# DYNAMIC SYMMETRY OF THE RELATIVISTIC SYMMETRIC COULOMB PROBLEM IN THE CASE OF THE CONTINUOUS SPECTRUM

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It is shown that in the relativistic symmetric Coulomb model the states of the continuous spectrum with fixed energy $E > m$ realize a single infinite-dimensional irreducible representation of the dynamic group $SL(2, c) \otimes SU_2$. Relative to its most important subgroup $SL_j(2, c)$ this representation can be decomposed into two irreducible representations with $m = -1, \rho = -i + 2\alpha ZE(E^2 - m^2)^{-1/2}$ and $m = 1, \rho = i + 2\alpha ZE(E^2 - m^2)^{-1/2}$. For $E = m$ the representation of $SL_j(2, c)$ is not completely reducible. Analogous results are obtained for a Pauli electron in a Coulomb field. It is also shown that the dynamic group connected with the quadratic relativistic equation is $SL(2, c) \otimes SU(2, 2)$; the solutions of the continuous spectrum are classified relative to the subgroups $SL_{j\pm}(2, c)$.

In recent papers [1-5] it has been shown that it is possible to use the infinite-dimensional unitary representations of the noncompact group $O(4, 2)$ for the classification of particles [3] as well as for the calculation of various form factors.[3,5] This group is the dynamic group of the nonrelativistic hydrogen atom.[6] The principle of its application, and of the possible application of other noncompact groups consists in investigating completely the action of the noncompact group in an exactly soluble model and in determining its irreducible representation which describes the spectrum of the model under consideration and the scattering amplitude. Then only the group structure of the problem is taken over, like a replica, to the physics of elementary particles. It is therefore useful to study the group structure of all available exactly soluble models, especially the relativistic ones. The relativistic Coulomb problem has been investigated from this point of view in a number of papers.[7,8] Biedenharn [9] has proposed a relativistic symmetric Coulomb Hamiltonian which differs from the usual one by a term which makes a contribution to the fine structure. This model of Biedenharn is of interest in itself, since in its discrete spectrum the symmetry of the nonrelativistic Coulomb problem $O(4)$ [11,12] is established,[10] which is violated in the exact relativistic Coulomb problem. The solutions of the symmetric model differ by terms of the order $(e^2Z/\hbar c)^2/|k|$ from the approximate Coulomb functions of Sommerfeld and Maue [13] and are very useful for the calculation of concrete effects.[14] In [10] the symmetry group $O(4)$ was found only for the discrete spectrum, and its generators were realized in matrix form.

In the present paper we construct the dynamic group for the continuous spectrum of the symmetric relativistic Coulomb problem—the Biedenharn model. It turns out that the dynamic group is the direct product of groups $SL(2, c) \otimes SU_2$ and the infinite number of states with given energy $E \geq m$ realize a single irreducible representation of this group. We also give a classification of the states with the help of the other dynamic groups $SL_j(2, c)$. It turns out that the states with a given energy $E > m$ realize its completely reducible representation which can be decomposed into two irreducible representations with

$$m = 1, \quad \rho = -i + 2\alpha ZE(E^2 - m^2)^{-1/2}$$

and

$$m = -1, \qquad \rho = i + 2\alpha ZE(E^2 - m^2)^{-1/2}.$$

We also obtain the dynamic group connected with the quadratic Hamiltonian H of the Biedenharn model, which is isomorphic to $SL(2, c) \otimes SU(2, 2)$. The states with energy $E > m$ form a single infinite-dimensional representation of this group. A classification of the states is given with the help of the subgroups $SL_{j\pm}(2, c)$.

We consider the special case $E = m$ (for an attractive potential). The dynamic group is in this case isomorphic to $SL(2, c) \otimes SU_2$. The states with $E = m$ form an irreducible representation of this group. In this special case we obtain a not completely reducible representation of the group $SL_j(2, c)$, which comprises all states in the same way as in the case of the free Dirac equation.[8] We also investigate the nonrelativistic limit of the Biedenharn model—a Pauli electron in a Coulomb field.

## 1. THE BIEDENHARN MODEL

The behavior of an electron in an attractive Coulomb field is described by the Hamiltonian [15]

$$H_D = \boldsymbol{\alpha}\mathbf{p} + \beta m - \frac{\alpha Z}{r}, \tag{1}$$

where $\boldsymbol{\alpha}$, $\beta$ are the standard Dirac matrices, and $\alpha$ is the fine structure constant.

