STOCHASTIC RESONANCE ACCELERATION OF CHARGED PARTICLES

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Submitted October 29, 1967

Zh. Eksp. Teor. Fiz. 48, 1264–1269 (April, 1968)

We consider the acceleration of charged particles in a randomly polarized field of transverse plane electromagnetic waves propagating in the direction of a uniform magnetic field. It is shown that in the resonance case, in which the characteristic field frequency is equal to the particle gyro-frequency, the mean kinetic energy of the particles can be increased appreciably. It is proposed that the effect may be important in cosmological situations.

CHARGED particles can be accelerated significantly by a resonance mechanism:[1,2] a transverse plane monochromatic wave propagates along a fixed magnetic field with phase velocity equal to the velocity of light in vacuum c and the frequency of the wave is approximately the gyro-frequency for the appropriate particle in the fixed magnetic field. It is obvious that this situation will not generally exist under natural conditions. In order to estimate the effectiveness of the acceleration mechanism it is necessary to consider an electromagnetic wave with random polarization. In this case the accelerated particles will have a random energy distribution in which one expects a preponderance of particles with high energies to appear close to the resonance region. A characteristic time for the variation of the field, which then serves as a correlation time for the random function that describes the wave, must be approximately equal to the Larmor period of the particles. In this case the mean energy of the particles increases, that is to say, the plasma is heated as a consequence of the cyclotron resonance. A similar effect has been considered in cyclic accelerators by Burshteĭn, Veksler, and Kolomenskiĭ.[3]

We shall neglect particle collisions, assuming that the plasma is of low density. This requirement is also necessary if we are to take the velocity of propagation of the wave equal to the velocity of light in vacuum. Under these conditions there is also no dispersion so that the wave is a function only of the time phase $\theta = t - z$. The role of dissipation in the resonance system being considered is played by the radiation reaction in the electromagnetic field. We shall also neglect effects that are only important in very intense fields. The radiation reaction effect has been considered in [4,5]. Both of these assumptions are obviously valid for the majority of cases that are encountered in cosmological situations.

Let the electromagnetic field be described by the vector potential **A**:

$$\mathbf{A} = \mathbf{F}(\theta) - {}^1\!/_2[\mathbf{Br}], \tag{1*}$$

where the fixed uniform magnetic field **B** is directed along the z-axis, the vector **F** lies in the xy plane, the radius vector **r** is expressed in light seconds, and $\theta = t - z$.

The equation of motion for particles in the field described by Eq. (1) is

$$\frac{dp_{\parallel}}{dt} = -\frac{e}{mc}\mathbf{u}\frac{d\mathbf{F}}{d\theta}, \tag{2}$$

$$\frac{d}{dt}\left(\mathbf{p}_{\perp} + \frac{e}{mc}\mathbf{F}\right) = [\boldsymbol{\Omega}\mathbf{u}]. \tag{3}$$

Here, **p** is the particle momentum in units of mc, $p_{\parallel}$ and $\mathbf{p}_{\perp}$ are the projections of this momentum on the z-axis and the xy plane respectively, **u** is the particle velocity divided by the velocity of light in vacuum, and $\boldsymbol{\Omega} = e\mathbf{B}/mc$ is a fixed vector.

From Eqs. (2) and (3) we obtain an integral of the motion

$$Y = \beta - p_{\parallel} = \beta(1 - u_{\parallel}) = \text{const}, \tag{4}$$

where β is the particle energy in units of the rest energy mc^2.

Assuming that $d\theta = dt(1 - u_{\parallel}) = Ydt/\beta$ in Eq. (3) we transform to a differential equation in terms of the time phase θ:

$$\frac{d\mathbf{p}_{\perp}}{d\theta} - \left[\frac{\boldsymbol{\Omega}}{Y}\mathbf{p}_{\perp}\right] = -\frac{e}{mc}\frac{d\mathbf{F}}{d\theta}.$$

