DETERMINATION OF THE REGIME OF ELECTRIC RELAXATION UNDER SHOCK LOADING

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Submitted November 25, 1967

Zh. Eksp. Teor. Fiz. 54, 1253—1258 (April, 1968)

The problem of calculating the current in a circuit of a parallel-plate capacitor with arbitrary load, subjected to shock compression, is solved under most general assumptions. A method is indicated for determining the law governing the decrease of the electric polarization, and also of the conductivity and the dielectric constant, of a shock compressed material from the experimental data.

A number of papers, a review of which is contained in[1], describes the occurrence of an emf following shock loading of a dielectric filling a parallel-plane capacitor, as a result of the polarization of the dielectric on the shock wave front (SWF). An approximate theory of this phenomenon is developed in[2], and an exact solution of the problem is given for a number of particular cases in[3-5]. We present below a solution of the problem under the most general assumptions, the main result being formulas (19) and (29), which make it possible to determine directly from the experimental data the law governing the decrease of the polarization with time, and also the dielectric constant and the conductivity of the shock-compressed matter.

Assume that the plates of the capacitor have an area S and the space between them is filled with a substance having a dielectric constant ϵ_0 and a conductivity λ_0. In the initial state, the distance between the plates is L. We denote by v_0 and v the velocity of the SWF respectively relative to the cold and compressed substances: $\delta = v_0/v$ is the compression of the substance. The shock wave moves from the left-hand plate to the right-hand one. The coordinate x is reckoned from the initial position of the left plate, so that the SWF corresponds to $x = x_f = v_0 t$. It is also convenient to introduce a system (y, t), in which the compressed matter is stationary, and on the left plate we have y = 0, and on the SWF we have $y = y_f + v^t$. The SWF will pass through the point with Lagrangian coordinate y at the instant $t_1 = y/v$. Therefore the time when a given particle is in the compressed state is $\xi = t - t_1 = t - y/v$. The shock wave produced in the compressed matter a polarization P(y, t), which by virtue of the symmetry is perpendicular to the SWF. We shall assume that the polarization in each particle depends only on the time that the particle stays in the compressed state:

$$P(y,t) = P(\xi) = P(t - y/v) \quad (0 < y < vt). \tag{1}$$

The inhomogeneous polarization leads to the appearance on the SWF of a volume density of the bound charges

$$\rho_0(y,t) = -\frac{\partial P}{\partial y} = \frac{1}{v}P'(\xi), \tag{2}$$

where the prime denote differentiation of the function P(t) with respect to its argument.

Let ρ_1 be the volume density of the charges produced by the conductivity, j the density of the conductivity current. Ahead of the SWF, where there are no bound charges, we have the system of equations (E—field intensity, D—induction, φ—potential):

$$\frac{\partial \rho_1}{\partial t} + \frac{\partial j}{\partial x} = 0, \quad j = \lambda_0 E = \frac{\lambda_0}{\varepsilon_0}D, \qquad \frac{\partial D}{\partial x} = 4\pi\rho_1, \quad \frac{\partial \varphi}{\partial x} = -\frac{1}{\varepsilon_0}D, \tag{3}$$

whence

$$\frac{\partial \rho_1}{\partial t} + \frac{\rho_1}{\theta_0} = 0 \quad \left(\theta_0 = \frac{\varepsilon_0}{4\pi\lambda_0}\right). \tag{4}$$

A solution of this equation is $\rho_1(x, t) = g(x)\exp(-t/\theta_0)$, where g(x) is an arbitrary function. Inasmuch as there are no charges in the entire region $0 < x < L$ at the initial instant t = 0, we get g(x) = 0 and $\rho_1(x, t) = 0$. We denote by $q_1(t)$ and $q_2(t)$ the density of the surface charges of the conductivity on the left and right plates, respectively. Then ahead of the SWF we have

$$D = -4\pi q_1, \quad j = -\frac{1}{\theta_0}q_1, \quad \varphi(x,t) = V(t) + \frac{4\pi}{\varepsilon_0}q_1(t)(x - L) \qquad (v_0 t < x < L), \tag{5}$$

where V(t) is the potential difference between the plates (the potential of the left-hand plate will be assumed to be zero). If I(t) is the current in the external circuit in the direction from the right plate to the left, we have

for: $dq_1/dt = j(x = L, t) - i(t) \quad (i = I/S)$.