Biedenharn [9] has introduced the symmetric Hamiltonian

$$H_B = \boldsymbol{\alpha}\mathbf{p} + \beta m - \frac{\alpha Z}{r} + \rho_2 \frac{\boldsymbol{\sigma}\mathbf{r}}{r^2} K \left\{\left[1 + \left(\frac{\alpha Z}{K}\right)^2\right]^{1/2} - 1\right\}, \tag{2}$$

where $K = \beta(\sigma L + 1)$ is the Dirac operator.

The operator H differs from $H_B$ by the perturbation term $H_{fs}$, $H = H_B + H_{fs}$ responsible for the fine structure of the hydrogen spectrum. The states of the discrete spectrum of the operator $H_B$ are characterized by the principal quantum number N, have the energy $E = m[1 + (\alpha Z/N)^2]^{-1/2}$, and are $2N^2$-fold degenerate (cf. Biedenharn and Swamy [10]).

Following [10], we go over to a new representation with the help of the operator

$$S = \exp\left[-\frac{1}{2}\rho_2\sigma n \operatorname{arsh}\frac{\alpha Z K}{K^2}\right] \quad \mathbf{n} = \frac{\mathbf{r}}{r} \tag{3}$$

The operator $H_B$ then goes over into $H = SH_BS^{-1}$:

$$H = S^2(\alpha p - \beta m). \tag{4}$$

Using (4), we write the equation for the eigenfunctions of H in the form

$$O_+\psi_E \equiv (\alpha p + \beta m - ES^{-2})\psi_E = 0. \tag{5}$$

To determine $\psi_E$ it is useful to go over to the squared equation of second order with the help of the factorization

$$\psi_E = O_-\Phi_E \equiv (\alpha p + \beta m + ES^2)\Phi_E. \tag{6}$$

Substituting (6) in (5), we find that $\Phi_E$ satisfies the equation

$$Q\Phi_E \equiv \left[\Delta - \frac{2\alpha ZE}{r} + E^2 - m^2\right]\Phi_E = 0. \tag{7}$$

We note that a solution of (5) is automatically a solution of (7).

Equation (7) can be regarded as a Schrödinger equation with a potential which depends on the energy. In the nonrelativistic limit it goes over into the Schrödinger equation for the hydrogen atom. The solutions of (7) have the form [7]

$$\Phi_E = F_l(kr)Y_{l,m}(\theta,\varphi)u, \tag{8}$$

where

$$F_l(kr) = C_l(\eta)(kr)^l \exp(-ikr)F(l+1-i\eta, 2l+2, 2ikr);$$
$$k = (E^2 - m^2)^{1/2}, \quad \eta = \alpha ZE/k,$$
$$C_l(\eta) = 2^l e^{-\pi\eta/r}|\Gamma(l+1+i\eta)|/\Gamma(2l+2),$$

$Y_{lm}(\theta, \varphi)$ is the spherical harmonic, and u is a four-component spinor.

It is easy to see that the operators $H_B$ and H commute with the operator of the total angular momentum $\mathbf{j} = \mathbf{r}\times\mathbf{p} + \Sigma/2$ and the Dirac operator K. We can therefore construct stationary states which are eigenvectors of the operators $j^2$, $j_z$, and K with the eigenvalues j, $\mu$, and K, where $j = |K| - 1/2$:

$$H\psi_{EK\mu} = E\psi_{EK\mu}, \quad K\psi_{EK\mu} = K\psi_{EK\mu}, \quad j_z\psi_{EK\mu} = \mu\psi_{EK\mu}.$$

For completeness we give the expressions for these functions:

$$\psi_{EK\mu} = c\begin{pmatrix} c_1F_{l(-K)} & \chi_{-K}{}^{\mu} \\ F_{l(K)} & \chi_K{}^{\mu}\end{pmatrix}, \tag{9}$$

$$c_1{}^{-1} = [E(K^2 + \alpha^2Z^2) - mK][\alpha^2Z^2E^2 + K^2(E^2 - m^2)]^{1/2},$$

where $\chi_K^{\mu}$ is a two-component spherical spinor and is given by the usual formula;[15] $\chi_K^{\mu}$ is an eigenvector of the operators $\sigma L + 1$ and $j_z$; $F_{l(K)}(kr)$ is given by (8), with the index

$$l(K) = |K| + 1/2[\operatorname{sign} K - 1].$$

The constant c is determined by the normalization condition. Thus we have an infinite number of states with $j = 1/2, 3/2, \ldots$ for a given energy, and to each value j there correspond two states $\psi_{EK\mu}$ with $K = \pm(j + 1/2)$.

