JOSEPHSON EFFECT IN SUPERCONDUCTORS SEPARATED BY A NORMAL METAL

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The penetration of electron pairs into a normal metal leads to the appearance of a superconducting current through a layer of normal metal bounding on the superconductors. The amplitude and frequency of this current are determined as functions of the applied voltage. The electrodynamic properties of such a system are investigated.

1. INTRODUCTION

Two superconductors separated by a thin layer of normal metal have the properties of a Josephson element[1]. The electron pairs penetrate into the normal metal, so that a superconducting current, proportional to the sine of the phase difference between the ordering parameters in the superconductors, can flow through the layer. The critical value of the current decreases with increasing thickness of the normal-metal layer. At a layer thickness on the order of the dimension of the pair, the critical current can be appreciably larger than in an ordinary Josephson element.

If a dc voltage is applied to such a system, the phase difference will depend on the time, and the superconducting current becomes alternating. The frequency of the current is proportional to the applied voltage. The amplitude of the alternating current consists of two parts. One part has the same dependence on the normal-layer thickness as the critical current in the stationary case. The other part vanishes in the absence of voltage on the layer, but can decrease slowly with decreasing thickness.

Since the resistance of the external circuit is large compared with the resistance of the element, the total current through an element connected to a dc source will be constant. Therefore, besides the superconducting current, there appears an alternating normal current, that cancels the superconducting current and produces an alternating voltage across the layer. The displacement current produced thereby in the metal is small compared with the superconducting or normal current. However, the displacement current and the associated radiation power can be large compared with the ordinary Josephson element.

2. ORDERING PARAMETER IN THE NORMAL METAL

The penetration of superconducting pairs into a normal metal leads to the appearance of an ordering parameter $\Delta(\mathbf{r})$ inside this metal. At sufficiently large normal-metal thickness, $\Delta(\mathbf{r})$ is small deep in the layer and therefore satisfies the linear equation. Near the critical temperature of the normal metal, the dependence of Δ on the transverse coordinate z is obtained from the Ginzburg-Landau equation and is given by

$$\Delta(z) = \Delta_1 \exp(-z/\xi) + \Delta_2 \exp((z-d)/\xi), \qquad (1)$$

where d—thickness of the layer of the normal metal and ξ is determined by the expression[2]

$$\xi = \left(\frac{T_c\eta}{T-T_c}\right)^{1/2}$$

$$\eta = \frac{v^2\tau_{tr}}{3\pi T}\left\{\frac{\pi^2}{8} + \pi T\tau_{tr}\left[\psi\left(\frac{1}{2}\right) - \psi\left(\frac{1}{2} + \frac{1}{4\pi T\tau_{tr}}\right)\right]\right\}, \qquad (2)$$

τ_{tr}—transport time between the collisions, v—Fermi velocity and $\psi(z)$—derivative of the logarithm of the Γ function.

The superconducting current is given by the ordinary formula

$$j = \frac{ie}{m}C\left(\Delta\frac{\partial\Delta^*}{\partial z} - \Delta^*\frac{\partial\Delta}{\partial z}\right), \qquad (3)$$

where the constant $C = m^3 v\eta/\pi^2$. Substituting (1) in (3), we get

$$j = -\frac{4e}{m\xi}C\,\mathrm{Im}\,(\Delta_1\Delta_2^*)\,e^{-d/\xi}. \qquad (4)$$

The parameters Δ_1 and Δ_2 should be obtained from the condition that they be continuous on the boundaries between the normal metal and the superconductors. Since the current is small, the phase Δ in the superconductors depends little on the coordinates. Therefore the phases Δ_1 and Δ_2 are the same as in the corresponding superconductors. It is seen here from (4) that, just as in the ordinary Josephson effect, the superconducting current is proportional to the sine of the phase difference Δ in the superconductors.

The quantities $|\Delta_1|$ and $|\Delta_2|$ can be obtained by a variational method when the temperature is close to the critical temperatures of the normal metal and of the superconductors[3]. In the general case it can only be stated that if the temperature is not too close to T_c of the normal metal, and the coefficient of transmission through the boundary is on the order of unity, then Δ_1 and Δ_2 are of the same order as the ordering parameters in the corresponding superconductors. Thus, the superconducting current can be much larger than in the ordinary Josephson element.

In the case of small transparency of the boundaries between the normal metal and the superconductors, this current can be obtained by perturbation theory for any temperature and thickness of the normal-metal layer. In order of magnitude, the obtained results are valid also when the coefficient of passage through the boundary is not small. In this case it is convenient to use the tunnel Hamiltonian[4]

$$\hat{H} = \hat{H}_0 + \hat{T}_1 + \hat{T}_2, \tag{5}$$

where

$$\hat{T}_{1,2} = \sum_{\nu\mu_{1,2}} (T^{(1,2)}_{\mu_{1,2}\nu} a_\nu^+ a_{\mu_{1,2}} + \text{э. с.}).$$

The index ν describes the state of the electron in the normal metal, and μ—in the superconductor.