The solution of this equation is

$$p_x = \sin\frac{\Omega}{Y}\theta \cdot \int_{\theta_0}^{\theta}\left(f_x \sin\frac{\Omega}{Y}\theta - f_y\cos\frac{\Omega}{Y}\theta\right)d\theta + \cos\frac{\Omega}{Y}\theta\cdot\int_{\theta_0}^{\theta}\left(f_x\cos\frac{\Omega}{Y}\theta + f_y\sin\frac{\Omega}{Y}\theta\right)d\theta + p_{\perp 0}\cos\left[\frac{\Omega}{Y}(\theta-\theta_0)+\alpha_0\right],$$

$$p_y = \cos\frac{\Omega}{Y}\theta \cdot \int_{\theta_0}^{\theta}\left(f_y \cos\frac{\Omega}{Y}\theta - f_x\sin\frac{\Omega}{Y}\theta\right)d\theta + \sin\frac{\Omega}{Y}\theta\cdot\int_{\theta_0}^{\theta}\left(f_x\cos\frac{\Omega}{Y}\theta + f_y\sin\frac{\Omega}{Y}\theta\right)d\theta + p_{\perp 0}\sin\left[\frac{\Omega}{Y}(\theta-\theta_0)+\alpha_0\right], \tag{5}$$

where the vector $\mathbf{f} = -(e/mc)d\mathbf{F}/d\theta$ is proportional to the electric field of the wave **E** and has the dimensions of frequency while the subscript zero denotes the values of quantities for $\theta = \theta_0$. We note that if the function $\mathbf{f}(\theta)$ is written in the form of a superposition of Fourier harmonics then, as is evident from Eq. (5), expressions for the transverse momentum will contain a characteristic resonance denominator $\omega^2 - (\Omega/Y)^2$.

The particle energy is determined only by the square of the transverse momentum. Using the identity $\beta^2 = p_{\parallel}^2 + p_{\perp}^2 + 1$ and the integral of the motion (4) we find

$$\beta = (p_{\perp}^2 + Y^2 + 1)/2Y.$$

*$[\mathbf{Br}] \equiv \mathbf{B}\times\mathbf{r}$.

Furthermore, for reasons of simplicity we assume that at time $t = 0$ the particle is at rest at the origin of coordinates. Thus

$$\theta_0 = 0, \qquad p_{\perp 0} = 0, \qquad Y = 1 \tag{6}$$

and the square of the transverse momentum is twice the kinetic energy of the particle:

$$2w = 2(\beta - 1) = p_\perp^2.$$

Let $\mathbf{f}$ be a vector whose components f_x and f_y are independent stochastic stationary functions of the phase time θ with normal Gaussian density distributions φ_x and φ_y with the same dispersion σ:

$$\varphi_x = \frac{1}{\sqrt{2\pi}\sigma}\exp(-f_x^2/2\sigma^2), \qquad \varphi_y = \frac{1}{\sqrt{2\pi}\sigma}\exp(-f_y^2/2\sigma^2),$$

and with corresponding autocorrelation functions ρ_x and ρ_y:

$$\rho_x = \rho_y = \rho_f = \sigma^2 \int_{\theta_0}^{\theta} a(\omega)\cos\omega(\theta_1 - \theta_2)\,d\omega.$$

Physically this assumption means that we are considering an electromagnetic wave which is a superposition of two independent linearly polarized waves whose planes of polarization are perpendicular to each other and whose intensities are random functions of the argument θ with the same statistical characteristics. According to the Wiener-Khinchin theorem the function $\sigma^2 a(\omega)$ is the power spectrum for the quantity f_x or f_y, that is to say, it is equal to the ensemble average of the square of the corresponding Fourier component of the quantity eE_x/mc or eE_y/mc:

$$\int_0^\infty a(\omega)\,d\omega = 1.$$

We will now seek the distribution of kinetic energy over the statistical ensemble of particles characterized by the initial conditions in (6). In general this distribution will depend on the time phase θ. We first write an expression for w:

$$2w = \left[\int_0^\theta (f_y\cos\Omega\theta - f_x\sin\Omega\theta)\,d\theta\right]^2 + \left[\int_0^\theta (f_y\sin\Omega\theta + f_x\cos\Omega\theta)\,d\theta\right]^2. \tag{7}$$

Since the dispersion for the quantities f_x and f_y is the same, the integrands in Eq. (7)

$$q_1 = f_y\cos\Omega\theta - f_x\sin\Omega\theta, \qquad q_2 = f_y\sin\Omega\theta + f_x\cos\Omega\theta$$

are also independent Gaussian random functions of θ with the same dispersion (cf. for example [6]) and autocorrelation function ρ_q, defined by the relation $\rho_q = \rho_f\cos\Omega(\theta_1 - \theta_2)$. Thus, the functions q_1 and q_2 are stationary.