The transition to a new representation allowed Biedenharn [7] to obtain an additional integral of the motion--the operator $R = \sigma\cdot p - i\alpha Z\sigma\cdot n\beta KH/K^2$, which commutes with H and j. We note that the additional integral of the motion connected with the twofold degeneracy was first discovered in the exact Coulomb problem by Johnson and Lippmann.[16] The meaning of this operator $R_1$ as the analogue of the helicity operator $\sigma\cdot\rho$ of the free Dirac equation was established by Biedenharn.[7] The existence of the operator R becomes evident when we write the operator $O_\pm$ in the form of a $2\times 2$ matrix:

$$O_\pm = \begin{pmatrix} \mp E[1 + (\alpha Z/K)^2]^{1/2} + m & \sigma p - i\alpha Z(\sigma n)(\sigma L + 1)E/K^2 \\ \sigma p - i\alpha Z(\sigma n)(\sigma L + 1)E/K^2 & \pm E[1 + (\alpha Z/K)^2]^{1/2} - m \end{pmatrix} \tag{10}$$

In the case of the continuous spectrum $E > m$ we introduce the three normalization operators

$$X_1 = (E^2 - m^2 + \alpha^2Z^2E^2/K^2)^{-1/2}\left(\sigma p - i\alpha Z\frac{(\sigma n)\beta KH}{K^2}\right), \tag{11}$$
$$X_2 = (\alpha^2Z^2E^2 + K^2E^2 - m^2K^2)^{-1/2}(\sigma pK - i\alpha Z\sigma n\beta H)$$
$$X_3 = (K^2)^{-1/2}K, \qquad \mathbf{n} = \mathbf{r}/r.$$

It is easy to see that these three operators commute with H and form the chiral algebra $SU_2$ [8] with the commutation relations

$$[X_i, X_k] = 2i\varepsilon_{ikl}X_l, \quad \{X_i, X_k\} = 2\delta_{ik}. \tag{12}$$

It is seen from (11) and (12) that the states with $K = \pm(j + 1/2)$ form the spinor representation of this group $SU_2$.

## 2. DYNAMIC GROUP OF THE SQUARED EQUATION

By the dynamic symmetry of equation (7) we shall understand the algebra $\mathfrak{S}$ of operators $S_\alpha$ belonging to $\mathfrak{S}$ such that $[Q, S_\alpha] = 0$. It is clear that the operators $S_\alpha$, acting on any solution of (7), give again a solution.

It is evident that the operator of the orbital angular momentum $\mathbf{L} = \mathbf{r}\times\mathbf{p}$ belongs to the algebra $\mathfrak{S}$ since $[\mathbf{L}, Q] = 0$.

In analogy with the nonrelativistic case we introduce the Runge-Lenz operator [17]*

$$\tilde{\mathbf{A}} = 1/2([\mathbf{Lp}] - [\mathbf{pL}]) + \frac{\alpha ZE\mathbf{r}}{r},$$

which commutes with Q. It is easy to verify that the operators $L_i$ and $\tilde{A}_i$ satisfy the following commutation relations:

$$[L_i, L_k] = i\varepsilon_{ikl}L_l, \qquad [L_i, \tilde{A}_k] = i\varepsilon_{ikl}\tilde{A}_l$$
$$[\tilde{A}_i, \tilde{A}_k] = -i(E^2 - m^2 - Q)\varepsilon_{ikl}L_l. \tag{13}$$

In the space $H_E$ of the solutions of (7) with fixed energy $E > m$ we define the normalized operators $A_i = (E^2 - m^2 - Q)^{-1/2}\tilde{A}_i$, which satisfy the commutation relations

$$[L_iL_k] = i\varepsilon_{ikl}L_l, \qquad [L_iA_k] = i\varepsilon_{ikl}A_l,$$
$$[A_iA_k] = -i\varepsilon_{ikl}A_l. \tag{13'}$$

The relations (13′) show that $L_i$, $A_i$ form a Lie algebra of operators which is isomorphic to the Lie algebra of the group SL(2, c). In the following we shall call the Lie algebra by the same name as the corresponding group, for brevity.

