STOCHASTIC INSTABILITY OF TRAPPED PARTICLES AND CONDITIONS OF APPLICABILITY OF THE QUASI-LINEAR APPROXIMATION

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We consider the motion of particles in the field of a plane wave under the influence of a perturbation. We show that near the separatrix dividing the "trapped" from the "free-flying" particles a region (ergodic layer) is formed in which the particle motion is stochastic in character; this leads for not too large times to an increase in the average energy of the trapped particles. We use this instability to study the motion of particles in a wave packet with a large discrete set of harmonics. We obtain the conditions under which the motion can be described by a Fokker–Planck type equation (quasi-linear equation) for any initial conditions with a priori assumptions about the presence of some random parameters in the system. We calculate the time for randomization of the phases of the particles.

INTRODUCTION

THE problem of studying the behavior of particles trapped in some potential well is of practical interest for us in connection with the following two plasma physics problems. The first one is connected with the evolution of an instability for trapped particles in toroidal systems, leading to an anomalous diffusion of the plasma.[1, 2] The second is connected with the study of the conditions under which it is possible to construct a quasi-linear plasma theory. The aim of the present paper is a detailed study of a specific non-linear instability of trapped particles and the solution of the second of the above-mentioned problems. We shall show that small perturbations lead to a randomization of the motion of trapped and free-flying particles which have an energy close to the edges of the potential well. In Sec. 1 of the paper we study the conditions that such a type of stochastic instability appears and we estimate the extent of the region (called in the following the ergodic layer) in which the instability develops, as a function of the parameters of the external perturbation. The very fact that a random acceleration of trapped particles occurs, and that it is possible that they leave the well, leads apparently to the necessity for looking in another way at problems where the plasma-containing field geometry is such that trapped particles may appear during a long time (see, e.g., [3]). The method of investigation used in Sec. 1 is similar to the one applied in [4], in which we studied the random decay of magnetic surfaces near a separatrix. This is connected with the common formulation of the two problems in the Hamiltonian formalism.

In Sec. 2 we study the motion of a particle in the field of a wave packet with a large discrete number of traveling waves. The particle may be trapped over some period by the field of the wave moving with a phase velocity close to the particle velocity. The presence of the stochastic instability described in Sec. 1 leads to the appearance of a limitation on the maximum time of particle trapping, depending on its velocity, and to the possibility of obtaining a criterion that the motion of a particle can be described by a Fokker–Planck type equation (quasi-linear equation[5]). Moreover, we also find in Sec. 2 the time after which the phase of the particle is randomized.

In Sec. 3 we consider the motion of a particle inside the ergodic layer. It turns out that in some part of the ergodic layer the kinetic equation describing the behavior of the particle has the form of a Fokker–Planck equation, and this enables us to find the stationary distribution inside the ergodic layer and the time necessary to establish the stationary distribution.

1. CONDITIONS FOR THE ESTABLISHMENT OF A STOCHASTIC INSTABILITY

Let us consider the one-dimensional motion of a charged particle in the field of a traveling wave under the influence of a perturbation which is also a traveling wave:

$$m\ddot{x} = -eE_0 \sin(k_0x - \omega_0 t) - eE_1 \sin(k_1 x - \omega_1 t),$$

where the index 0 refers to the unperturbed force and the index 1 to the perturbation, which is assumed to be small:

$$\varepsilon = E_1 / E_0 \ll 1.$$

It is convenient to change to a reference frame moving with the phase velocity of the unperturbed wave ω_0/k_0, and to introduce the variables

$$\xi = k_0 x - \omega_0 t, \quad V = \tau_0 (k_0 v - \omega_0), \quad \tau_0 = \sqrt{m / eE_0 k_0} \tag{1.1}$$

The equation of motion of the particle then becomes

$$\tau_0 \frac{dV}{dt} = -\sin\xi - \varepsilon \sin\left(\frac{k_1}{k}\xi + k_1 \Delta V_{ph} t\right), \tag{1.2}$$

where

$$\Delta V_{ph} = \frac{\omega_0}{k_0} - \frac{\omega_1}{k_1}. \tag{1.3}$$

The unperturbed motion ($\varepsilon = 0$) is a motion in a po-

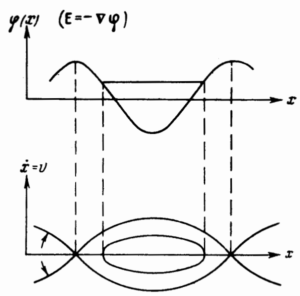

FIG. 1.

tential that depends periodically on the coordinate (Fig. 1, upper part), and finite trajectories are separated in the phase plane from the infinite ones by a separatrix (indicated by the arrows in the lower part of Fig. 1). We shall be interested, in first instance, in the effect of the perturbation on the motion of trapped particles which correspond to finite trajectories inside the separatrix in the phase plane. We shall show below that an investigation of the "free-flying" particles (with infinite unperturbed trajectories in the phase plane) proceeds analogously.