Using (5), we get

$$\frac{dq_1}{dt} + \frac{q_1}{\theta_0} = -i(t). \tag{6}$$

In the region of the compressed matter, the field equations have the form (ϵ—dielectric constant, λ—conductivity of the compressed matter).

$$\frac{\partial \rho_1}{\partial t} + \frac{\partial j}{\partial y} = 0, \quad j = \lambda E = \frac{\lambda}{\varepsilon}D, \qquad \frac{\partial D}{\partial y} = 4\pi(\rho_0 + \rho_1), \quad \frac{\partial \varphi}{\partial y} = -\frac{1}{\varepsilon}D. \tag{7}$$

It follows therefore that $\rho_1(y, t)$ satisfies the equation

$$\frac{\partial \rho_1}{\partial t} + \frac{\rho_1}{\theta} = -\frac{\rho_0}{\theta} = -\frac{1}{\theta v}P'(\xi) \quad \left(\theta = \frac{\varepsilon}{4\pi\lambda}\right). \tag{8}$$

Let us multiply both halves of (8) by $e^{t/\theta}$ and integrate with respect to time from t_1 to t; we obtain

$$\rho_1(y,t) = f(y)e^{-t/\theta} - \frac{P(\xi)}{\theta v} + \frac{e^{-\xi/\theta}}{\theta v}\left[P(0) + \frac{1}{\theta}\int_0^{\xi} e^{\eta/\theta}P(\eta)\,d\eta\right], \tag{9}$$

where $f(y) = \rho_1(t, t_1)\exp(t_1/\theta)$. On both boundaries of the compressed matter there is a bound charge with surface density P(0) at $y = y_f$ and $-P(t)$ at y = 0. Therefore

$$\frac{1}{4\pi}D(0,t)=q_2(t)-P(t),\quad \frac{1}{4\pi}[D(x_f,t)-D(y_f,t)]=P(0). \qquad (10)$$

In the last of the equations (7) we substitute (2) and (9) integrating, taking into account the first of the boundary conditions (10):

$$\frac{1}{4\pi}D(y,t)=q_2(t)+e^{-t/\theta}\int_0 f(z)\,dz-P(\xi)+P(0)\,(e^{-\xi/\theta}-e^{-t/\theta})$$

$$+\frac{1}{\theta}\Big[e^{-\xi/\theta}\int_0^{\xi}e^{\eta/\theta}P(\eta)\,d\eta-e^{-t/\theta}\int_0^{t}e^{\eta/\theta}P(\eta)\,d\eta\Big]. \qquad (11)$$

Hence

$$\frac{1}{4\pi}D(y_f,t)=q_2(t)+e^{-t/\theta}\int_0^{y_f}f(z)\,dz-e^{-t/\theta}\Big[P(0)+\frac{1}{\theta}\int_0^{t}e^{\eta/\theta}P(\eta)\,d\eta\Big]. \qquad (12)$$

The condition for charge conservation on passage of the SWF

$$j(y_f,t)-v\rho_1(y_f,t)=j(x_f,t)-v_0\rho_1(x_f,t) \qquad (13)$$

is transformed into

$$e^{-t/\theta}\Big[\int_0^{y_f}f(z)\,dz-\theta v f(y_\Phi)\Big]+q_2(t)+\frac{\theta}{\theta_0}q_1(t) \qquad (14)$$

$$-e^{-t/\theta}\Big[P(0)+\frac{1}{\theta}\int_0^{t}e^{\eta/\theta}P(\eta)\,d\eta\Big]=0.$$

The second condition of (10) yields

$$q_2(t)+q_1(t)+e^{-t/\theta}\int_0^{y_f}f(z)\,dz+P(0) \qquad (15)$$

$$-e^{-t/\theta}\Big[P(0)+\frac{1}{\theta}\int_0 e^{\eta/\theta}P(\eta)\,d\eta\Big]=0.$$

From the last two equations we get

$$f(y_\Phi)=\frac{1}{\theta v}e^{t/\theta}\Big[\Big(\frac{\theta}{\theta_0}-1\Big)q_1(t)-P(0)\Big]. \qquad (16)$$

The distribution of the potential $\varphi(y, t)$ is obtained from (7) and (11), putting $\varphi(0, t) = 0$:

$$\varphi(y,t)=-\frac{4\pi}{\varepsilon}\Big\{\Big[q_2(t)-P(0)e^{-t/\theta}-\frac{1}{\theta}e^{-t/\theta}\int_0 e^{\eta/\theta}P(\eta)\,d\eta\Big]y$$

$$+e^{-t/\theta}\int_0^{y}(y-z)f(z)\,dz+v\Big[\theta P(0)\,(e^{-\xi/\theta}-e^{-t/\theta})$$

$$+e^{-\xi/\theta}\int_0^{\xi}e^{\eta/\theta}P(\eta)\,d\eta-e^{-t/\theta}\int_0 e^{\eta/\theta}P(\eta)\,d\eta\Big]\Big\}. \qquad (17)$$