In second order in the transparency, the ordering parameter in the normal metal consists of two terms proportional respectively to T_1^2 and T_2^2. Each term satisfies the equation

$$\Delta(z) = -g \int_0^d \Pi(z, z')\Delta(z')dz' \tag{6}$$
$$+ g \sum_{\mu\nu\rho} \sum_\omega F_\nu(\omega) T_{\mu\nu} G_\mu(\omega)[\delta(\hat{\mathbf{r}} - \mathbf{r})]_{\mu\rho} G_\rho(-\omega) T_{\rho\nu}.$$

as was found by de Gennes and Shapoval[5,6]

$$\Pi(z, z') = T \sum_\omega \int \overline{G_\omega(z, z')}\,\overline{G_{-\omega}(z, z')}\,dx\,dy$$
$$= T \sum_\omega \frac{mp_0}{\pi} w(z, z', 2i|\omega|), \tag{7}$$

where

$$w(z, z', \tau) = 1/2\pi \int w(z, z', \omega) \exp(-i\omega\tau)\,d\omega$$

is the probability of the passage of the particle from the point with coordinate z to the point with coordinate z′ within a time τ. The second term in the right side of (6) can be written in the form[7]

$$g\frac{mp_0}{\pi} T \sum_\omega \int \langle \hat{T}(\tau_1)\delta[\hat{\mathbf{r}}(\tau_2) - \mathbf{r}]\hat{T}(\tau)\rangle$$
$$\times F_\omega(\tau - \tau_1) G_\omega(\tau_1 - \tau_2) G_{-\omega}(\tau_2 - \tau)\,d\tau_1\,d\tau_2, \tag{8}$$

where the operators are written in the Heisenberg representation

$$\hat{T}(\tau) = \exp(-i\hat{H}_0\tau)\hat{T}\exp(i\hat{H}_0\tau),$$

and

$$G_\omega(\tau) = \frac{1}{2\pi}\int G_\omega(\xi_\nu)\exp(-i\xi_\nu\tau)\,d\xi_\nu;$$

$F_\omega(\tau)$ is defined in similar fashion. The brackets $\langle\,\rangle$ denote averaging over all states on the Fermi surface.

With quasi-classical accuracy, the average of the product of the operators in formula (8) can be replaced by the expression

$$\frac{mp_0}{\pi}\langle \hat{T}(\tau_1)\delta[\hat{\mathbf{r}}(\tau_2) - \mathbf{r}]\hat{T}(\tau)\rangle$$
$$= \int_{p_z>0} \frac{1}{4\pi}\delta(\tau_1 - \tau) D^2(p_z) w(p_z, \mathbf{r}, \mathbf{r}_s, \tau_2 - \tau)\,dp_z^2\,dS. \tag{9}$$

Here $w(p_z, \mathbf{r}, \mathbf{r}_s, \tau)$—probability that the particle emitted from the point $\mathbf{r}_s$ on the wall with energy ϵ_F and with transverse momentum p_z will reach the point $\mathbf{r}$ after a time τ, and $D^2(p_z)$—coefficient of transmission through the boundary.

Using formulas (7), (8), and (9), we transform Eq. (6) for $\Delta(z)$ into

$$\Delta(z) = -\frac{gmp_0}{\pi} T \sum_\omega \int_0^d w(z, z', 2i|\omega|)\Delta(z')\,dz' \tag{10}$$
$$+ T\sum_\omega \frac{g\Delta_2 e^{i\varphi_2}}{8\pi\sqrt{\omega^2 + \Delta_2^2}} \int_{p_z>0} w(z, d, p_z, 2i|\omega|) D^2(p_z)\,dp_z^2.$$

In the most interesting case, when the electron free path is much shorter than the pair dimension ξ_0, the probability $w(z, z', 2i|\omega|)$ satisfies the diffusion equation and does not depend on p_z:

$$\left[2|\omega| - \frac{1}{3}v^2\tau_{tr}\frac{\partial^2}{\partial z^2}\right] w(z, z', 2i|\omega|) = \delta(z - z'), \quad \left.\frac{\partial w}{\partial z}\right|_{z=0,d} = 0. \tag{11}$$

From (11) we get

$$w = \frac{1}{d}\sum_{n=-\infty}^{\infty} \frac{\cos(\pi nz/d)\cos(\pi nz'/d)}{2|\omega| + {}^1/_3(\pi n/d)^2 v^2\tau_{tr}}. \tag{12}$$

Solving Eq. (10) for $\Delta(z)$ with kernel (12), we get

$$\Delta(z) = \sum_{n=-\infty}^{\infty} \Delta_n \cos\left(\frac{\pi nz}{d}\right),$$
$$\Delta_n = \frac{g}{4\pi d}(-1)^n \Delta_2 e^{i\varphi_2}\left(T\sum_{\omega>0}\frac{1}{\sqrt{\omega^2 + \Delta_2^2}}\,\frac{1}{\omega + {}^1/_6(\pi n/d)^2 v^2\tau_{tr}}\right)$$
$$\times\left(1 + g\frac{mp_0}{\pi} T\sum_{\omega>0}\frac{1}{\omega + {}^1/_6(\pi n/d)^2 v^2\tau_{tr}}\right)^{-1}\int D^2(p_z)\,dp_z^2. \tag{13}$$

The obtained solution is the first term of the expansion of Δ in powers of D^2. In order for this expansion to be valid, it is necessary that the nonlinear effects be small. The strongest limitation is apparently the condition $\Delta^2 \ll T(T - T_c)$, where T_c—temperature of the transition of the normal metal into the superconducting state. Estimating Δ by means of formula (13), we obtain the inequality

$$D^2 \ll \sqrt{T\tau_{tr}}(T - T_c)/T_c,$$

which yields a limitation on the transmission coefficient D^2 at temperatures close to critical.