Using the Hamiltonian of the unperturbed motion

$$H = {}^1\!/_2 V^2 - \cos\xi,$$

we introduce the action (I) and angle (ϑ) variables in the usual way:

$$I = I(H) = \frac{1}{2\pi}\oint V\,d\xi = \frac{8}{\pi}\left\{E\left(\frac{\pi}{2}, q\right) - (1-q^2)F\left(\frac{\pi}{2}, q\right)\right\}, \quad (1.4)$$

$$\vartheta(\xi, I) = \frac{\partial S(\xi, I)}{\partial I}, \quad S(\xi, I) = \frac{1}{2\pi}\int^{\xi} V\,d\xi,$$

where

$$q^2 = {}^1\!/_2(1 + H),$$

and F and E are first and second kind elliptic integrals, respectively.

The equation of motion (1.2) has for $\varepsilon = 0$ in the new variables the standard form

$$dI/dt = 0, \quad d\vartheta/dt = \Omega(I),$$

where the frequency Ω is determined by the expression

$$\tau_0\Omega = \frac{dH(I)}{dI} = \frac{\pi}{2F(\pi/2, q)}. \quad (1.5)$$

When there is a perturbation we have

$$\frac{dI}{dt} = \frac{dI}{dH}\left\{\frac{\partial H}{\partial V}\frac{dV}{dt} + \frac{\partial H}{\partial \xi}\frac{d\xi}{dt}\right\} = -\frac{\varepsilon}{\Omega\tau_0^2}V\sin\left(\frac{k_1}{k_0}\xi + k_1\Delta V_{\rm ph}\right), \quad (1.6)$$

where ξ in the argument of the sine is assumed to be expressed as a function of I and ϑ. The change in the phase ϑ under the influence of the perturbation will be studied below.

We have shown earlier[4, 5] that near the separatrix a stochastic kind of particle instability, leading to the departure of particles from the potential well, may occur. In the following we shall study the problem of the stability of the particles near the separatrix in the example considered, using the method of [4].

Let us consider the spectrum of the unperturbed particle velocity V(t). When $\varepsilon = 0$ the solution of (1.2) is

$$F(q, \eta) = F(q_0, \eta_0) + t/\tau_0,$$

where

$$\sin(\xi/2) = q\sin\eta$$

while q_0 and η_0 are initial values. Hence

$$V = 2q\cos\xi = 2q\,\mathrm{cn}\,[F(q_0, \eta_0) + t/\tau_0]. \quad (1.7)$$

Omitting, for the sake of simplicity, the initial conditions we have the following Fourier series expansion of the velocity:

$$V = 2q\,\mathrm{cn}\frac{t}{\tau_0} = \frac{4\pi}{F}\sum_{n=1}^{\infty}\frac{a^{n-1/2}}{1+a^{2n-1}}\cos\left[(2n-1)\frac{\pi t}{2\tau_0 F}\right]$$

$$= 8\Omega\tau_0\sum_{n=1}^{\infty}\frac{a^{n-1/2}}{1+a^{2n-1}}\cos[(2n-1)\Omega t],$$

$$a = \exp(-\pi F'/F),$$

$$F \equiv F(\pi/2, q), \quad F' \equiv F(\pi/2, \sqrt{1-q^2}). \quad (1.8)$$

Near the separatrix $H \to 1$, $q \to 1$, $\Omega \to 0$, and

$$F \approx \frac{1}{2}\ln\frac{32}{1-H}, \quad \Omega\tau_0 \approx \pi\Big/\ln\frac{32}{1-H}, \quad F' \approx \pi/2,$$

$$a \approx \exp\left(-\pi^2\Big/\ln\frac{32}{1-H}\right) = \exp(-\pi\Omega\tau_0). \quad (1.9)$$

It follows from (1.8) and (1.9) that the characteristic width of the spectrum of V(t) is determined by the number n_0 of the Fourier harmonic which is equal to

$$n_0 \sim \frac{\pi}{|\ln a|} = \frac{1}{\Omega\tau_0} \gg 1. \quad (1.10)$$

This means that for particles moving near the separatrix V(t) has an almost constant value $V_0 \sim 2\pi/F = 2$, which is localized in a very narrow range of time of the order of τ_0, and which recurs periodically with a frequency Ω, with $\Omega\tau_0 \ll 1$. This fact is extremely important, since the sharpness of the momenta on the right-hand side of (1.6), connected with the described behavior of V(t), may lead to an appreciable change in the action I of the particle.

