible to calculate the damping decrement of the longitudinal oscillations of the average field. The longitudinal oscillations of the average field must inevitably attenuate as a result of scattering by the inhomogeneities of the medium. The fact that the function $\epsilon_0(\omega)$ in such a medium is complex is obvious.

5. ASYMPTOTIC EXPANSION OF THE EFFECTIVE CONSTANT AS $k \to \infty$ IN THE CASE OF SMALL SCALE FLUCTUATIONS: INFLUENCE OF SPATIAL DISPERSION ON THE RADIATION OF SOURCES PLACED IN A RANDOMLY INHOMOGENEOUS MEDIUM

We shall henceforth consider the case when $\epsilon_0(\omega)$ is a real function. If the medium is small-scale ($k_0 l \ll 1$, where l—radius of the fluctuations $\Delta\epsilon$) and $|\xi^{\mathrm{eff}}_{ij}(\omega, \mathbf{k})| \ll 1$, then the approximation in which we can confine ourselves to the first term of the infinite series for the mass operator Q_0 is valid. In this case

$$\varepsilon^l_{\mathrm{eff}}(\omega, k) = \varepsilon_0(\omega)[1 + \xi^l_{\mathrm{eff}}(\omega, k)],$$

$$\varepsilon^{tr}_{\mathrm{eff}}(\omega, k) = \varepsilon_0(\omega)[1 + \xi^{tr}_{\mathrm{eff}}(\omega, k)],$$

$$\xi^l_{\mathrm{eff}}(\omega, k) = -2\langle\xi^2\rangle q(p, p_0),$$

$$\xi^{tr}_{\mathrm{eff}}(\omega, k) = \frac{p_0^2\langle\xi\rangle^2}{p}\int_0^\infty\Gamma_\xi(x)e^{ip_0x}\sin px\,dx + \langle\xi^2\rangle q(p, p_0),$$

$$q(p, p_0) = \int_0^\infty\Gamma_\xi(x)\left[\frac{3}{p^2x^2}\left(\frac{\sin px}{px} - \cos px\right) - \frac{\sin px}{px}\right]\frac{1}{x}dx$$

$$+ \frac{p_0}{2p}\int_0^\infty\Gamma_\xi(x)\left[\frac{1}{px}\left(\frac{\sin px}{px} - \cos px\right) - \sin px\right]dx$$

$$- i\frac{p_0^3}{3p}\int_0^\infty\Gamma_\xi(x)x\sin px\,dx, \quad (27)$$

where we have introduced in lieu of the correlation function B_ξ the correlation coefficient

$$\varepsilon_0(\omega) = \langle\varepsilon\rangle - \frac{1}{3}\frac{\langle\Delta\varepsilon^2\rangle}{\langle\varepsilon\rangle} - \frac{4}{27}\frac{(\langle\Delta\varepsilon^2\rangle)^2}{(\langle\varepsilon\rangle)^3} + \ldots,$$

$$\langle\xi^2\rangle \approx \langle\Delta\varepsilon^2\rangle/\langle\varepsilon\rangle^2.$$

We note that these expressions are valid at arbitrary p_0 if the fluctuations are small ($\Delta\epsilon/\langle\epsilon\rangle \ll 1$). Then

$$B_\xi(r) = \langle\xi^2\rangle\Gamma_\xi(r/l),\ p = kl,\ p_0 = k_0\sqrt{\varepsilon_0}l \ll 1.$$

The imaginary parts of $\epsilon^l_{\mathrm{eff}}$ and $\epsilon^{tr}_{\mathrm{eff}}$ are determined by the quantity $\operatorname{Im} q(p, p_0)$ ($\langle \xi^2 \rangle$—real function of ω at real $\epsilon_0(\omega)$). The asymptotic behavior of $\operatorname{Im} q(p, p_0)$ as $p \to \infty$ determines the character of the convergence of the integrals (20) and (23), which express the correlation function of the thermal field, the average power flux from the point source, and the Green's function of the average field (the real part of which determines the thermal fluctuations in the medium[14]).