3. CRITICAL CURRENT OF THE ELEMENT

The current through the contact is determined by the change of the number of particles in one superconductor:

$$j = -e\langle\dot{N}_1\rangle = ie\left\langle \exp(-\hat{H}_0/T) T_\tau \exp\left(-\int_0^{1/T}\hat{T}(\tau)\,d\tau\right)[\hat{T}\hat{N}_1]\right\rangle$$
$$\times\left\langle \exp(-\hat{H}_0/T) T_\tau \exp\left(-\int_0^{1/T}\hat{T}(\tau)\,d\tau\right)\right\rangle^{-1}, \tag{14}$$

where

$$\hat{H}_0 = \sum_{i=1,2,3}(\varepsilon_i a_i^+ a_i + a_i a_i \Delta_i^+ + \Delta_i a_i^+ a_i^+).$$

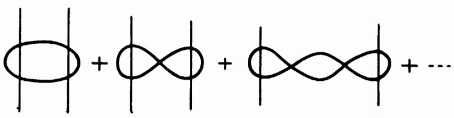
Expanding this expression in powers of $\hat{T}$ and using the Matsubara technique[8], we get

$$j = 4e\,\mathrm{Im}\Big\{T\sum_\omega F_{\mu_1}^+(\omega) T^{(1)}_{\mu_1\nu_1} G_{\nu_1}(\omega)$$
$$\times T^{(2)}_{\nu_1\mu_2} F_{\mu_2}(\omega) T^{(2)}_{\mu_2\nu_2} G_{\nu_2}(-\omega) T^{(1)}_{\mu_1\nu_2}$$
$$- T\sum_\omega\int F_{\mu_1}^+(\omega) T^{(1)}_{\mu_1\nu_1} G_{\nu_1}(\omega) T^{(1)}_{\mu_1\nu_2} G_{\nu_2}(-\omega)\Delta(\mathbf{r})[\delta(\hat{\mathbf{r}} - \mathbf{r})]_{\nu_1\nu_2}\,d\mathbf{r}\Big\}. \tag{15}$$

This expression corresponds to the diagrams shown in the figure. Using, as above, the quasiclassical approximation, we find

$$j = eS\,\mathrm{Im}\Big\{T\sum_\omega \frac{\Delta_1\Delta_2\exp[i(\varphi_2 - \varphi_1)]}{16\pi mp_0\sqrt{\omega^2 + \Delta_1^2}\sqrt{\omega^2 + \Delta_2^2}}$$
$$\times\int D_1^2(p_{1z}) D_2^2(p_{2z}) w(0, d, p_{1z}, p_{2z}, 2i|\omega|)\,dp_{1z}^2\,dp_{2z}^2$$
$$- T\sum_\omega\int_0^d \frac{\Delta_1 e^{-i\varphi_1}}{2\pi\sqrt{\omega^2 + \Delta_1^2}}\int dp_z^2 D_1^2(p_{1z}) w(0, z, p_{1z}, 2i|\omega|)\Delta(z)\,dz\Big\}. \tag{16}$$

Substituting in (16) the expressions for w and Δ from



(12) and (13), get for the case when the electron free path is much smaller than the dimension of the pair

$$j = \frac{eS\Delta_1\Delta_2 \sin(\varphi_2-\varphi_1)}{16\pi d m p_0} \sum_{n=-\infty}^{\infty} (-1)^n$$
$$\times \Bigg\{ T\sum_{\omega>0} \frac{1}{\sqrt{\omega^2+\Delta_1^2}}\, \frac{1}{\omega + {}^1/_6(\pi n/d)^2 v^2\tau_{tr}} \Bigg[\frac{1}{\sqrt{\omega^2+\Delta_2^2}} - \frac{T\sum\limits_{\omega_1>0}(\omega_1^2+\Delta_2^2)^{-1/2}(\omega_1+{}^1/_6(\pi n/d)^2v^2\tau_{tr})^{-1}}{\pi/gmp_0 + T\sum\limits_{\omega_2>0}(\omega_2+{}^1/_6(\pi n/d)^2v^2\tau_{tr})^{-1}} \Bigg]\Bigg\}$$
$$\times \int D_1^2(p_{1z})\,dp_{1z}^2 \int D_2^2(p_{2z})\,dp_{2z}^2. \qquad (17)$$