Let us now consider the right-hand side of Eq. (1.6) and separate in it the resonance terms. Using the expansion (1.8) for V(t) one can easily write down the resonance condition

$$\Omega(I_s) \approx k_1\Delta V_{\rm ph}/s, \quad (1.11)$$

where s is some integer ($s \gg 1$) which we shall call the number of the resonance, I_s the value of the action corresponding to the resonance frequency. It follows from (1.11) that the distance between two neighboring resonances with numbers s and s + 1 is equal to

$$\delta = \Omega(I_s) - \Omega(I_{s+1}) \approx \frac{k_1\Delta V_{\rm ph}}{s^2} = \frac{\Omega^2(I_s)}{k_1\Delta V_{\rm ph}}. \quad (1.12)$$

Each s-th resonance is enclosed by a local separatrix of dimension ΔI_s and the frequency width of the resonance is equal to $(d\Omega(I_s)/dI)\Delta I_s$. The condition for the appearance of a stochastic instability of a particle is the inequality[7, 8]

$$\left|\frac{d\Omega(I_s)}{dI}\right|\Delta I_s \gg \delta = \frac{\Omega^2(I_s)}{k_1\Delta V_{\rm ph}}. \quad (1.13)$$

In that case the spectrum of the particle motion becomes very close to a continuous one and the motion of the particle is similar to Brownian motion. This leads to the fact that the particle energy increases on the average with time.

To determine ΔI_s we integrate (1.6) over a time interval $\gtrsim \tau_0$ in the vicinity of the s-th resonance. Bearing in mind that at sufficiently low frequencies (i.e., large s) the correction to the frequency $\Omega(I_s)$ satisfies both (1.13) and the inequality

$$\left|\frac{d\Omega(I_s)}{dI}\right|\Delta I_s \ll \Omega(I_s), \tag{1.14}$$

we get from (1.6) and (1.8)

$$\Delta I_s \approx \frac{\varepsilon}{\tau_0}\frac{a^{s-1/2}}{1+a^{2s-1}}\left[s\left|\frac{d\Omega(I_s)}{dI}\right|\Delta I_s\right]^{-1}. \tag{1.15}$$

Hence we find after substituting s from (1.11)

$$\Delta I_s \approx \left[\frac{4\varepsilon a^{s-1/2}/(1+a^{2s-1})}{|d\Omega(I_s)/dI|\tau_0}\frac{\Omega(I_s)}{k_1\Delta V_{ph}}\right]^{1/2}. \tag{1.16}$$

When the number s of the resonance is less than the limit of the spectrum n_0 (see (1.10)) we get $a^s \sim 1$ and the instability condition (1.13) becomes

$$K = 2\varepsilon\frac{k_1\Delta V_{ph}}{\tau_0\Omega^3}\left|\frac{d\Omega}{dI}\right| \gg 1. \tag{1.17}$$

Using (1.19) to evaluate $|d\Omega/dI|$ near the separatrix,

$$\left|\frac{d\Omega}{dI}\right| = \left|\frac{d\Omega}{dH}\right|\frac{dH}{dI} = \frac{\tau_0^2\Omega^3}{32\pi}\exp\left(\frac{\pi}{\Omega\tau_0}\right),$$

we can obtain from the condition $K \gtrsim 1$ a limiting value for the frequency Ω_0 starting from which the instability develops:

$$\Omega_0\tau_0 \lesssim \pi \Big/ \ln\frac{16\pi}{\varepsilon\tau_0 k_1\Delta V_{ph}}. \tag{1.18}$$

In the case when the detuning of the phase velocities of the main wave and of the perturbation is comparable with the average vibrational frequency of the particle τ_0^{-1}, i.e.,

$$\left|\frac{\omega_0}{k_0} - \frac{\omega_1}{k_1}\right| \sim \frac{1}{k_1\tau_0}, \quad k_1\Delta V_{ph}\tau_0 \sim 1, \tag{1.19}$$

the limit of instability can be found from the condition

$$\Omega_0\tau_0 \lesssim \pi \Big/ \ln\frac{16\pi}{\varepsilon}, \tag{1.20}$$

which according to (1.9) corresponds to a limit in energy H_0:

$$1 - H_0 \lesssim 2\varepsilon/\pi. \tag{1.21}$$

If, however, $\tau_0 k_1 \Delta V_{ph} \ll 1$, i.e.,

$$\left|\frac{\omega_0}{k_0} - \frac{\omega_1}{k_1}\right| \ll \frac{1}{k_1\tau_0} \tag{1.22}$$

the frequency limit of instability Ω_0 is determined by Eq. (1.18) and the energy limit by the expression

$$1 - H_0 \lesssim \varepsilon k_1 \Delta V_{ph}\tau_0.$$

Under condition (1.22) the instability region is thus compressed compared to (1.19) and tends to zero as $k_1\Delta V_{ph} \to 0$. It is also interesting to note that in the most unstable case (1.19) the main contribution to ΔI_s comes from the harmonic in the expansion of V(t), which lies on the edge of the spectrum and has $s \sim n_0$, while in the case (1.22) the contribution to ΔI_s is determined by the "interior" harmonics.