Using a well known method, based on multiple integration by parts[5], we can write

$$\operatorname{Im} q(p, p_0) = \frac{2}{3} p_0^3 \sum_{n=0}^{N/2-1} (-1)^n (n+1) \frac{\partial^{2n+1}\Gamma_\xi(0)}{\partial x^{2n+1}} p^{-2(n+1)} + 0(p^{-N-2}),$$

$$\operatorname{Re} q(p, p_0) = \frac{\pi}{4} \frac{\partial \Gamma_\xi(0)}{\partial x} \frac{1}{p} + \frac{\pi}{4} \frac{\partial^3 \Gamma_\xi(0)}{\partial x^3} \frac{1}{p^3}$$

$$+ \sum_{n=0}^{(N-1)/2} \frac{(-1)^n}{(2n-1)(2n-3)} \frac{\partial^{2n}\Gamma_\xi(0)}{\partial x^{2n}} p^{-2n} + 0(p^{-N+1}). \tag{28}$$

The derivatives of the correlation coefficient $\Gamma_\xi(x)$ at $x = 0$, which take part in the expansion, are assumed to exist. Substituting the expression (28) in formula (27), we get as $p \to \infty$

$$\operatorname{Im} \varepsilon_{\text{eff}}^{l}(\omega, k) = \operatorname{Im} \varepsilon_{\text{eff}}^{tr}(\omega, k) = -2\varepsilon_0 \langle \xi^2 \rangle \operatorname{Im} q(p, p_0)$$
$$= -\frac{4}{3} \varepsilon_0(\omega) \langle \xi^2 \rangle p_0^3 \left[\frac{\partial \Gamma_\xi(0)}{\partial x} \frac{1}{p^4} - 2 \frac{\partial^3 \Gamma_\xi(0)}{\partial x^3} \frac{1}{p^6} + \ldots \right],$$

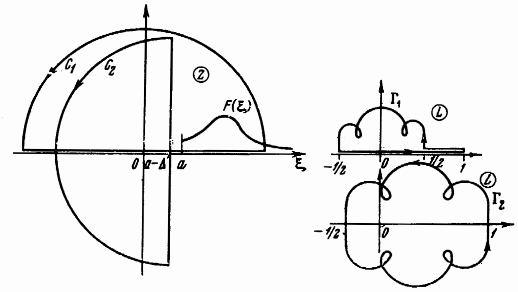
$$\operatorname{Re} \varepsilon_{\text{eff}}^{l}(\omega, k) = \varepsilon_0 \left(1 - \frac{2}{3} \langle \xi^2 \rangle \right) - \frac{1}{2} \varepsilon_0 \langle \xi^2 \rangle \frac{\partial \Gamma_\xi(0)}{\partial x} \frac{1}{p} + \ldots \tag{29}$$

A similar formula can be written also for $\operatorname{Re} \epsilon_{\text{eff}}^{tr}(\omega, k)$ as $k \to \infty$.

The most interesting is the first formula of (29). As follows from (20), (23), and (29), the spatial dispersion due to the random inhomogeneities ensures convergence of the integrals that express the intensity of the thermal fluctuations of the field in the inhomogeneous medium, the average flux of power from the point dipole, etc. The expansion coefficients of $\operatorname{Im} \epsilon_{\text{eff}}^{l}$ and $\operatorname{Im} \epsilon_{\text{eff}}^{tr}$ are the odd derivatives of the correlation coefficient $\Gamma_\xi(x)$ at $x = 0$. Inasmuch as $\Gamma_\xi(x)$ is an even function, the odd derivatives differ from zero when $x = 0$ only if these derivatives are discontinuous at the point $x = 0$, and in (28) and (29) we have in mind in this case the limit of the derivatives at $x = 0$ on the $x > 0$ side.

The character of the convergence of the integrals (20) in (23) is closely connected with those characteristics of the medium, which determine the behavior of the correlation function $B_\xi(x)$ (or $B_\epsilon(x)$) at $x = 0$. If the function $\Gamma_\xi(x)$ is analytic, then $\operatorname{Im} \epsilon_{\text{eff}}^{l}$, $\operatorname{Im} \epsilon_{\text{eff}}^{tr} \to 0$ as $k \to \infty$ at no slower a rate than $e^{-\alpha k}$ (for example, when $\Gamma_\xi(x) = \exp(-x^2)$ we get $\operatorname{Im} q(p, p_0) \sim \exp(-p^2/4)$).