Let us investigate the dependence of the superconducting current on the temperature and on the thickness of the layer of the normal metal. At high temperatures $T \gg T_c$ and $T \gg T^*$, where $T^* = v^2\tau_{tr}/6\pi d^2$, the current decreases exponentially with temperature and with normal-layer thickness:

$$j = \frac{eS}{8\pi p_0^2}\Delta_1\Delta_2\sin(\varphi_2-\varphi_1)\sqrt{\frac{6T}{\pi\tau_{tr}}}(\pi^2T^2+\Delta_1^2)^{-1/2}(\pi^2T^2+\Delta_2^2)^{-1/2}$$
$$\times \exp\left\{-\frac{d}{v}\sqrt{\frac{6\pi T}{\tau_{tr}}}\left[1+\left(\ln\frac{\omega_D}{2\pi T}+\frac{2\pi^2}{gmp_0}\right)^{-1}\right]\right\}\times$$
$$\times \int D_1^2(p_{1z})\,dp_{1z}^2 \int D_2^2(p_{2z})\,dp_{2z}^2. \qquad (18)$$

If the critical temperature of the normal metal is not too small, $T_c \gg T^*$, then the exponential dependence is retained also in the case when $T \gtrsim T_c$. The argument of the exponential is in this case $-kd$, where k is determined from the equation

$$1+\frac{gmp_0}{2\pi^2}\left[\ln\frac{2\omega_D\gamma}{\pi T} + \psi\left(\frac{1}{2}\right) - \psi\left(\frac{1}{2} + \frac{\pi\tau_{tr}}{12T}(kv)^2\right)\right] = 0. \qquad (19)$$

In the direct vicinity of the critical point, $T - T_c \ll T_c$, the expression for the current is

$$j = e\Delta_1\Delta_2\sin(\varphi_2-\varphi_1)\frac{\sqrt{6\pi}\,TS}{4\pi p_0^2\sqrt{(T-T_c)\tau_{tr}}}\left(T\sum_{\omega>0}\frac{1}{\omega\sqrt{\omega^2+\Delta_1^2}}\right)$$
$$\times \left(T\sum_{\omega>0}\frac{1}{\omega\sqrt{\omega^2+\Delta_2^2}}\right)\mathrm{sh}^{-1}\left(\frac{d}{v}\sqrt{\frac{24(T-T_c)}{\pi\tau_{tr}}}\right)$$
$$\times \int D_1^2(p_{1z})\,dp_{1z}^2 \int D_2^2(p_{2z})\,dp_{2z}^2. \qquad (20)$$

It follows therefore that when $T - T_c \ll T^*$ the current depends on the temperature and thickness of the normal-metal layer in power-law fashion. Formula (20) can also be obtained by substituting in Eq. (3) the asymptotic expression for Δ from (13) at temperatures close to critical.

The expression for the current has a different form if $T_c \ll T^*$. In this case when $T \ll T^*$ the dependence of the current on the thickness of the normal layer and on the temperature ceases to be exponential:

$$j = \frac{eS\sin(\varphi_2-\varphi_1)}{32\pi^2 mp_0}\left\{\ln\left(\frac{4v^2\tau_{tr}\gamma}{3\pi T d^2}\right)\right.$$
$$- \left(\ln\frac{4\Delta_1\gamma}{\pi T}\right)\left(\ln\frac{4\Delta_2\gamma}{\pi T}\right)\left(\frac{2\pi^2}{gmp_0}+\ln\frac{2\omega_D\gamma}{\pi T}\right)^{-1}$$
$$\left.+ \left(\ln\frac{3\Delta_1 d^2}{v^2\tau_{tr}}\right)\left(\ln\frac{3\Delta_2 d^2}{v^2\tau_{tr}}\right)\left(\frac{2\pi^2}{gmp_0}+\ln\frac{3\omega_D d^2}{2v^2\tau_{tr}}\right)^{-1}\right\}$$
$$\times \int D_1^2(p_{1z})\,dp_{1z}^2 \int D_2^2(p_{2z})\,dp_{2z}^2. \qquad (21)$$

This formula gives the expression for the current at $T \ll T^*$ also in the case when the electrons in the normal metal are repelled ($g > 0$), so that the metal does not go over into the superconducting state when the temperature is decreased. When $T \gg T^*$, formula (18) is valid. By measuring the temperature dependence of the critical current, it is possible to estimate the interaction constant g, even if it is positive.

4. AMPLITUDE OF SUPERCONDUCTING CURRENT IN THE NONSTATIONARY CASE

If we apply a dc voltage to the system, then the superconducting current becomes alternating. At sufficiently small voltage, the adiabatic approximation is valid:

$$j = j_s \sin\left(2e\int^t V(t')\,dt'\right), \qquad (22)$$

where j_s is given by (17).

In order to find the region of applicability of the adiabatic approximation and to estimate the non-adiabatic terms, let us consider the simplest case, when the electrons do not interact in the normal metal, and the voltage is applied essentially to the contacts between the normal metal and the superconductors. In this case the Hamiltonian (5) takes the form

$$\hat H = \hat H_0 + eV_1\sum a_{\nu_1}^+ a_{\nu_1} + eV_2 \sum a_{\nu_2}^+a_{\nu_2} + eV_3\sum a_{\nu_3}^+a_{\nu_3} + \hat T, \qquad (23)$$

where V_1 and V_3 are the potentials of the superconductors, and V_2 is the potential of the normal metal.