The integrals that appear in (7) are also Gaussian stochastic functions of the time phase θ; however they are not stationary. We shall call these functions s_1 and s_2. These functions have the same dispersion:

$$\tau = \sigma^2 \int_0^\infty \frac{a(\omega)}{(\omega^2 - \Omega^2)^2}\{[1 - \cos(\omega - \Omega)\theta](\omega + \Omega)^2 + [1 - \cos(\omega + \Omega)\theta](\omega - \Omega)^2\}\,d\omega. \tag{8}$$

It is evident from Eq. (5) that s_1 and s_2 are the projections of the vector $p_\perp$ on orthogonal axes which are rotated about the z axis by an angle $\Omega\theta$ with respect to the x and y axes. These projections differ from p_x and p_y in that they are statistically independent. This last feature has a simple physical significance. The components p_x and p_y are coupled quantities since there is a magnetic field $\mathbf{B}$. By introducing the functions q_1 and q_2 in place of f_x and f_y and the functions s_1 and s_2 in place of p_x and p_y we have converted to a reference system that rotates about the z axis with an angular velocity Ω; thus we have eliminated the fixed magnetic field. It is clear that in this reference system the projections s_1 and s_2 of the vector $\mathbf{p}_\perp$ are not statistically coupled quantities. This statistical independence makes it possible to describe the joint probability that the values of the components of the transverse momentum s_1 and s_2 lie within the intervals ds_1 and ds_2 respectively; this probability is given by

$$P(s_1, s_2)\,ds_1 ds_2 = 1/2\pi\tau\exp(-p_\perp^2/2\tau)\,ds_1 ds_2.$$

We can now easily determine the distribution of transverse momentum $p_\perp^2$ and thus the particle kinetic energy. The density distribution is given by

$$R(w) = \tau^{-1}\exp(-w/\tau). \tag{9}$$

The mean kinetic energy τ can be interpreted as the temperature of the particles that move in the wave and are characterized by a phase difference θ. When the power of the harmonic field $a(\omega)$ is not small close to the gyro-frequency Ω this temperature can reach rather large values. We shall assume that the function $a(\omega)$ is non-zero only within some frequency range $\omega_0 - \Delta\omega \le \omega \le \omega_0 + \Delta\omega$ about the frequency ω_0 and that it is constant over this range. Making use of this assumption in the integration in Eq. (8), we have

$$\tau = 2a\sigma^2 \frac{\Delta\omega[1 - \cos(\omega_0 - \Omega)\theta\cos\Delta\omega\theta] - (\omega_0 - \Omega)\sin(\omega_0 - \Omega)\theta\sin\Delta\omega\theta}{(\omega_0 - \Omega)^2 - (\Delta\omega)^2}$$
$$+ \frac{\Delta\omega[1 - \cos(\omega_0 + \Omega)\theta\cos\Delta\omega\theta] - (\omega_0 + \Omega)\sin(\omega_0 + \Omega)\theta\sin\Delta\omega\theta}{(\omega_0 + \Omega)^2 - (\Delta\omega)^2}$$
$$+ \frac{\theta}{2}\int_{\omega_0 - \Delta\omega}^{\omega_0 + \Delta\omega}\left[\frac{\sin(\omega + \Omega)\theta}{\omega + \Omega} + \frac{\sin(\omega - \Omega)\theta}{\omega - \Omega}\right]d\omega\Big\}. \tag{10}$$

Assuming that the interval $\Delta\omega$ is small, $|\omega_0 \pm \Omega| \gg \Delta\omega$, in the nonresonance case we find

$$\tau = 4a\sigma^2\left[\frac{\omega_0^2 + \Omega^2}{(\omega_0^2 - \Omega^2)^2}\Delta\omega - \frac{\sin\Delta\omega\theta}{\theta}\,\frac{(\omega_0^2 + \Omega^2)\cos\omega_0\theta\cos\Omega\theta - 2\omega_0\Omega\sin\omega_0\theta\sin\Omega\theta}{(\omega_0^2 - \Omega^2)^2}\right].$$

For particles that have large phase differences in the wave $\Delta\omega\theta \gg \pi$ the temperature is independent of the time phase:

$$\tau = 4a\sigma^2\Delta\omega\frac{\omega_0^2 + \Omega^2}{(\omega_0^2 - \Omega^2)^2}. \tag{11}$$

This quantity is of the order of the square of the ratio of the wave field to the fixed magnetic field $\tau \sim (E/B)^2$.