*$[\mathbf{Lp}] \equiv \mathbf{L}\times\mathbf{p}$.

Let us calculate the Casimir operators $C_1$ and $C_2$:

$$C_1 = (L_i + iA_i)^2 = -1 - \alpha^2 Z^2 E^2 / (E^2 - m^2),$$
$$C_2 = (L_i - iA_i)^2 = -1 - \alpha^2 Z^2 E^2 / (E^2 - m^2). \qquad (14)$$

Thus the infinite number of states (8) with energy $E > m$ and $L = 0, 1, 2, \ldots$ form a single infinite-dimensional representation of the group SL(2, c) with $m = 0$, $\rho = 2\alpha ZE(E^2 - m^2)^{-1/2}$, $2\alpha Z < \rho < \infty$.

The operators $C_1$ and $C_2$ are expressed through m and $\rho$ in the following way:

$$C_1 = \left(\frac{m}{2} - 1 - \frac{i\rho}{2}\right)\left(\frac{m}{2} + 1 - \frac{i\rho}{2}\right),$$
$$C_2 = \left(\frac{m}{2} - 1 + \frac{i\rho}{2}\right)\left(\frac{m}{2} + 1 + \frac{i\rho}{2}\right).$$

The mathematical theory of the representations of the group SL(2, c) is presented in the monograph of Naimark.[18]

It is evident that the Dirac matrices $\gamma_i$ commute with the operator Q. Thus the 15 Dirac matrices forming the Lie algebra SU(2, 2)[19,20] are contained in the algebra S. Since the spin variables in (7) are separated from the space variables, the full symmetry algebra $\mathfrak{S}$ is clearly the direct product SL(2, c) $\otimes$ SU(2, 2). We note that the four-component spinors are an irreducible representation of the group SU(2, 2) with the largest weight (1, 0, 0).

Thus we see that the states with a given energy E form a single irreducible representation of the group SL(2, c) $\oplus$ SU(2, 2) which is a tensor product of the representation of the group SL(2, c) with $m = 0$, $\rho = 2\alpha ZE(E^2 - m^2)^{-1/2}$ and the four-dimensional representation SU(2, 2).

It is of interest to study a certain symmetry algebra N one of whose generators is the total angular momentum operator $j_i$. The algebra $\mathfrak{N}$ is the direct sum of the subalgebras $SL_{j\pm}(2, c)$ determined by the generators

$$j_i^\pm = {}^1\!/_2(1 \pm \beta)(L_i + {}^1\!/_2\sigma_i), \qquad K_i^\pm = {}^1\!/_2(1 \pm \beta)(A_i + {}^1\!/_2\sigma_i) \quad (15)$$

(where $L_i$, $A_i$ were defined above, and $\sigma_i$ are the Pauli matrices) with the commutation relations

$$[j_i^\pm, j_k^\pm] = i\varepsilon_{ikl} j_l^\pm, \quad [j_i^\pm, K_j^\pm] = i\varepsilon_{ijl} K_l^\pm, \quad [K_i^\pm, K_j^\pm] = -i\varepsilon_{ijl} j_l^\pm. \ (16)$$

In (16) either all the upper or all the lower signs are to be taken. Operators with different signs belonging to $SL_{j\pm}(2, c)$ and $SL_{j-}(2, c)$, respectively, commute with each other.

In order to find out which representations of the algebra are realized on the solutions of (7) with a given energy, we must calculate the Casimir operators of the algebras $SL_{j\pm}(2, c)$ and determine their eigenvalues and the corresponding invariant spaces.

Using (15) and (16), we obtain the Casimir operators

$$C_1^\pm = (j_i^\pm + iK_i^\pm)^2 = {}^1\!/_2(1 \pm \beta)(-1 - \alpha^2 Z^2 E^2 / (E^2 - m^2)),$$
$$C_2^\pm = (j_i^\pm - iK_i^\pm)^2 = {}^1\!/_2(1 \pm \beta)(-2iN - \alpha^2 Z^2 E^2 / (E^2 - m^2)), \quad (17)$$

where $N = i(\sigma \cdot L - i\sigma \cdot A + 1)$. In order to find the eigenvalues of the operators $C_i^\pm$, $i = 1, 2$ and the corresponding invariant subspaces, it suffices, by virtue of (17), to find them for the operators N and $\beta$.