In the interaction representation, the current j(t) is best written in the form proposed by Keldysh[9]:

$$\langle \hat j(t)\rangle = \mathrm{Sp}\left\{\exp\left(\frac{(\Omega-\hat H_0)}{T}\right) T_c\left[\exp\left(-i\int_\Gamma \hat T_0(t_1)\,dt_1\right)\hat j_0(t)\right]\right\}, \qquad (24)$$

where the contour Γ passes over the entire time axis from $-\infty$ to $+\infty$, and then back from $+\infty$ to $-\infty$, T_c is the ordering operator on this contour, and the current operator and the tunnel Hamiltonian are written in the interaction representation:

$$\hat T_0(t) = \exp(i\hat H_0 t)\sum\{\hat T^{(1)}_{\nu_1\nu_2}a_{\nu_1}^+a_{\nu_2}\exp[ie(V_1-V_2)t]$$
$$+ \hat T^{(2)}_{\nu_2\nu_3}a_{\nu_2}^+a_{\nu_3}\exp[ie(V_2-V_3)t] + \text{э. c.}\}\exp(-i\hat H_0 t),$$
$$\hat j_0(t) = ie\exp(i\hat H_0 t)\sum\{T^{(1)}_{\nu_1\nu_2}a^+_{\nu_1}a_{\nu_2}\exp[ie(V_1-V_2)t] - \text{э. c.}\}\exp(-i\hat H_0 t). \qquad (25)$$

When expression (24) for the current is expanded in powers of $\hat T$, one obtains retarded, advanced, and chronologically-ordered Green's functions. Following Keldysh[9], it is convenient to use the matrix notation. As a result, in fourth order in the transparency, we obtain for the superconducting current the expression

$$j(t) = 2e\,\mathrm{Re}\left\{\exp[-2it(V_3-V_1)]\right.$$
$$\times \mathrm{Sp}\left[\int_{-\infty}^{+\infty}\frac{d\omega}{2\pi} T^{(1)}_{\mu\nu_1}T^{(2)}_{\nu_1\lambda}T^{(2)}_{\lambda\nu_2}T^{(1)}_{\nu_2\mu}\right.$$
$$\times \hat G_{\nu_1}(\omega+V_3-V_2)\hat F_\lambda(\omega)\hat G^T_{\nu_2}(-\omega+V_3-V_2)$$
$$\left.\left.\times \hat F^+_\mu(\omega-V_3+V_1)(1+\hat\sigma_z)\right]\right\}, \qquad (26)$$

where

$$\hat{G}_\nu(\omega)=\frac{1}{2}\begin{pmatrix}1 & -1\\ 1 & 1\end{pmatrix}\begin{pmatrix}(\omega-\xi-i\delta)^{-1}; & 0\\ 2\pi i\delta(\omega-\xi)\,\mathrm{th}\frac{\omega}{2T}; & (\omega-\xi+i\delta)^{-1}\end{pmatrix}\begin{pmatrix}1 & 1\\ -1 & 1\end{pmatrix},$$

$$\hat{G}_\nu{}^T(\omega)=\frac{1}{2}\begin{pmatrix}1 & 1\\ -1 & 1\end{pmatrix}\begin{pmatrix}(\omega-\xi+i\delta)^{-1}; & 0\\ -2\pi i\delta(\omega-\xi)\,\mathrm{th}\frac{\omega}{2T}; & (\omega-\xi-i\delta)^{-1}\end{pmatrix}\begin{pmatrix}1 & -1\\ 1 & 1\end{pmatrix},$$

$$\hat{F}_\nu(\omega)=\frac{i\Delta}{2}\begin{pmatrix}1 & -1\\ 1 & 1\end{pmatrix}$$

$$\times\begin{pmatrix}-(\xi^2+\Delta^2-(\omega-i\delta)^2)^{-1}; & 0\\ 2\pi i\,\mathrm{th}\frac{|\omega|}{2T}\delta(\xi^2+\Delta^2-\omega^2); & (\xi^2+\Delta^2-(\omega+i\delta)^2)^{-1}\end{pmatrix}\begin{pmatrix}1 & -1\\ 1 & 1\end{pmatrix},$$

$$\hat{F}_\nu{}^+(\omega)=\frac{\Delta^*}{2}\begin{pmatrix}1 & 1\\ -1 & 1\end{pmatrix}$$

$$\times\begin{pmatrix}-(\xi^2+\Delta^2-(\omega-i\delta)^2)^{-1}; & 0\\ -2\pi i\,\mathrm{th}\frac{|\omega|}{2T}\delta(\xi^2+\Delta^2-\omega^2); & (\xi^2+\Delta^2-(\omega+i\delta)^2)^{-1}\end{pmatrix}\begin{pmatrix}1 & 1\\ -1 & 1\end{pmatrix}.$$