The highest acceleration is obviously expected in the resonance case $\omega_0 = \Omega$. In this case the mean kinetic energy of particles that are characterized by a time phase θ is

$$\tau = 2a\sigma^2\left[\frac{\cos\Delta\omega\theta - 1}{\Delta\omega} + \theta\int_0^{\Delta\omega\theta}\frac{\sin\xi}{\xi}d\xi + \frac{\Delta\omega(1 - \cos 2\Omega\theta\cos\Delta\omega\theta) - 2\Omega\sin 2\Omega\theta\sin\Delta\omega\theta}{4\Omega^2 - (\Delta\omega)^2} + \frac{\theta}{2}\int_{(2\Omega - \Delta\omega)\theta}^{(2\Omega + \Delta\omega)\theta}\frac{\sin\xi}{\xi}d\xi\right].$$

Now let $\Delta\omega \ll 2\Omega$. In this case, when $\Delta\omega\theta \ll \pi$ we find

$\tau = a\sigma^2\Delta\omega[\theta^2 + (1 - \cos 2\Omega\theta)/2\Omega^2]$; furthermore, if $\Omega\theta \gg \pi$ then $\tau = \sigma^2 a\Delta\omega\theta^2$. Assuming that $a\Delta\omega = 1$ and $\sigma^2 = \overline{(eE/mc)^2}$ we find

$$\tau = \overline{(eE/mc)^2}\theta^2. \tag{12}$$

In the other limit $\Delta\omega\theta \gg \pi$, using the asymptotic expression for the sine integral we find

$$\tau = 2a\sigma^2(\pi\theta/2 - 1/\Delta\omega) \approx \pi\sigma^2 a\theta = \pi\overline{(eE/mc)^2}\theta/\Delta\omega. \tag{13}$$

Assuming that $\Delta\omega \sim 0.1\Omega$ and $\Delta\omega\theta \sim 0.1$ we find that the mean kinetic energy of particles as given by Eq. (12) is of the same order as that for the nonresonance case given by Eq. (11): $\tau \sim (E/B)^2$. The only important difference arises in a very narrow range $\Delta\omega$. Thus, even when $\Delta\omega \sim 10^{-2}\,\Omega$ and $\Delta\omega\theta \sim 0.1$ it follows from Eq. (12) that $\tau \sim 10^2(E/B)^2$. In the other limit $\Delta\omega\theta \gg \pi$, we use Eq. (13); introducing the values $\Delta\omega \sim 0.1\Omega$ and $\Delta\omega\theta \sim 10$ we find that $\tau \sim 10^3(E/B)^2$.

An important feature in the derivation of Eqs. (9) and (10) for the energy distribution was the assumption of statistical independence for the quantities s_1 and s_2. This assumption is justified only when the quantities f_x and f_y are noncorrelated random functions with the same dispersion. However, if we assume that each of the quantities f_x and f_y is autocorrelated, then in general it will be reasonable to assume a correlation between f_x and f_y, which are projections of the same vector. Furthermore, a random wave can have a dominant direction of polarization, a situation that is frequently observed for cosmic radio emission. In this case the dispersion of the quantities f_x and f_y is different and this feature also leads to a violation of the statistical independence of s_1 and s_2. Without stopping to resolve the problem based on this more general formulation, we present here an expression for the case of one linearly polarized random wave. At resonance, with $\Omega\theta \gg \pi$, the energy density distribution is given by

$$R(w) = (2\pi\tau w)^{-1/2}\exp(-w/2\tau),$$

where the mean kinetic energy for $\Delta\omega/\Omega \ll 1$ is given by

$$\tau = \frac{a\sigma^2}{2}\left(\theta\int_0^{\Delta\omega\theta}\frac{\sin\xi}{\xi}d\xi + \frac{\cos\Delta\omega\theta - 1}{\Delta\omega}\right)$$

The mechanism considered here can operated under much less stringent limitations than the conventional exact resonance acceleration mechanism. Thus, we do not require a monochromatic wave; a weak homogeneity in the fixed magnetic field is allowed and lack of coincidence between the direction of this field and the direction of propagation of the wave can be tolerated. In addition any difference in the velocity of the wave from the velocity of light in vacuum does not have a strong effect on the results that have been obtained.

For these reasons it is proposed that stochastic resonance acceleration can be effective in a number of cosmological cases. For example, it can be important in the explanation of effects such as the heating of plasma in the emission of supernovae and the acceleration of electrons that is characteristic of this process; also, the heating of the solar corona and the acceleration of particles in the interplanetary medium. It is possible that this mechanism also serves as an injection mechanism for the Fermi acceleration of charged particles associated with moving magnetic inhomogeneities in the interstellar space.

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Translated by H. Lashinsky
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