It is easy to verify that $N^2 = \alpha^2 Z^2 E^2/(E^2 - m^2)$. Thus the eigenvalues of the operator N are equal to $\pm\alpha ZE(E^2 - m^2)^{-1/2}$. The projection operator on the invariant space corresponding to the value $|N|$ is equal to $P_+ = (N + |N|)/2|N|$, and that corresponding to $-|N|$ is equal to $P_- = (N - |N|)/2|N|$. Thus the whole space $H_E$ of states with a given energy $E > m$ [of which the solutions (8) form the basis] is decomposed into the direct sum of four invariant spaces $H^\pm_{\pm|N|}$

$$H = H^+_{|N|} \oplus H^+_{-|N|} \oplus H^-_{|N|} \oplus H^-_{-|N|},$$

the projection operators on the corresponding spaces are given by

$$P_\pm^\pm = (1 \pm \beta)(N \pm |N|)/4|N|.$$

In the table we give the values of the Casimir operators of the algebra $\mathfrak{N}$ in the invariant spaces $H^\pm_{\pm|N|}$ [$|N| = \alpha ZE(E^2 - m^2)^{-1/2}$].

Thus, in $H^+_{|N|}$ we have the infinite-dimensional representation $SL_{j+}(2, c)$ with $m = -1$, $\rho = i + 2\alpha ZE(E^2 - m^2)^{-1/2}$ and in $H^+_{-|N|}$ we have that with $m = 1$, $\rho = -i + 2\alpha ZE(E^2 - m^2)^{-1/2}$. In the spaces $H^-_{\pm|N|}$ the operators (16) of the algebra $SL_{j+}(2, c)$ give zero. Analogously, $SL_{j-}(2, c)$ gives zero in the spaces $H^+_{\pm|N|}$; in $H^-_{|N|}$ we have the representation with $m = -1$, $\rho = -i + 2\alpha ZE(E^2 - m^2)^{-1/2}$, and in $H^-_{-|N|}$ we have that with $m = 1$, $\rho = i + 2\alpha ZE(E^2 - m^2)^{-1/2}$.

## 3. DYNAMIC GROUP OF THE HAMILTONIAN H

Any solution of (7) is projected by the operator $O_-$ of (6) into a solution of (5). If the operator $O_-^{-1}$, the inverse of $O_-$ existed, then one could establish the symmetry group of (5) from the symmetry group $\mathfrak{S}$ of (7). Let us define its operators by the formula $S^1_\alpha = O_- S_\alpha O_-^{-1}$, where $S_\alpha$ belongs to $\mathfrak{S}$. However, there exist functions $\Phi_{EK\mu}$ which the operator $O_-$ transforms to zero. This means that there exists no operator which is the inverse of $O_-$.

We now extend our concept of symmetry, as was done, for example, in [6]. The symmetry of (5) will be understood to be given by the algebra of operators $S^1_\alpha$ such that $[S'_\alpha, O]\psi_{EK\mu} = 0$. Thus the basic property of the operators $S'_\alpha$ is preserved: any solution $\psi_E$ of (5) is again transformed into a solution by the operators $S'_\alpha$.

Let us consider functions $\Phi^-_{EK\mu}$ of the form $\Phi^-_{EK\mu} = (0, \varphi_{EK\mu})$, where $\varphi_{EK\mu}$ is a two-component spinor which satisfies (7) and is an eigenvector of the operators $(\sigma \cdot L + 1)$ and $L_z + \sigma_z/2$:

$$(\sigma L + 1)\varphi_{EK\mu} = K\varphi_{EK\mu}, \qquad (L_z + {}^1\!/_2\sigma_z)\varphi_{EK\mu} = \mu\varphi_{EK\mu}.$$

From the explicit form (10) of the operator O it is clear that $\psi_E = O\Phi^- \neq 0$.

We note that one obtains a unique function $\Phi^-_{EK\mu}$ from $\psi_{EK\mu}$. Indeed, since the function $\psi_{EK\mu}$ itself satisfies (7), we have $\Phi^-_{EK\mu} = (1/2)(1 - \beta)\psi_{EK\mu}$.