Let us consider now the case of a very strong detuning of the phase velocities of the main and the perturbing oscillations:

$$\left|\frac{\omega_0}{k_0} - \frac{\omega_1}{k_1}\right| \gg \frac{1}{k_1\tau_0}, \quad \tau_0 k_1\Delta V_{ph} \gg 1. \tag{1.23}$$

The harmonic with $s \gg n_0$ which lies in the "tail" of the spectrum gives the main contribution to the integral for ΔI_s. In this case it follows from (1.9), (1.11), and (1.16) that

$$a^s = \exp(-\pi\tau_0 k_1\Delta V_{ph}),$$

$$\Delta I_s \approx \left[4\varepsilon\frac{\Omega(I_s)}{\tau_0 k_1\Delta V_{ph}}\frac{\exp(-\pi\tau_0 k_1\Delta V_{ph})}{|d\Omega(I_s)/dI|}\right]^{1/2}$$

The limit of instability is found from the condition

$$K = 4\varepsilon\frac{k_1\Delta V_{ph}}{\tau_0\Omega^3}\left|\frac{d\Omega}{dI}\right|\exp(-\pi\tau_0 k_1\Delta V_{ph}) = \varepsilon\frac{k_1\Delta V_{ph}\tau_0}{8\pi}\exp\left(-\pi\tau_0 k_1\Delta V_{ph} + \frac{\pi}{\Omega\tau_0}\right) \gtrsim 1. \tag{1.24}$$

The case (1.23) is certainly the most stable one, since the change in the action ΔI_s is exponentially small. From (1.24) we find for the frequency limit of instability

$$\Omega_0\tau_0 \lesssim \pi\left[\ln\frac{8\pi}{\varepsilon\tau_0 k_1\Delta V_{ph}} + k_1\Delta V_{ph}\tau_0\right]^{-1}.$$

In the case when

$$\ln\varepsilon^{-1} \gg \tau_0 k_1\Delta V_{ph},$$

we have the result (1.20) and for the opposite inequality

$$\Omega_0\tau_0 \lesssim \frac{\pi}{k_1\Delta V_{ph}\tau_0} \ll 1. \tag{1.25}$$

The instability region in the energy scale turns out to be exponentially small under the condition (1.25):

$$1 - H_0 \lesssim 32\exp(-\tau_0 k_1\Delta V_{ph}). \tag{1.26}$$

To study the behavior of a particle in such a small vicinity of the separatrix we need, generally speaking, other methods than those used. However, we can assume that the region of instability under condition (1.23) is certainly much smaller than in the case $\tau_0 k_1 \Delta V_{ph} \sim 1$.

To determine the conditions that a stochastic instability may arise one could use a somewhat different approach from the one described above. For this we remember that V(t) is localized in a very narrow interval $\sim\tau_0$ (relative to all other time scales of the problem) and that the effect of the perturbation on the particle proceeds therefore by jumps from resonance to resonance with an interval between jumps $\sim \delta^{-2} = k_1\Delta V_{ph}/\Omega^2$. Thanks to this the unperturbed particle equations of motion hold between jumps, while during a jump the action I changes and correspondingly also the phase ϑ. This enables us to write down the perturbed equations of motion as a set of finite-difference equations:

$$I_{n+1} = I_n + \Delta I_n,$$
$$\vartheta_{n+1} = \left\{\vartheta_n + \frac{d\Omega(I_n)}{dI}\Delta I_n\frac{k_1\Delta V_{ph}}{\Omega^2(I_n)}\psi_n + \frac{k_1\Delta V_{ph}}{\Omega(I_n)}\right\}, \tag{1.27}$$

where the phase ϑ_n is measured for the sake of convenience in units 2π, the braces $\{\ldots\}$ indicate the fractional part of the argument,

$$\Delta I_n = \frac{\varepsilon}{\tau_0\Omega(I_n)}\int V(t)\sin\left(\frac{k_1}{k_0}\xi + k_1\Delta V_{ph}t\right)dt \sim \frac{\varepsilon}{\tau_0\Omega(I_n)}\psi_n, \tag{1.28}$$

while the integral in (1.28) is taken in the vicinity of the moment t_n. The instant t_n is taken immediately before the n-th jump when the particle action I_n and phase ϑ_n change. We noted earlier that the interval between t_{n+1} and t_n is equal to $k_1\Delta V_{ph}/\Omega^2(I_n)$. The function $\psi_n(\vartheta_n, I_n, t_n)$ depends periodically on ϑ_n and t_n; the quantity $|\psi_n| \leq 1$ and the explicit form of ψ_n are not needed in the following. We can write the equation for the phases more concisely:

$$\vartheta_{n+1} = \left\{\vartheta_n + \frac{k_1\Delta V_{ph}}{\Omega(I_n)} + K_n\psi_n\right\}, \tag{1.29}$$

where

$$K_n = \left|\frac{d\Omega(I_n)}{dI}\right|\Delta I_n \frac{k_1\Delta V_{ph}}{\Omega^2(I_n)}\frac{1}{\psi_n}$$

which agrees with the definition of K in Eq. (1.17) when we take (1.28) into account. A set such as (1.27) has been studied before[9] and we shall give the main result without proof.