As in the stationary case, it is convenient to change from the energy variables ξ_ν to the time representation. As a result we obtain, with quasiclassical accuracy,

$$j(t)=\frac{eS}{8\pi mp_0}\mathrm{Re}\Big\{\exp[-2it(V_3-V_1)]\int_{-\infty}^{+\infty}\frac{d\omega}{2\pi}$$
$$\times\,\mathrm{Sp}[\hat{G}_{\omega+V_3-V_2}(\tau)\hat{F}_\omega(0)\hat{G}^{+T}_{-\omega+V_3-V_2}(-\tau)\hat{F}_{\omega-V_3+V_1}(0)$$
$$\times(1+\hat{\sigma}_z)]\,w(0,d,p_{1z},p_{2z},\tau)D_1^2(p_{1z})D_2^2(p_{2z})\,dp_{1z}^2\,dp_{2z}^2\,d\tau\Big\}. \qquad (27)$$

Here

$$\hat{G}_\omega(\tau)=\frac{1}{2\pi}\int\hat{G}_\omega(\xi)\exp(-i\xi\tau)\,d\xi.$$

Multiplying the matrices in (27) and integrating with respect to τ, we get

$$j(t)=-\frac{eS\Delta_1\Delta_2}{32\pi mp_0}\mathrm{Re}\Big\{e^{2i(V_1-V_3)t}\int_{-\infty}^{+\infty}\frac{d\omega}{2\pi}w(0,d,p_{1z},p_{2z},2\omega) \qquad (28)$$
$$\times\Big\{\frac{\mathrm{th}[(\omega+V_3-V_2)/2T]}{\sqrt{(\omega-i\delta)^2-\Delta_2^2}\sqrt{(\omega-V_3+V_1-i\delta)^2-\Delta_1^2}}$$
$$+\frac{\mathrm{th}[(\omega+V_3-V_1)/2T]}{\sqrt{(\omega-i\delta)^2-\Delta_2^2}\sqrt{(\omega+V_3-V_1-i\delta)^2-\Delta_1^2}}$$
$$+\frac{\mathrm{th}[(\omega+V_3-V_1)/2T]-\mathrm{th}[(\omega+V_3-V_2)/2T]}{\sqrt{(\omega-i\delta)^2-\Delta_2^2}\sqrt{(\omega+V_3-V_1+i\delta)^2-\Delta_1^2}}\Big\}$$
$$\cdot D_1^2(p_{1z})D_2^2(p_{2z})\,dp_{1z}^2\,dp_{2z}^2,$$

where

$$w(\omega)=\int_0^\infty\exp(-i\omega\tau)\,w(\tau)\,d\tau$$

is an analytic function of ω in the lower half plane.

The terms in the curly brackets of (28) have essentially different analytic singularities in the lower half plane of ω. In the first two terms, the singularities appear only in the poles of the tangents. In the last term, in addition, there is a cut between the points $\pm\Delta_1+V_1-V_3-i\delta$. Terms of this type arise in the general theory of nonstationary phenomena in superconductors[10].

The dependence of the current on the layer thickness d at large metal thicknesses is determined by the singularities of the integrand in (28) that are closest to the real axis. The integral of the first two terms reduces to a sum over ω_n, which goes over when $V \ll T$ into expression (17) for the critical current. The last term vanishes for the stationary case. Its dependence on the thickness of the normal-metal layer is determined by the singularities of the root and is proportional to $w(0, d, 2(\pm\Delta_1+V_1-V_3))$. This term can therefore decrease more slowly with increasing thickness than the critical current, and it may become more significant at not too small voltages across the layer.

The effect is most noticeable for a pure metal, when the electron mean free path is large compared with the layer thickness. In this case the non-adiabatic term decreases with thickness in power-law fashion and will make the main contribution to the amplitude when

$$V\gg T\,\mathrm{ch}^2\Big(\frac{\Delta}{2T}\Big)\Big(\frac{\Delta_1 d}{v}\Big)^{1/2}\exp\Big(-\frac{2\pi Td}{v}\Big).$$

Such a slow decrease of the amplitude is a result of the fact that in the pure metal the electron pairs retain their correlation after passing through the normal metal. In a contaminated metal, however, the violation of the correlation of the electrons upon scattering by impurities leads to an exponential dependence of the current amplitude on the layer thickness. The probability $w(0, d, 2\omega)$ is obtained in this case from formula (12) and is given by

$$w(0,d,2\omega)=-\frac{\sqrt{3}(1+i)}{2v\sqrt{\omega\tau_{tr}}}\Big/\sin\Big[(1-i)\frac{d}{v}\sqrt{\frac{3\omega}{\tau_{tr}}}\Big]. \qquad (29)$$

Therefore the non-adiabatic term is proportional to $\exp(-dv^{-1}\sqrt{3\Delta_1/\tau_{tr}})$, and determines the decrease of the amplitude with increasing layer thickness of the normal metal only when $\Delta_1 < 2\pi T$. It can be assumed that in more complicated cases, when an interaction takes place between the electrons in the normal metal or when the voltage drop is across the entire layer, there exists a wide range of voltages and temperatures for which the adiabatic approximation is valid.