Let us define the operators $L'_i$, $A'_i$, and $\sigma'_i$ by

$$L_i' = \begin{pmatrix} RL_iR^{-1} & 0 \\ 0 & CL_iC^{-1} \end{pmatrix}, \quad A_i' = \begin{pmatrix} RA_iR^{-1} & 0 \\ 0 & CA_iC^{-1} \end{pmatrix},$$
$$\sigma_i' = \begin{pmatrix} R\sigma_iR^{-1} & 0 \\ 0 & C\sigma_iC^{-1} \end{pmatrix} \qquad (18)$$

where

$$R = \sigma p + i(\sigma n)(\sigma L + 1)E/K^2, \quad C = E(1 + \alpha^2 Z^2/K^2)^{1/2} - m,$$
$$R^{-1} = [E^2 - m^2 + \alpha^2 Z^2 E^2/K^2]^{-1} R,$$
$$C^{-1} = \left[E^2 - m^2 + \frac{\alpha^2 Z^2 E^2}{K^2}\right]^{-1}\left[E\left(1 + \frac{\alpha^2 Z^2}{K^2}\right)^{1/2} + m\right].$$

| $H^+_{\|N\|}$ | $H^+_{-\|N\|}$ | $H^-_{\|N\|}$ | $H^-_{-\|N\|}$ |
|---|---|---|---|
| $C_1^- = C_2^- = 0$<br>$C_1^+ = -1 - N^2$<br>$C_2^+ = -2i\|N\| - N^2$ | $C_1^- = C_2^- = 0$<br>$C_1^+ = -1 + N^2$<br>$C_2^+ = 2i\|N\| - N^2$ | $C_1^+ = C_2^+ = 0$<br>$C_1^- = 1 + N^2$<br>$C_2^- = 2i\|N\| + N^2$ | $C_1^+ = C_2^+ = 0$<br>$C_1^- = 1 + N^2$<br>$C_2^- = -2i\|N\| + N^2$ |

The operators $L_i$ and $A_i$ are defined above, and $\sigma_i$ are the Pauli matrices. The operators (18) satisfy the same commutation relations (13′) as the operators $A_i$,

Let us verify that any of the operators $L_i'$, $A_i'$, and $\sigma_i'$ applied to a solution $\psi EK\mu$ of (5) leads again to a solution. Using (8), (6), and (10), we find

$$\tilde{\psi}_E = L_i'\psi_{EK\mu} = L_i'O_-\Phi^-_{EK\mu} \tag{19}$$

$$= \begin{pmatrix} RL_i & \varphi_{EK\mu} \\ CL_i & \varphi_{EK\mu} \end{pmatrix} = O_- L_i \Phi^-_{EK\mu}.$$

Applying now the operator $O_+$ to $\tilde{\psi}_E$, we find, owing to (10), $O_+\tilde{\psi}_E = 0$. An analogous proof holds for the operators $A_i'$ and $\sigma_i'$. Thus we find that the operators (18) have the property

$$[L_i'O_+]\psi_E = 0, \quad [A_i'O_+]\psi_E = 0, \quad [\sigma_i'O_+]\psi_E = 0$$

and hence, form the dynamic symmetry group of (5).

We note that (5) goes over into the free Dirac equation for $\alpha \to 0$, and the operators (18) go over into the operators constructed in [8].

The dynamic group with the generators $L_i'$, $A_i'$, and $\sigma_i'$ is isomorphic to the direct product $SL(2, c) \otimes SU_2$. The Casimir operators of this group commute with $O_+$ identically, not only when applied to solutions. Calculating the Casimir operators (14) by substituting in them $L_i$ and $A_i$ for $L_i'$ and $A_i'$, we find $C_1 = -1 - \alpha^2Z^2E^2/(E^2 - m^2)$, $C_2 = -1 - \alpha^2Z^2E^2/(E^2 - m^2)$ and $\sigma_i'^2 = 3$.

The states with a given energy $E > m$ form a single infinite-dimensional representation of the group $SL(2, c) \otimes SU(2)$. This representation is the tensor product of the representation $SL(2, c)$ with $m = 0$, $\rho = 2\alpha ZE(E^2 - m^2)^{-1/2}$ and the spinor representation of the group $SU_2$.