We put here $y = y_f$ (i.e., $\xi = 0$), and equate it to the value of the potential from (5) at $x = x_f$; we eliminate $q_2(t)$ with the aid of (15) and obtain

$$\frac{\varepsilon}{4\pi}V(t)-q_1(t)\Big[vt+\frac{\varepsilon}{\varepsilon_0}(L-v_0t)\Big]+v\theta P(0)\Big[1-\frac{t}{\theta}-e^{-t/\theta}\Big]$$

$$-ve^{-t/\theta}\int_0^{t}e^{\eta/\theta}P(\eta)\,d\eta-e^{-t/\theta}\int_0^{y_f}zf(z)\,dz=0. \qquad (18)$$

We multiply all the terms of this equation by $e^{t/\theta}$ and differentiate; with the aid of (6) and (16) we get

$$P(t)=\frac{\alpha TC_0}{S}\Big(\frac{dV}{dt}+\frac{V}{\theta}\Big)+[\alpha T-(\alpha-1)t]i(t)$$

$$+\Big[\alpha\Big(\frac{1}{\theta_0}-\frac{1}{\theta}\Big)(T-t)+\alpha-1\Big]q_1(t), \qquad (19)$$

where $\alpha = (\epsilon/\epsilon_0)\delta$, $T = L/v_0$, $C_0 = \epsilon_0 S/4\pi L$. We express $q_1(t)$ in terms of $i(t)$, integrating Eq. (6) under the initial condition $q_1(0) = 0$:

$$F(t)=-q_1(t)=e^{-t/\theta_0}\int_0^{t}i(t')e^{t'/\theta_0}dt'. \qquad (20)$$

This relation makes it possible F(t) from the experimentally known value of the current i(t). If the external circuit includes an ohmic resistance R, such that $V(t) = RI(t)$, then (19) takes the form

$$\alpha Tt_0\frac{d^2F}{dt^2}+\Big[\alpha T\Big(\frac{t_0}{\theta_0}+\frac{t_0}{\theta}+1\Big)-(\alpha-1)t\Big]\frac{dF}{dt}$$

$$+\Big[\frac{\alpha T}{\theta}\Big(\frac{t_0}{\theta_0}+1\Big)+\Big(\frac{1}{\theta_0}-\frac{\alpha}{\theta}\Big)t-(\alpha-1)\Big]F=P(t), \qquad (21)$$

where $t_0 = RC_0$. From this equation we obtain (for R = 0' all the previously known particular cases[3-5]. When $R \neq 0$, the solution of (21) is expressed in terms of parabolic-cylinder functions. However, if the function P(t) and all the coefficients of (21) are known, then it is more convenient to use numerical integration, using the initial condition

$$F(0)=\frac{dF}{dt}(0)=0.$$

Equation (19) can be written in the form

$$P(t)=\alpha X(t)+\beta Y(t)+Z(t), \qquad (22)$$

where $\beta = \alpha/\theta = \lambda \cdot 4\pi\delta/\epsilon_0$ and

$$X(t)=\frac{TC_0}{S}\frac{dV}{dt}+(T-t)i+\Big(\frac{T-t}{\theta_0}+1\Big)q_1,$$

$$Y(t)=\frac{TC_0}{S}V-(T-t)q_1$$

$$Z(t)=ti-q_1. \qquad (23)$$

In (19) and (22), the external load can be of most general form. Let us assume that two experiments can be performed, in which the states of the matter behind the SWF were the same, but the parameters of the external circuit and the time T were different. In each of the experiments we determine a different set of functions X(t), Y(t), and Z(t). We apply relation (22) to each of these measurements, obtain their difference, and eliminate thereby the function P(t):

$$\psi(t)=\alpha\Delta X+\beta\Delta Y+\Delta Z=0, \qquad (24)$$

where $\Delta X = X_1 - X_2$, $\Delta Y = Y_1 - Y_2$, and $\Delta Z = Z_1 - Z_2$ are functions of time and are known from experiment (it is assumed that the parameters of the substance in the uncompressed state and the degrees of compression on the SWF are known). To determine the constants α and β we can use the method of least squares. Then α and β are chosen from the condition that the integral

$$H(\alpha,\beta)=\int_0^{t}\psi^2dt$$

have the minimum value. We set up the equations

$$\partial H/\partial\alpha=0,\qquad \partial H/\partial\beta=0,$$

from which we get

$$\alpha=A^{-1}\Big\{\int_0^{t}\Delta X\,\Delta Y dt\int_0^{t}\Delta Y\,\Delta Z dt-\int_0^{t}\Delta X\,\Delta Z dt\int_0^{t}(\Delta Y)^2dt\Big\},$$

$$\beta=A^{-1}\Big\{\int_0^{t}\Delta X\,\Delta Y dt\int_0^{t}\Delta X\,\Delta Z dt-\int_0^{t}\Delta Y\,\Delta Z dt\int_0^{t}(\Delta X)^2dt\Big\},$$

$$A=\int_0^{t}(\Delta X)^2dt\int_0^{t}(\Delta Y)^2dt-\Big(\int_0^{t}\Delta X\,\Delta Y dt\Big)^2. \qquad (25)$$

After determining the parameters α and β, we reconstruct the function P(t) from (22). An estimate of the accuracy can in this case be, for example, the difference of the values of P(t) obtained in the first and second measurements. If some additional hypotheses predict an analytic form of P(t) with a certain number of parameters, for example, $P(t) = P_0 e^{-t/\tau}$, then if one measurement is sufficient for the determination of P_0, τ, α, and β, if we bear in mind the least-squares method. However, the system of equations obtained in this case is nonlinear with respect to the sought quantities, and its solution is difficult.