Using (1.27), we can find by iteration $\vartheta_n = \vartheta_n(\vartheta_0)$, where ϑ_0 is the initial phase, and evaluate the correlation function of the phases

$$R_m = \frac{1}{\mathcal{N}}\int_0^1 \left(\vartheta_{n+m} - \frac{1}{2}\right)\left(\vartheta_n - \frac{1}{2}\right)d\vartheta_n,$$

$$\mathcal{N} = \int_0^1 \left(\vartheta_n - \frac{1}{2}\right)^2 d\vartheta_n.$$

When $K \gg 1$ it turns out that $R_m \sim \exp(-m \ln K)$ (K is some average value of the quantities K_n) or, changing from a discrete to a continuous time scale,

$$R(t) \sim \exp\left(-\frac{\Omega^2 t}{k_1\Delta V_{ph}}\ln K\right).$$

This means that after a time

$$\tau \sim k_1\Delta V_{ph}/\Omega^2 \ln K \tag{1.30}$$

the phase correlation of the particle is uncoupled and a Brownian-motion type instability appears and leads to an effective increase in the particle energy and its emergence from the potential well. When $K \ll 1$ the particle motion is stable against perturbations ($R \sim 1$) at least over a very long time interval. In that case the particle turns out to be trapped in the region near a single resonance. We can thus assume that the lower limit of the instability range can be found from the condition $K \gtrsim 1$, and we are led to the result (1.17).

Sinaĭ[10] proved that when a certain "stretching" condition is satisfied the motion takes on a mixing character and the phase correlation R is uncoupled. In our case this condition has the form

$$\frac{\delta\vartheta_{n+1}}{\delta\vartheta_n} \sim K_n\frac{d\psi_n}{d\vartheta_n} > 1. \tag{1.31}$$

From this it follows at once that there may exist in phase space a region of dimensions $\sim K^{-1}$ in which inequality (1.31) is not valid and the corresponding phase trajectories are stable over a very long time. Mel'nikov[6] asserts that trajectories stable over an arbitrarily long time exist near the separatrix. In the following we shall call the region near the separatrix, for which the randomness criterion $K \gtrsim 1$ is satisfied, the ergodic layer. We shall now show that free-flying particles near the separatrix are randomized in a way similar to trapped particles.

We define the variable I for free-flying particles as follows:

$$I = I(H) = \frac{1}{2\pi}\int_0^{2\pi} V(H,\xi)\,d\xi = \frac{8}{\pi}qE\left(\frac{\pi}{2}, \frac{1}{q}\right),$$

$$q^2 = \frac{1+H}{2}, \quad \vartheta(\xi, I) = \frac{\partial S(\xi, I)}{\partial I},$$

$$S(\xi, I) = \frac{1}{2\pi}\int V d\xi = \frac{2q}{\pi}E\left(\frac{\xi}{2}, \frac{1}{q}\right),$$

$$\dot{\vartheta} = \Omega(I) = dH(I)/dI, \quad V(t) = 2q\,\mathrm{dn}\frac{t}{\tau_0}. \tag{1.32}$$

Equations (1.32) are completely analogous to the corresponding formulae (1.4), (1.5) for the trapped particles. The equations of motion have the same form (1.6) and near the separatrix

$$\Omega\tau_0 \approx \pi\sqrt{\frac{1+H}{2}}\Big/\ln\frac{16(1+H)}{H-1}, \tag{1.33}$$

while the velocity V(t) is a succession of pulses localized in an interval $\sim\tau_0$. The free-flying particles do therefore not need a special study.

Let us summarize the investigation made here. Because of the symmetry just described, for free-flying particles near the separatrix there is also formed an ergodic layer of the same width as for trapped particles. The separatrix is thus surrounded on both sides by a region of stochastic instability of particles (shaded part in Fig. 2), which has apparently no sharp boundary. The largest width of the ergodic layer occurs for a perturbation which has a phase velocity of the same order as the phase velocity of the main wave, while condition (1.19) is satisfied

$$k_1\Delta V_{ph}\tau_0 \sim 1.$$

In that case the thickness of the ergodic layer is determined by inequality (1.20) and for not too small ε it turns out to be of the order of unity in frequency and of order ε in energy.