Let us consider the algebra $SL_j(2, c)$, where the total angular momentum operator $j_i$ is taken as one of the generators. It follows from (18) that $j_i = L_i' + \sigma_i'/2$, since $j_i$ commutes with R and C. Let us define the operator $K_i' = L_i' + i\sigma_i'/2$. These operators are the generators of the algebra $SL_j(2, c)$ with the commutation relations (13′). We calculate the Casimir operators

$$C_1 = (j_i + iK_i')^2 = -1 - \alpha^2Z^2E^2/(E^2 - m^2),$$

$$C_2 = (j_i - iK_i')^2 = -2i\tilde{N} - \alpha^2Z^2E^2/(E^2 - m^2),$$

where

$$\tilde{N} = -i[1 + (E^2 - m^2)^{-1/2}R]K.$$

We obtain for $\tilde{N}'^2$, taking into account of the fact that R and K commute,

$$\tilde{N}'^2 = \alpha^2Z^2E^2/(E^2 - m^2).$$

We introduce the projection operators $P_+$ and $P_-$ on the invariant spaces $H_\pm$ with $\tilde{N} = |\tilde{N}|$ and $\tilde{N} = -|\tilde{N}|$, correspondingly:

$$P_\pm(\tilde{N} \pm |\tilde{N}|)/2|\tilde{N}|.$$

The whole space of states with energy E is decomposed into two spaces $H_+ \otimes H_-$ in each of which we have an irreducible representation of the group $SL_j(2, c)$. In $H_+$ we have the representation with $m = -1$, $\rho = i + 2\alpha ZE(E^2 - m^2)^{-1/2}$, in $H_-$ we have that with $m = 1$, $\rho = -i + 2\alpha ZE(E^2 - m^2)^{-1/2}$.

## 4. SYMMETRY OF THE "FREE" MOTION

A special case is the point of the spectrum $E = m$ for an attractive potential. Equation (7) with $E = m$ goes over into the nonrelativistic equation for the hydrogen atom with $E = 0$. Because of (13), the operators $\tilde{A}_i$ commute with each other on the solutions $\Phi_E$ with $E = m$. The group $SL(2, c)$ degenerates into a group which is isomorphic to the group of motions in three-dimensional space. Degeneracies of this type have been considered by Iononü and Wigner.[21]

In this case one can, however, construct a dynamic group which is isomorphic to $SL(2, c)$. The generators of this group $\mathbf{L} = \mathbf{r} \times \mathbf{p}$ and $\mathbf{M} = (1/2)\alpha Zm(\mathbf{L} \times \mathbf{A} - \mathbf{A} \times \mathbf{L})$ satisfy the commutation relations (13′). The Casimir operators are $C_1 = -1$ and $C_2 = -1$. Thus we find that the solutions with $E = m$, $L = 0, 1, 2, \ldots$ form a single infinite-dimensional representation of $SL(2, c)$ with $m = \rho = 0$.

The full symmetry group is evidently the direct product $SL(2, c) \otimes SU(2, 2)$, where $SU(2, 2)$ is the algebra of the 15 Dirac matrices. The states with $E = m$ transform like the tensor product of the representation $SL(2, c)$ with $m = \rho = 0$ and the representation of $SU(2, 2)$ with the largest weight $(1, 0, 0)$.

Let us now consider the algebra $\mathfrak{N} = SL_{j^+}(2, c) + SL_{j^-}(2, c)$ with the generators

$$j_i^\pm = 1/2(1 \pm \beta)(L_i + 1/2\sigma_i), \quad K_i^\pm = 1/2(1 \pm \beta)(L_i + 1/2 i\sigma_i).$$