5. ELECTRODYNAMIC PROPERTIES OF THE ELEMENT

The influence of the current field on the phase of the ordering parameter in superconductors leads to a dependence of the phase on the time and coordinates. If the phase varies sufficiently slowly, then the current in turn is expressed in terms of the phase difference, and this relation determines the electrodynamic properties of the element.

At zero voltage across the layer, as in the ordinary Josephson effect, the current penetrates inside the contact only at distances on the order of the Josephson depth of penetration[11]

$$\lambda_j^2=\hbar c^2S/8\pi ej_s(d+\lambda_1+\lambda_2). \qquad (30)$$

Owing to the large value of the critical current, λ_j will be much smaller than in the Josephson element. However, the condition $\lambda_j^2 \gg d\xi$ is always satisfied, and therefore the connection between the current and the phase difference can be regarded as local, just as in the ordinary Josephson effect.

In the nonstationary case, the behavior of the system depends on the relations between the following parameters: λ_j, the width l of the normal-metal layer, and the depth δ of the skin layer. If $l \ll \lambda_j, \delta$, then the current does not depend on the coordinates, and the equation for the conservation of the total current yields an equation for the phase difference φ of the ordering parameters in superconductors:

$$j = \frac{1}{2eR}\frac{\partial\varphi}{\partial t} + j_s \sin\varphi, \tag{31}$$

where j is the total current flowing through the contact in the external circuit. The first term, which equals V/R, is the normal current flowing through the contact. The displacement current can be neglected.

We shall assume that the current in the external circuit is maintained constant. Then when $j \leq j_S$ the solution of Eq. (31) is φ = const, and a dc superconducting current flows through the system. When $j > j_S$, the solution of the equation is

$$\mathrm{tg}\frac{\varphi}{2} = \frac{j_s}{j} + \sqrt{1 - \left(\frac{j_s}{j}\right)^2}\, \mathrm{tg}\left(eR\sqrt{j^2 - j_s^2}\, t\right). \tag{32}$$

We see therefore that the voltage across the layer $V = \dot{\varphi}/2e$ is a periodic function of the time with period

$$T = \pi / eR\sqrt{j^2 - j_s^2} \tag{33}$$

and changes from a maximum value $V_{max} = (1/2)R(j + j_S)$ to a minimum value $V_{min} = (1/2)R(j - j_S)$. The average voltage across the contact is given by the expression

$$\bar{V} = R\sqrt{j^2 - j_s^2}. \tag{34}$$

We see therefore that the current-voltage characteristic of the element is a hyperbola, which goes over into the ordinary Ohm's law at large currents. It follows from (33) and (34) that the frequency of the voltage oscillations is connected with the average voltage across the layer by the ordinary Josephson relation $\omega = 2e\bar{V}$.

At j close to j_S, the phase $\varphi(t)$ differs little from a step function. In this case the dependence of the voltage on the time consists of narrow periodic pulses that follow each other with a frequency $2\pi/T$ and have a Lorentz form:

$$V(t) = Rj / [1 + (eRjt)^2]. \tag{35}$$

Owing to the small width of the peak, the number of voltage harmonics is large:

$$N \sim [1 - (j_s / j)^2]^{-1/2}. \tag{36}$$

In the other limiting case $j \gg j_S$, the voltage varies in accordance with

$$V = R[j + j_s \sin(2eRjt)]. \tag{37}$$

The amplitudes of the higher harmonics of the voltage will in this case be small.

With increasing current j through the contact, the average voltage, and consequently also the frequency of the alternating voltage, increases. Therefore, the depth of the skin layer can become smaller than the width of the contact l. The distribution of the current over the contact becomes inhomogeneous and in place of Eq. (31) it is necessary to use the more general equation

$$\frac{\hbar}{2eR}\frac{\partial\varphi}{\partial t} + j_s \sin\varphi = \frac{Sc^2\hbar}{8\pi e(\lambda_1 + \lambda_2 + d)}\frac{\partial^2\varphi}{\partial x^2}. \tag{38}$$

The boundary condition for this equation is the equation for the total current j through the contact, which in the absence of an external field takes the form

$$\frac{\hbar c^2 S}{8\pi l(\lambda_1 + \lambda_2 + d)e}\frac{\partial\varphi}{\partial x}\bigg|_{x=\pm l/2} = \mp\frac{j}{2}, \tag{39}$$

We shall solve (35) by iteration with respect to j_S/j:

$$\varphi = \varphi_0 + \varphi_1, \tag{40}$$

where

$$\varphi_0 = \omega t + (x/\delta)^2,$$

$$\varphi_1 = \left(\frac{\delta}{2\lambda_j}\right)^2 \sqrt{\frac{\pi}{2e}}\, \mathrm{Im}\, e^{i\omega t}\Big\{C\left(e^{(1+i)x/\delta} + e^{-(1+i)x/\delta}\right)$$
$$- e^{(1+i)x/\delta}\,\Phi\left(-\frac{i}{\sqrt{2}} - \frac{1-i}{\delta\sqrt{2}}x\right) - e^{-(1+i)x/\delta}\,\Phi\left(-\frac{i}{\sqrt{2}} + \frac{1-i}{\delta\sqrt{2}}x\right)\Big\} \tag{41}$$

The constant C is obtained from the boundary condition $\hbar\omega = 2ejR$, $\Phi(z)$ is the error integral, and $\delta^2 = SRc^2/2\pi\omega\,(\lambda_1 + \lambda_2 + d)$. If the contact resistances are small compared with the resistance of the layer of the normal metal, then the parameter δ coincides with the ordinary depth of the skin layer: $\delta^2 = c^2/2\pi\sigma\omega$.