2. BROWNIAN MOTION OF A PARTICLE IN THE FIELD OF A WAVE PACKET

As a first application of the instability discussed in the preceding section we shall consider the motion of a particle in the field of a relatively wide wave packet:

$$\dot{v} = \sum_k (V_k e^{i\theta_k} + V_{-k}e^{-i\theta_k}),$$

$$\dot{\theta}_k = kv - \omega_k \equiv \Omega_k(v),$$
$$\theta_k = kx - \omega_k t, \quad V_k = eE_k/m, \quad v = \dot{x}, \tag{2.1}$$
$$V_k = V_{-k}^*, \quad \omega_{-k} = -\omega_k.$$

We shall assume that the spectrum of the packet is discrete with a characteristic distance Δk between the

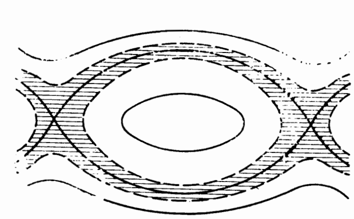

FIG. 2.

wave numbers and with a characteristic distance δ_k between the frequencies:

$$\delta_k = \Omega_{k+\Delta k}(v) - \Omega_k(v) = \Delta k (v - d\omega_k / dk). \quad (2.2)$$

We noted already before that under the conditions most conducive for instability the ergodic layer has a width in frequency of order τ_0^{-1} for not too small ε. This means that if we take a packet in which the harmonics are spaced so closely to one another that

$$\tau_0 \delta_{k_0} = \delta_{k_0} \sqrt{\frac{m}{eE_0 k_0}} \ll 1, \quad (2.3)$$

where E_0 and k_0 are some values of the field and the wave number averages over the packet, the ergodic layers from the different waves coalesce and there will be no trapped particles. In fact, such an estimate is not very rigorous since the amplitudes of all the waves in the packet may be comparable and the estimate of the width of the ergodic layer given in Sec. 1 is inapplicable in the present case. In the following we give a more correct derivation of inequality (2.3), using a method developed in [1].

We introduce a distribution $f(v, \theta, t)$ satisfying the Liouville equation

$$\frac{\partial f}{\partial t} + \sum_k \Omega_k(v) \frac{\partial f}{\partial \theta_k} + \frac{\partial}{\partial v} \sum_k (V_k e^{i\theta_k} + V_{-k} e^{-i\theta_{+k}}) f = 0 \quad (2.4)$$

and containing no other information than what follows from the equations of motion. For the sake of simplification we write

$$f(v, \theta_{k_1}, \theta_{k_2}, \ldots, t) \equiv f(v, \theta, t).$$

We assume the distribution f to satisfy also the normalization condition

$$\int f d\Gamma = 1, \quad d\Gamma = d\theta dv.$$

We expand f in a series:

$$f(v, \theta, t) = \sum_{(n)} [f^{(n)}(v, t) e^{i(n,\theta)} + f^{(-n)}(v, t) e^{-i(n,\theta)}],$$
$$f^{(-n)} = f^{(n)*}, \quad (n, \theta) \equiv n_1 \theta_{k_1} + n_2 \theta_{k_2} + \ldots \quad (2.5)$$

Changing in (2.4) to the interaction representation and retaining in the expansion in the perturbation only the main terms as $t \to \infty$, one gets easily the following equation for $f^{(0)}(v, t)$:

$$\frac{\partial f^{(0)}}{\partial t} = 2 \operatorname{Re} \frac{\partial}{\partial v} \sum_k V_k f^{(1)}(0) e^{i\Omega_k t} + 2 \frac{\partial}{\partial v} \sum_k |V_k|^2 \int^t \cos \Omega_k t' dt' \frac{\partial f^{(0)}}{\partial v} + O(V^3), \quad (2.6)$$

where

$$f^{(n)}(0) \equiv f^{(n)}(v, 0).$$

We make two remarks about Eq. (2.6). The first is connected with the choice of initial conditions for the problem. If we assume random phases at t = 0, i.e.,

$$f^{(n)}(0) = 0, \quad n \neq 0,$$

then the first term in (2.6) vanishes and we are led to the well-known Fokker–Planck type quasi-linear equation:[5]

$$\frac{\partial f^{(0)}}{\partial t} = 2\pi \frac{\partial}{\partial v} \sum_k |V_k|^2 \delta(\omega_k - kv) \frac{\partial f^{(0)}}{\partial v}. \quad (2.7)$$

We shall, however, be interested in the derivation of an equation such as (2.7) under arbitrary initial conditions. If, in particular, it turns out that $f^{(1)}(0) = 0$, we must retain in (2.6) higher-order terms that preserve a memory of the initial phases. The second remark is connected with the fact that each differentiation of the factor $\exp(i\Omega_k t)$ with respect to v leads to the appearance of secular terms proportional to powers of t which are the higher the higher higher the order of the terms in (2.6) are in the perturbation. We shall discuss below the possibility of cutting off the series in (2.6).