If we make not two but three measurements, in which the state of the matter behind the SWF is the same, then the relation (22) leads to a system of three linear equations, from which α, β, and P(t) are determined uniquely. The criterion for the reliability of the results can in this case, in particular, be the changes in $\alpha(t)$ and $\beta(t)$.

We now consider the stage $t > T$, when the SWF has already reached the right-hand plate. The region of applicability of (7) then remains constant: $0 < y < y_0 = vT = L/\delta$. We multiply both halves of (8) by $e^{t/\theta}$ and integrate with respect to time from T to t:

$$\rho_1(y,t) = w(y)e^{-t/\theta} + \frac{1}{\theta v}[P(T - y/v)e^{-(t-T)/\theta} - P(\xi)] + \frac{1}{\theta^2 v}e^{-\xi/\theta}\int_{T-y/v}^{\xi} e^{\eta/\theta}P(\eta)\,d\eta \quad (\xi = t - y/v), \tag{26}$$

where $w(y) = \rho_1(y, T)e^{T/\theta}$. Putting $t = T$ in (9), we obtain w(y) and substitute in (26); as a result we find that (9) is valid also for $t > T$. Inasmuch as the first of the boundary conditions (10) is valid for all instants of time, formulas (11) and (17) are valid also for $t > T$. We put $y = y_0$ in (17); the difference of the potentials between the plates is

$$V(t) = \varphi(y_0, t) = -\frac{4\pi}{\varepsilon}\Big\{y_0\Big[q_2(t) - P(0)e^{-t/\theta} - \frac{1}{\theta}e^{-t/\theta}\int_0^t e^{\eta/\theta}P(\eta)\,d\eta\Big] + e^{-t/\theta}\int_0^{y_0}(y_0 - z)f(z)\,dz + \theta vP(0)(e^{T/\theta} - 1)e^{-t/\theta} + ve^{-t/\theta}\Big[e^{T/\theta}\int_0^{t-T}e^{\eta/\theta}P(\eta)\,d\eta - \int_0^t e^{\eta/\theta}P(\eta)\,d\eta\Big]\Big\}. \tag{27}$$

The condition for the conservation of a charge on the left-hand plate leads,

$$dq_2/dt = \iota - j(y = 0, t)$$

when account is taken of (7) and (10), to the equation

$$\frac{dq_2}{dt} + \frac{q_2}{\theta} = i(t) + \frac{P(t)}{\theta}. \tag{28}$$

We multiply both halves of (27) by $e^{t/\theta}$ and differentiate, using (28). We then obtain in lieu of (19), for $t > T$,

$$\frac{\alpha TC_0}{S}\Big(\frac{dV}{dt} + \frac{V}{\theta}\Big) + T\iota(t) = P(t) - P(t - T). \tag{29}$$

Relation (29) is useful, in particular, because at sufficiently T the first term in the right side becomes negligibly small, owing to relaxation, compared with the second term. This makes it possible to determine the initial value of P(t) by measuring the current i(t) and voltage V(t) during a later stage of the experiment.

The author thanks A. G. Ivanov, Yu. V. Lisytsin, V. N. Mineev, and E. Z. Novitskiĭ for a discussion of the results.

[1] A. G. Ivanov, E. Z. Novitskiĭ, V. N. Mineev, Yu. V. Lisytsin, Yu. N. Tyunyaev, and V. I. Bezrukov, Zh. Eksp. Teor. Fiz. 53, 41 (1967) [Sov. Phys.-JETP 26, 28 (1968)].

[2] A. G. Ivanov and E. Z. Novitskiĭ, PMTF (Appl. Math. and Theor. Phys. 5, 104 (1966).

[3] F. E. Allison, J. Appl. Phys. 36, 2111 (1965).

[4] Ya. B. Zel'dovich, Zh. Eksp. Teor. Fiz. 53, 237 (1967) [Sov. Phys.-JETP 26, 159 (1968)].

[5] A. G. Ivanov, Yu. V. Lisitsyn, and E. Z. Novitskiĭ, ibid. 54, 285 (1968) [27, 153 (1968)].

Translated by J. G. Adashko
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