For a thin contact ($l \ll \delta, \lambda_j$) we obtain formula (37). Let us investigate the behavior of the phase φ for a thick contact $l \gg \delta$. Inside the contact, at distances larger than δ from its edges, the phase φ_1 no longer depends on the boundary conditions and is determined by the last two terms of (41). The amplitude of the phase in the center of the contact reaches its maximum value, which is of the order of δ^2/λ_j^2. At distances larger than δ from the center of the contact, the change of phase φ_1 follows the law

$$\varphi_1 = -\frac{\delta^4}{4\lambda_j^2 x^2}\sin\left(\omega t + \frac{x^2}{\delta^2}\right).$$

Near the edges of the contact, the solution is determined by the first term in formula (41), which decreases exponentially with increasing distance from the edge. The voltage V_c on the edge of the contact is given by the formula

$$V_c = \frac{\hbar}{2e}\frac{\partial\varphi}{\partial t} = jR\left(1 + \frac{\delta^3}{\sqrt{2}\,\lambda_j^2 l}\sin(2eRjt)\right). \tag{42}$$

Thus, the amplitude of the alternating voltage on the edge of the contact, which determines the radiation power of the element, decreases with increasing element width.

The obtained solution is valid provided the maximum value is $\varphi_1 \ll 1$. This condition is always satisfied if $\delta \ll \lambda_j$. Thus, for thin contacts ($l \ll \lambda_j$) the regions of applicability of formulas (32) and (41) overlap. For broad contacts, the dependence of the phase on the time and on the coordinates was obtained only in the region of large frequencies, when $\delta \ll \lambda_j$.

6. DISCUSSION OF RESULTS

In conclusion, let us stop to discuss the possibilities of experimentally investigating the contact under consideration. In the study of the critical current through the elements, the shape of the contact can be the same as that of an ordinary Josephson element. In this case there is no essential limitation on the contact area S. The thickness of the contact d should not greatly exceed the parameter $\xi = [\pi v^2 \tau_{tr}/24(T - T_c)]^{1/2}$, in order that the critical current through the contact not be too small. The main difference between such a contact and an ordinary Josephson element is that the critical current can in this case be much larger. In addition, notice should be taken of its temperature dependence: the critical current increases sharply on approaching the temperature at which the normal metal goes over into the superconducting state. If the normal metal does not become superconducting at any temperature, then the critical current increases sharply when the temperature approaches zero.

Just as in an ordinary Josephson element, the critical current begins to depend on the external magnetic field only in fields stronger than the intrinsic magnetic field. Owing to the large value of the critical current, these fields are much larger than in the ordinary Josephson effect:

$$H \sim \Phi_0 / \lambda_j (d + \lambda_1 + \lambda_2).$$

In the study of the nonstationary effect with sufficiently high frequency, an appreciable limitation arises not only on the thickness of the contact d but also on its length l. This limitation is connected with the fact that the resistance of the contact should be large, so that when the total current through the contact is smaller than the critical current of the superconductor, the voltage on the contact be sufficiently large. If the width of the contact is larger than or of the order of the depth of penetration of the field into the superconductor, then the maximum voltage on the contact is of the order of $eV_{max} \sim \Delta d \lambda_{tr} / v \tau_{tr} l$. In addition, to obtain the maximum amplitude of the alternating voltage on the contact, its width l should be smaller than the depth δ of the skin layer of the normal metal. Such contacts can be produced by sputtering over a thin superconducting film with a narrow slit a thin layer of normal metal at the location of the slit. The width of the slit plays the role of the thicknsss of the contact d, and the thickness of the superconducting current plays the role of the width of the contact l.

The length of the electromagnetic waves radiated by such a contact can be of the order of 1 cm. At a slit width d on the order of ξ, the amplitude of the alternating voltage eV can reach $\sim \Delta$ in the superconductor. The intensity of radiation from the element, determined by the square of the amplitude of the alternating voltage, will be larger in this case than in an ordinary Josephson contact.

It should be noted that the gain in the amplitude of the alternating voltage is much smaller than in the amplitude of the superconducting current. This is connected with the fact that in the Josephson element the alternating superconducting current is balanced out by the displacement current in the capacitor, and in the case under consideration it is balanced out by the normal current through the metal. The effective reactance of the capacitor to the displacement current $(\omega C)^{-1}$ is much larger than the resistance of the normal layer R. The large value of the critical current makes it possible to investigate the phenomenon in the case when the current through the contact is close to its critical value. Then the voltage across the contact has an unusual dependence on the time, and constitutes narrow periodic pulses, the repetition frequency of which decreases when the current approaches the critical value.

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