Let us consider the non-linear effects connected with the resonance between the particle and the wave. Let there occur for some value of the velocity v a resonance with the k-th harmonic of the field

$$\Omega_k(v_r) = kv_r - \omega_k t = 0.$$

Because of the resonance, the particle velocity is changed by an amount

$$\delta v_k \sim V_k \Big/ \left| \frac{d\Omega_k(v)}{dv} \right| \delta v_k,$$

whence

$$\delta v_k \sim \sqrt{\frac{V_k}{k}} = \frac{1}{k\tau_k}, \quad \frac{1}{\tau_k} \equiv \sqrt{\frac{eE_k k}{m}}. \quad (2.8)$$

If the resonance width (2.8) is much larger than the distance between the resonances, the motion of the particle remains random as in Sec. 1. The condition for this is

$$K_k = k\delta v_k / \delta_k \gg 1, \quad (2.9)$$

which is the same as (2.3). In the case $K_k \ll 1$ the particle is trapped by the wave with which it is in resonance and a consideration of this case is of no interest. When (2.9) is satisfied the frequency $\Omega_k(v)$ is a function of the time and this fact must reflect on Eq. (2.6).

For the following we consider the correlation function

$$R_k = \langle\langle e^{i\theta_k(t) - i\theta_k(0)} \rangle\rangle = \langle\langle e^{i\Omega_k(v)t} \rangle\rangle, \quad (2.10)$$

where we have written

$$\langle\langle \ldots \rangle\rangle \equiv \left(\frac{1}{2\pi}\right)^N \int d\theta_{k_1,(0)} \ldots d\theta_{k_N(0)},$$
$$\theta_{k,(0)} \equiv \theta_k(v, t = 0) \quad (2.11)$$

while N is the number of harmonics in the wave packet. Writing the equation for the phase $\theta_k(v, t)$ as a finite differences equation similar to (1.29) (see also [11]) we have

$$\theta_{k,(n+1)} = \left\{ \theta_{k,(n)} + \frac{\Omega_k}{\delta_k} + \sum_{k'} K^2_{kk',(n)} \cos \theta_{k,(n)} \right\}, \quad (2.12)$$

where we have introduced a discrete time scale $t_1, t_2, \ldots$, with intervals

$$t_{n+1} - t_n \sim \delta_k^{-1}$$

and written

$$\theta_{k,(n)} = \theta_k(v_n, t_n), \quad K_{kk'} = \frac{d\Omega_k(v)}{dv} \delta v_{k'}.$$

Iterating Eq. (2.12) and finally expressing $\theta_{k(n)}$ in terms of $\theta_{k(\theta)}$ we find

$$R_k \sim e^{i\Omega_k t - t/t_r}, \quad 1/t_r = N_0 \delta_0 \ln K, \quad (2.13)$$

where N_0 is the number of harmonics occurring in the sum over k' in (2.12). Since this number is simply the number of resonances which the particle traverses in a separate jump, we have

$$N_0 = K$$

and, using (2.9) and (2.8) we find finally for the time t_r of the decoupling of the phase correlation

$$\frac{1}{t_r} = \frac{1}{\tau_0} \ln \frac{1}{\tau_0 \delta_0}, \tag{2.14}$$

where τ_0, δ_0 are some values of τ_k, δ_k averaged over the packet.

The further treatment of Eq. (2.6) is now not very difficult. We let the operator (2.11) act upon (2.6). This gives

$$\frac{\partial f^{(0)}}{\partial t} = 2 \operatorname{Re} \frac{\partial}{\partial v} \sum_k V_k f^{(1)}(0) \langle\!\langle e^{i\Omega_k t} \rangle\!\rangle + 2 \frac{\partial}{\partial v} \sum_k |V_k|^2 \int^t dt' \langle\!\langle \cos \Omega_k t' \rangle\!\rangle \frac{\partial f^{(0)}}{\partial v}. \tag{2.15}$$

When $K \ll 1$ we have

$$R_k = \langle\!\langle e^{i\Omega_k t} \rangle\!\rangle \approx e^{i\Omega_k t}[1 + O(K)],$$

and Eq. (2.15) is exactly the same as (2.6). However, under the condition (2.9) the first term in (2.15) vanishes according to (2.13) over a time t_r and we obtain when $t \gg t_r$:

$$\frac{\partial f^{(0)}}{\partial t} = 2 \frac{\partial}{\partial v} \sum_k |V_k|^2 \frac{t_r}{1 + \Omega_k^2 t_r^2} \frac{\partial f^{(0)}}{\partial v} \tag{2.16}$$

i.e., the quasi-linear equation, taking into account the fact that the time of phase memory is finite.[12] One can show that all other terms which contain a memory of the initial conditions and which are proportional to higher powers of the perturbation vanish similarly.

In order that the secular terms that occur in the derivation of (2.6) be unimportant it is apparently necessary that they remain small over the cut-off time t_r. This leads to the inequality

$$t_r / \tau_0 = |\ln \tau_0 \delta_0|^{-1} \ll 1, \tag{2.17}$$

which is automatically satisfied by virtue of (2.9) (or (2.3)). It is then, of course, necessary that the number N of harmonics in the packet be not less than N_0, and this leads to the well-known condition on the packet width[5]

$$N\delta_0 \gg N_0 \delta_0 = K\delta_0 = 1/\tau_0. \tag{2.18}$$

In the more general case when the packet is insufficiently wide so that $N < N_0$ the expression for t_r is changed as follows:

$$1/t_r = N\delta_0 \ln K \tag{2.19}$$

and the condition for the cut-off of the secular terms (2.17) which becomes

$$N\delta_0 \gg \frac{1}{\tau_0} \ln \frac{1}{\tau_0 \delta_0} \tag{2.20}$$

is no longer so trivial because of the factor $\ln K = \ln(1/\tau_0\delta_0)$.