The Casimir operators have the form

$$C_1^\pm = 1/2(1 \pm \beta), \quad C_2^\pm = -(1 \pm \beta)iN,$$

where $N = i(\sigma \cdot \mathbf{L} - i\sigma \cdot \mathbf{M} + 1)$. Calculating the square of N, taking account of the fact that $\mathbf{M} \cdot \sigma$ and $\sigma \cdot \mathbf{L} + 1$ anticommute, we obtain $N^2 = 0$. Hence the operator N cannot be brought into diagonal form but can only be written in the form

$$\begin{pmatrix} 0 & N_{12} \\ 0 & 0 \end{pmatrix}.$$

Thus the representation of $SL_{j^+}(2, c)$ acting in the space $H^+$ of states with $\beta = +1$ is not completely reducible. Let us separate from $H^+$ the subspace of states $H_0^+$ on which N gives zero. In $H_0^+$ an irreducible representation of the group $SL_{j^+}(2, c)$ with $m = 1$, $\rho = -i$ is realized; in the factor space $H^+/H_0^+$ we have an equivalent representation with $m = 1$, $\rho = -i$. We note that an analogous result is obtained for the free Dirac equation in [8]. The theory of not completely reducible representations of the group $SL(2, c)$ has been worked out by Zhelobenko.[24]

Using (18), where $A_i$ is replaced by the operator $M_i$, we obtain the set of operators $L_i'$, $M_i'$, and $\sigma_i'$ which commute with the operator $O_+$. These operators define the dynamic symmetry group of (5) in the case $E = m$. Calculating the Casimir operators of this group isomorphic to $SL(2, c) \otimes SU_2$, we find that the states with $E = m$ transform like the tensor product of the infinite-dimensional representation of the group $SL(2, c)$ with $m = \rho = 0$ and the spinor representation of $SU_2$.

Let us consider the algebra $SL_j(2, c)$ with the generators $j_i = L_i' + \sigma_i'/2$, $K_i' = M_i' + i\sigma_i'/2$. The Casimir operators are

$$C_1{}^+ = (j_i + iK_i')^2 = -1, \quad C_2 = (j_i - iK_i')^2 = -2iN',$$

where $N' = -i(1 - X_2)K$. It follows easily from (11) with $E = m$ that $N'^2 = 0$. This means that the states $E = m$ transform according to a not completely reducible representation of the group $SL_j(2, c)$. In the invariant space of states $H_0$ where $N$ is equal to zero, we have the representation with $m = 1$, $\rho = -1$; in the invariant factor space we have an equivalent representation.

The existence of not completely reducible representations of the simple group $SL(2, c)$ in the present problem is not entirely accidental, since these describe the state of free motion of a particle with spin $1/2$.[8]

## 5. NONRELATIVISTIC PAULI EQUATION

Let us go to the nonrelativistic limit in (7) by replacing $E$ by $m$ and $E^2 - m^2$ by $2mE'$ and regarding the spinors as two-component objects. Then (7) goes over into the usual Pauli equation for a particle with spin $1/2$ and energy $E'$ in a Coulomb field. The operators $\tilde{A}_i$ go over into the usual Runge-Lenz vector.[17]

Changing the energy in (13), (13′), and (14) according to the above-given rule, we find that the dynamic symmetry group is the group $SL(2, c) \otimes SU_2$ and the states with given energy $E'$ transform like the tensor product of the infinite-dimensional unitary representation $\rho = \sqrt{2}\alpha Z E^{-1/2}$, $m = 0$ of the group $SL(2, c)$ and the two-dimensional representation of $SU(2)$; this agrees with the result obtained earlier in [22,23]. We note that we have $0 < \rho < \infty$ in the nonrelativistic case, whereas $2\alpha Z < \rho < \infty$ in the relativistic case.

If we use the algebra $SL_{j+}(2, c)$ of the operators $j_i^+$ and $K_i^+$ [cf. (16)] we find that the states with energy $E$ belong to two infinite-dimensional nonunitary representations of the group $SL_{j+}(2, c)$ with $m = -1$, $\rho = i + 2\alpha Z E(E^2 - m^2)^{-1/2}$ and $m = 1$, $\rho = -i + 2\alpha Z E(E^2 - m^2)^{-1/2}$.

We note that the special case $E' = 0$ in an attractive potential corresponds exactly to the case $E = m$ discussed above. Thus the states with $E' = 0$ transform according to an irreducible representation of the group $SL(2, c) \otimes SU_2$ with the generators **L**, **M**, and $\sigma$ which is a tensor product of the representation $SL(2, c)$ with $m = \rho = 0$ and a spinor representation of $SU_2$. With the choice of the algebra $SL_{j+}(2, c)$ with the generators $j_iK_j$ we find that the states with $E' = 0$ form a not completely reducible representation with the Casimir operators $C_1 = -1$, $C_2 \neq 0$ but $C_2^2 = 0$.

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