The discontinuities of the derivatives $\Gamma_\xi(x)$ at $x = 0$ are connected in turn with the discontinuities of the function $\epsilon(\mathbf{r})$ and its derivatives (i.e., the discontinuities of the very realizations of the process). Thus, notice should be taken of the strong connection between the character of the scattering of the near-field of the sources placed in an inhomogeneous medium with the degree of smoothness of the functions $\epsilon(\mathbf{r})$, which



are realizations of the process $\epsilon(\mathbf{r})$. It follows from (18) and (19) that the contribution of the scattering of the quasi-static field by near-zone inhomogeneities can be very large, In particular, if we neglect the spatial dispersion due to the inhomogeneity of matter, then the density of the "effective heat" equals

$$Q_0 = \frac{\omega}{2\pi} \operatorname{Im} \varepsilon^{\text{eff}}(\omega) \langle \mathbf{E}(\omega, \mathbf{r}) \rangle \langle \mathbf{E}^*(\omega, \mathbf{r}) \rangle.$$

In the field of a point dipole, the total heat (the integral $\int Q_0 d\mathbf{r}$) is infinitely large[2)].

6. CONCLUSION

It was concluded in[8] that spatial dispersion due to inhomogeneities medium plays an insignificant role in the process of electromagnetic wave propagation. In problems involving radiation, the situation changes radically, since near the sources, where the field can strongly change over distances that are small compared correlation radius of the medium, we deal with a strong spatial dispersion, allowance for which is essential. We note that the expression for the correlation function of small-scale fluctuations in the form $\langle \Delta\epsilon^2 \rangle l^3 \delta(\mathbf{r})$, which is sometimes used in scattering problems, is equivalent to neglecting spatial dispersion. Allowance for spatial dispersion due to the inhomogeneities is also essential in problems of thermal fluctuations of the electromagnetic in randomly inhomogeneous media. In particular, owing to the inhomogeneity of the medium, the intensity of the thermal fluctuations becomes finite even if we neglect the true spatial dispersion of the medium.

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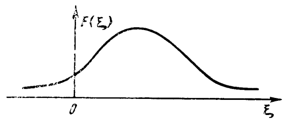
APPENDIX

To prove the statement that the solution of the equation $\langle \xi \rangle = 0$ is unique (Sec. 1) we can use the argument principle, according to which

$$N = (2\pi)^{-1} \Delta_C \arg F(z). \tag{A.1}$$

Here N—number of zeroes of the function $F(z)$, which is analytic in the region bounded by the contour C. The right side of (A.1) equals the number of revolutions of the vector L on going around the curve Γ, which cor-

2) Analogously, in a homogeneous absorbing medium the heat released in the field of a point dipole is infinite if we neglect spatial dispersion.



responds to the contour C in the mapping $L = F(z)$. The function $F(z)$, defined by expression (26), is analytic in the entire z plane with the exception of the points lying on the integration contour.

Let the function $W(\xi) = 0$ at $\xi < a$, $a > 0$ (Fig. 1), i.e., let the dielectric constant assume only positive values. We consider the contour C_1, which consists of a straight line $z = x + i0$ and an infinite half-circle in the upper half-plane. On the line $z = x + i\sigma$ ($\sigma > 0$) we have

$$\lim_{\sigma \to 0} \operatorname{Im} F(x + i\sigma) = {}^3/_2 \pi x W(x). \quad \text{(A.2)}$$

Making the transformation $L = F(z)$, we note that the half-circle with infinite radius corresponds to the point $L = -1/2$. The straight line $x + i0$ is transformed into a certain contour Γ_1 of the upper half-plane of L, which does not cross the real axis, since $\operatorname{Im} F \geq 0$ for any x. We verify similarly that $N = 0$ also in the lower half-plane of z. Considering the contour C_2, we find that it corresponds to the contour Γ_2. In this case $N = 1$, and the fact that Δ tends to zero does not change the situation. In the case under consideration we have one root $v_0 = x_0$, which lies on the real axis. It can be shown that $x_0 < -2a$. In fact,

$$F(-2a) = \int_a^{+\infty} \frac{\xi - a}{\xi + 2a} W(\xi) d\xi > 0.$$

On the other hand, $F(x \to -\infty) = -1/2$. Taking into account the uniqueness of the root, we see that the function $F(x)$ crosses the x axis once in the interval $[-2a, -\infty]$. From this we find that $\epsilon_0 = -x_0/2 > a$.

In the case when ϵ assumes only negative values, ϵ_0 is also negative.

Finally, in the case corresponding to Fig. 2, the equation $F(z) = 0$ has two complex roots, as can be verified in a perfectly similar manner.

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67