3. PARTICLE DISTRIBUTION FUNCTION IN THE ERGODIC LAYER

We now turn to the particle equation of motion (1.2) and obtain a kinetic equation for the distribution function f(I, t). According to (1.28) the jump in the action ΔI_n in one jolt is much less than unity under the condition $\varepsilon \ll \Omega\tau_0$. This means that except for a very small region in the frequency $\sim\varepsilon$ near the separatrix we can always use the Fokker–Planck equation for f

$$\frac{df}{dt} = -\frac{\partial}{\partial I}\left[\frac{\langle \Delta I \rangle}{\Delta t} f\right] + \frac{1}{2}\frac{\partial^2}{\partial I^2}\left[\frac{\langle (\Delta I)^2 \rangle}{\Delta t} f\right], \tag{3.1}$$

where Δt is the average time between jolts which is equal to

$$\Delta t = k_1 \Delta V_{ph} / \Omega^2. \tag{3.2}$$

Bearing in mind that the length of the jolt is $\sim \tau_0 \ll \Delta t$, we find from (1.2) and (1.6)

$$V(t) \approx -\sin\xi - \varepsilon \sin\left(\frac{k_1}{k_0}\xi + k_1 \Delta V_{ph} t\right),$$

$$\dot I \approx \frac{\varepsilon}{\Omega \tau_0^2} \sin\xi \cdot \sin\left(\frac{k_1}{k_0}\xi + k_1 \Delta V_{ph} t\right) + \frac{\varepsilon^2}{\Omega\tau_0^2} \sin^2\left(\frac{k_1}{k_0}\xi + k_1 \Delta V_{ph} t\right),$$

$$\langle \Delta I \rangle \approx {}^1/_2\, \varepsilon^2 / \Omega\tau_0,$$

$$\langle (\Delta I)^2 \rangle \approx \frac{\varepsilon^2}{\Omega^2 \tau_0^2} \left\langle V^2 \sin^2\left(\xi \frac{k_1}{k_0} + k_1 \Delta V_{ph} t\right)\right\rangle = \frac{\varepsilon^2 H}{\Omega^2 \tau_0^2}. \tag{3.3}$$

From (3.2) and (3.3) it follows that

$$\frac{1}{2}\frac{\partial}{\partial I}\frac{\langle (\Delta I)^2 \rangle}{\Delta t} = \frac{\langle \Delta I \rangle}{\Delta t},$$

and Eq. (3.1) takes the usual form

$$\frac{\partial f}{\partial t} = \frac{\partial}{\partial I} D \frac{\partial f}{\partial I}, \quad D = \frac{1}{2}\frac{\varepsilon^2 H}{\tau_0^2 k_1 \Delta V_{ph}}. \tag{3.4}$$

One can choose the boundary conditions for f as follows. Since a particle moving in the ergodic layer cannot penetrate into the region where the motion is stable the boundary of the stochasticity region (1.18) must be considered to be a reflecting wall.[13] The boundary condition is thus the absence of a particle current

$$D \frac{\partial f}{\partial I}\bigg|_{I=I_0} = 0, \tag{3.5}$$

where I_0 is the solution of the equation

$$\Omega(I_0)\tau_0 = \pi \Big/ \ln \frac{16\pi}{\varepsilon k_1 \Delta V_{ph}}. \tag{3.6}$$

From (3.4) to (3.6) we get at once a stationary solution

$$f(I) = 1/2\Delta, \quad \Delta = I(H = 1) - I_0. \tag{3.7}$$

From (3.7) we find

$$F(H) = 1/2\Delta\tau_0\Omega(H), \tag{3.8}$$

where

$$F(H)\,dH = f(I)\,dI.$$

Using the connection (1.9) between Ω and H and expressing H in terms of V and ξ we find finally

$$F(H) = \frac{1}{2\pi\Delta} \ln \frac{32}{1 + \cos\xi - V^2/2}. \tag{3.9}$$

The distribution (3.9) is valid everywhere except in the already mentioned small region where the Fokker–Planck approximation is not valid. This region is of order ε in the frequency and thus of order $\exp(-1/\varepsilon)$ in energy near the separatrix.

The time t_c of establishing the stationary distribution (3.9) can easily be found from (3.4), (3.6), and (1.4) and is of the following order:

$$t_c \sim \tau_0^2 k_1 \Delta V_{ph} / \varepsilon^2.$$

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